

Several special solutions of Eq. (111) have been given by Hui and Hamilton (1979). Denoting the angle between the direction of the carrier wave and the direction for which solutions are sought for by ϑ , the ξ, η plane is split into regions according to the sign of the quantity ψ defined by:

$$\psi = \cos^2 \vartheta - 2 \sin^2 \vartheta. \quad (112)$$

For $\psi > 0$ ($\tan^2 \vartheta < 1/2$), solutions for the group envelope in terms of the elliptic functions dn and cn always exist, i.e., groups of permanent waves and of infinite extent exist, which also vary periodically in space and time. The common limit ($m \rightarrow 1$) is the sech profile.

For the case that $\psi \leq 0$, the situation is much more complicated. In the critical direction ϑ_c such that $\tan^2 \vartheta_c = 1/2$ (or $\psi = 0$), only constant amplitude plane waves are possible. For the case $\psi < 0$, we refer to Hui and Hamilton (1979).

6. Higher-Order Modulation Equations

The equations discussed in the previous sections govern modulation of gravity waves valid up to $O(\varepsilon^3)$. These equations have been found to be capable of producing several broad features of nonlinear modulation. However, comparisons with experiments have also revealed features like deviation of the predicted growth rate of unstable modes for steeper waves ($\varepsilon > 0.15$) and asymmetric growth which lie beyond the NLS approach. These limitations of the NLS equation have drawn attention to the necessity of higher-order modulation. In this section, we will first discuss a higher-order modulation due to Dysthe (1979) which is valid only in deep water. Modification of this set of equations due to an ambient current will be treated following the recent work of Stocker and Peregrine (1999). Finally, the section will be closed by a description of “the Zakharov equation”. Zakharov’s set has two distinct features of being derived from an alternative principle and being more general, encompassing Dysthe’s equation as a special case.

6.1. The Dysthe equation

To express the potential and free surface elevation, we use the form (Dysthe’s form has been modified slightly for consistency),

$$\zeta = \bar{\zeta} + \frac{1}{2} [Ae^{i\vartheta} + A_2e^{i2\vartheta} + \dots + CC], \quad (113a)$$

$$\Phi = \bar{\Phi} + \frac{1}{2}[B e^{kz} e^{i\vartheta} + B_2 e^{2kz} e^{i2\vartheta} + \dots + CC], \tag{113b}$$

with $\bar{\Phi}$ and $\bar{\zeta}$ denoting the potential and elevation respectively of the slowly varying mean flow and where $\vartheta = \mathbf{k} \cdot \mathbf{x} - \omega t$ and $k = |\mathbf{k}|$. The governing equations for the modulation of B corresponding to Eqs. (2.19), (2.20) and (2.10) of Dysthe (1979) are given in dimensional variables by:

$$\begin{aligned} i \left(\frac{\partial B}{\partial t} + \frac{\omega}{2k} \frac{\partial B}{\partial x} \right) - \frac{\omega}{8k^2} \frac{\partial^2 B}{\partial x^2} + \frac{\omega}{4k^2} \frac{\partial^2 B}{\partial y^2} - \frac{k^4}{2\omega} |B|^2 B = \\ - \frac{i}{16} \frac{\omega}{k^3} \left(6 \frac{\partial^3 B}{\partial x \partial y^2} - \frac{\partial^3 B}{\partial x^3} \right) + i \frac{3k^3}{4\omega} B \left(B \frac{\partial B^*}{\partial x} - B^* \frac{\partial B}{\partial x} \right) \\ - i \frac{k^3}{4\omega} |B|^2 \frac{\partial B}{\partial x} + kB \left(\frac{\partial \bar{\Phi}}{\partial x} - i \frac{\partial \bar{\Phi}}{\partial z} \right) \Big|_{z=0}, \end{aligned} \tag{114a}$$

$$\frac{\partial \bar{\Phi}}{\partial t} + g \bar{\zeta} = 0 \quad \text{at } z = 0, \tag{114b}$$

$$\frac{\partial \bar{\Phi}}{\partial z} - \frac{\partial \bar{\zeta}}{\partial t} = \frac{\omega}{2g} \mathbf{k} \cdot \nabla (|B|^2) \quad \text{at } z = 0. \tag{114c}$$

Equation (114a) incorporates the correction of a misprint in the original Eq. (2.19) of Dysthe (1979) as pointed out by Janssen (1983). Another misprint appearing in Eq. (2.17) of the same article is the factor 3 of the second term of Γ which should be 8 and has been noted by Brinch-Nielsen and Jonsson (1986).

The terms on the left-hand side of the evolution equation (114a) are all of $\mathcal{O}(ka)^3$ while the terms on the right-hand side are all of fourth order. In other words, the usual NLS equation for deep water is retrieved if the higher-order correction terms contained in the right-hand side are set equal to zero. As $\bar{\zeta}$ is of third order, the term $\partial \bar{\zeta} / \partial t$ in Eq. (114c) may be neglected. This simplifies the substitution of $\partial \bar{\Phi} / \partial z$ in Eq. (114a). In that case, Eq. (114a) reduces to:