

### 5. Some Solutions of the NLS-type Equations

We consider the Nonlinear Schrödinger equation in its standard form,

$$i \frac{\partial A}{\partial \tau} + \lambda_1 \frac{\partial^2 A}{\partial \xi^2} - \nu_1 |A|^2 A = 0, \tag{97}$$

where  $A$  is the envelope of the free surface elevation  $\zeta(x, t)$ ,  $\lambda_1 = \omega''(k)/2 < 0$  and  $\nu_1 > 0$  for  $kh > 1.363$ . For  $\nu_1 > 0$ , so-called soliton solutions are possible, while for  $\nu_1 < 0$ , decaying solution can be found.

#### 5.1. Decaying solutions

A specific decaying solution with an oscillatory tail was given by Benney and Newell (1967), see also Segur and Ablowitz (1976). This decaying solution of Eq. (97) reads,

$$A(\xi, \tau) = (-\lambda_1 \tau)^{-\frac{1}{2}} \left( \frac{\lambda_1}{\nu_1} \right)^{\frac{1}{2}} \Lambda \exp \left[ -i \left( -\frac{\xi^2}{4\lambda_1 \tau} + |\Lambda|^2 \log(-\lambda_1 \tau) + \phi \right) \right], \tag{98}$$

where  $\Lambda$  and  $\phi$  are constants. Segur and Ablowitz take this form of solution as starting point for the case of the NLS equation for uneven bottom and they let the constants  $\Lambda$  and  $\phi$  then be slowly varying in  $\xi$  and  $\tau$ .

In 2D, a decaying solution which satisfies Eqs. (42) reads,

$$B(\xi, \eta, \tau) = \frac{\Lambda}{\tau} \exp \left[ i \left\{ \frac{\xi^2 + \frac{\eta^2}{\mu_1}}{4\tau} + \chi_1 \frac{\Lambda^2}{\tau} + \chi_2^2 b(\tau) + \theta \right\} \right], \tag{99a}$$

$$\phi^{(1,0)}(\xi, \eta, \tau) = -\chi_2 \frac{\partial b}{\partial \tau} \xi + c(\tau)\eta + d(\tau). \tag{99b}$$

For scaled versions of Eqs. (97) and (42), Ablowitz and Segur (1979) gave similar solutions.

#### 5.2. Soliton-type solutions

We now consider the case  $\nu_1 > 0$  so that soliton-type solutions are possible. The free-surface elevation is given by:

$$\zeta(x, t) = \frac{1}{2} A e^{i(k_0 x - \omega_0 t)} + CC. \tag{100}$$

Since we look for stationary solutions, we put,

$$A(\xi, \tau) = b(X)e^{i(\ell\xi - s\tau)} \quad \text{where} \quad X = \xi - v\tau, \tag{101}$$

is a moving frame with respect to the frame  $(\xi, \tau)$  and  $v$  and  $\ell$  are constants. Substituting Eq. (101) in the NLS equation (97) leads to an ordinary differential equation in  $b(X)$ ,

$$\lambda_1 \frac{\partial^2 b}{\partial X^2} + \alpha b - \nu_1 b^3 = 0 \quad \text{with} \quad \alpha = s - \frac{v^2}{4\lambda_1}, \tag{102}$$

and where has been substituted  $\ell = v/(2\lambda_1)$  in order to obtain real solutions for the amplitude  $b$ .

Imposing the condition that  $b(X)$  and  $db(X)/dx \rightarrow 0$  for  $X \rightarrow \mp\infty$ , the following solution is obtained (for details, see Dingemans, 1997, p. 919),

$$A(\xi, \tau) = a_0 \operatorname{sech} \left[ \left( -\frac{\nu_1}{\lambda_1} \right)^{1/2} a_0 X \right] \exp \left[ i \frac{v}{2\lambda_1} \xi - i \left( \frac{1}{2} \nu_1 a_0^2 + \frac{v^2}{4\lambda_1} \right) \tau \right], \tag{103}$$

where  $v$  and the amplitude  $a_0$  are still two free parameters. The parameter  $v$  is usually taken to be zero. Notice that this solution is valid only under the condition that  $\nu_1 \lambda_1 < 0$ , or, otherwise stated,  $\nu_1 > 0$  and thus  $kh > 1.363$ .

For the case that period conditions on the amplitude  $b$  are imposed, more possible stationary solutions are found. The conditions imposed are now:  $db/dX \rightarrow 0$  for  $b \rightarrow a_0$  as  $X \rightarrow X_\ell$ . With the notation  $r = b^2$ , the differential equation is:

$$\left( \frac{\partial r}{\partial X} \right)^2 = 2 \frac{\nu_1}{\lambda_1} r(r-r_0)(r-r_3) \quad \text{with} \quad r_0 = a_0^2 \quad \text{and} \quad r_3 = 2 \frac{\alpha}{\nu_1} - r_0. \tag{104}$$

In this case three cases for viable solutions have to be considered: (1)  $\nu_1 > 0$  and  $r_3 > 0$  leading to the dn-solution, (2)  $\nu_1 > 0$  and  $r_3 < 0$  leading to the cn-solution, and (3)  $\nu_1 < 0$  and  $r_3 > 0$ , giving the sn-solution. These solutions are:

**(1)  $\nu_1 > 0$  and  $r_3 > 0$**

$$A(\xi, \tau) = b_3 \operatorname{dn} \left[ \left( -\frac{r_3 \nu_1}{2\lambda_1} \right)^{1/2} X \middle| m \right] \cdot \exp \left[ i \frac{v}{2\lambda_1} \xi - i \left( \frac{1}{2} \nu_1 r_3 (2 - m) + \frac{v^2}{4\lambda_1} \right) \tau \right], \tag{105a}$$

$$\text{with } m = \frac{r_3 - r_0}{r_3} = \frac{\frac{2\alpha}{\nu_1} - 2r_0}{\frac{2\alpha}{\nu_1} - r_0} < 1. \tag{105b}$$

**(2)  $\nu_1 > 0$  and  $r_3 < 0$**

$$A(\xi, \tau) = a_0 \text{cn} \left[ \left( -\frac{a_0^2 \nu_1}{2m\lambda_1} \right)^{1/2} X \middle| m \right] \cdot \exp \left[ i \frac{v}{2\lambda_1} \xi - i \frac{1}{2} a_0^2 \nu_1 \left( \frac{1}{m} + 1 + \frac{v^2}{4\lambda_1 \nu_1 a_0^2} \right) \tau \right], \tag{106a}$$

$$\text{with } m = \left( 2 - \frac{2\alpha\lambda_1}{\nu_1 a_0^2} \right)^{-1}. \tag{106b}$$

**(3)  $\nu_1 < 0$  and  $r_3 > 0$**

$$A(\xi, \tau) = a_0 \text{sn} \left[ \mp \left( \frac{r_3 \nu_1}{2\lambda_1} \right)^{1/2} X \middle| m \right] \cdot \exp \left[ i \frac{v}{2\lambda_1} \xi - i \frac{1}{2} a_0^2 \nu_1 \left( \frac{1}{m} + 1 + \frac{v^2}{4\lambda_1 \nu_1 a_0^2} \right) \tau \right], \tag{107a}$$

$$\text{with } m = \left( \frac{2s}{a_0^2 \nu_1} - 1 - \frac{v^2}{4\lambda_1 \nu_1 a_0^2} \right)^{-1}, r_3 = \frac{a_0^2}{m} - \frac{v^2}{4\lambda_1 \nu_1}. \tag{107b}$$

Because for  $m \rightarrow 1$ , we have  $\text{dn} \rightarrow \text{sech}$ ,  $\text{sn} \rightarrow \tanh$  and  $\text{cn} \rightarrow \text{sech}$ , we see that the limiting values for  $m \rightarrow 1$  of the solutions (105)–(107) are:

$$A(\xi, t) = b_3 \text{sech} \left[ \left( -\frac{\alpha}{\nu_1 \lambda_1} \right)^{1/2} X \right] \exp \left[ i \frac{v}{2\lambda_1} \xi - is\tau \right],$$

$$\text{with } r_3 = b_3^2 = 2 \frac{\alpha}{\nu_1} > 0 \text{ and } \nu_1 > 0, \tag{108}$$

$$A(\xi, t) = a_0 \text{sech} \left[ \left( -\frac{\nu_1 a_0^2}{2\lambda_1} \right)^{1/2} X \right] \exp \left[ i \frac{v}{2\lambda_1} \xi - i \left( \nu_1 a_0^2 + \frac{v}{8\lambda_1} \right) \tau \right],$$

$$\text{with } \alpha < 0, \tag{109}$$

$$A(\xi, t) = a_0 \tanh \left[ \left( \frac{\alpha}{\lambda_1} \right)^{1/2} X \right] \exp \left[ i \frac{v}{2\lambda_1} \xi - i \left( \nu_1 a_0^2 + \frac{v^2}{8\lambda_1} \right) \tau \right],$$

$$\text{with } \nu_1 < 0 \text{ and } \alpha < 0. \tag{110}$$

Some examples of these solutions have been considered in Dingemans (1997, p. 925). We replot solutions for the dn, cn and sn-solutions given there.

In two dimensions, let us consider the deep-water case Eq. (61) because of its simplicity. In nondimensional quantities such that  $\omega = 1$  and  $\mathbf{k} = (\ell, m)^T = (1, 0)^T$  and  $g = 1$ , we then have,

$$2i \frac{\partial A}{\partial \tau} - \frac{1}{4} \left( \frac{\partial^2 A}{\partial \xi^2} - 2 \frac{\partial^2 A}{\partial \eta^2} \right) = |A|^2 A. \quad (111)$$

We note that for NLS-type of equations for water waves, the coordinate along the propagation direction  $\xi$  and the lateral one  $\eta$  are not interchangeable.

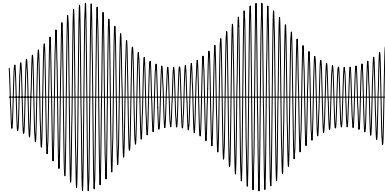


Fig. 2. A dn-solution.

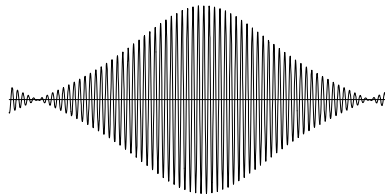


Fig. 3. A cn-solution.

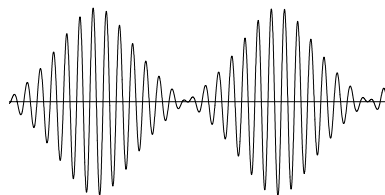


Fig. 4. A sn-solution.

Several special solutions of Eq. (111) have been given by Hui and Hamilton (1979). Denoting the angle between the direction of the carrier wave and the direction for which solutions are sought for by  $\vartheta$ , the  $\xi, \eta$  plane is split into regions according to the sign of the quantity  $\psi$  defined by:

$$\psi = \cos^2 \vartheta - 2 \sin^2 \vartheta. \quad (112)$$

For  $\psi > 0$  ( $\tan^2 \vartheta < 1/2$ ), solutions for the group envelope in terms of the elliptic functions  $\text{dn}$  and  $\text{cn}$  always exist, i.e., groups of permanent waves and of infinite extent exist, which also vary periodically in space and time. The common limit ( $m \rightarrow 1$ ) is the sech profile.

For the case that  $\psi \leq 0$ , the situation is much more complicated. In the critical direction  $\vartheta_c$  such that  $\tan^2 \vartheta_c = 1/2$  (or  $\psi = 0$ ), only constant amplitude plane waves are possible. For the case  $\psi < 0$ , we refer to Hui and Hamilton (1979).

## 6. Higher-Order Modulation Equations

The equations discussed in the previous sections govern modulation of gravity waves valid up to  $O(\varepsilon^3)$ . These equations have been found to be capable of producing several broad features of nonlinear modulation. However, comparisons with experiments have also revealed features like deviation of the predicted growth rate of unstable modes for steeper waves ( $\varepsilon > 0.15$ ) and asymmetric growth which lie beyond the NLS approach. These limitations of the NLS equation have drawn attention to the necessity of higher-order modulation. In this section, we will first discuss a higher-order modulation due to Dysthe (1979) which is valid only in deep water. Modification of this set of equations due to an ambient current will be treated following the recent work of Stocker and Peregrine (1999). Finally, the section will be closed by a description of “the Zakharov equation”. Zakharov’s set has two distinct features of being derived from an alternative principle and being more general, encompassing Dysthe’s equation as a special case.

### 6.1. The Dysthe equation

To express the potential and free surface elevation, we use the form (Dysthe’s form has been modified slightly for consistency),

$$\zeta = \bar{\zeta} + \frac{1}{2} [Ae^{i\vartheta} + A_2e^{i2\vartheta} + \dots + CC], \quad (113a)$$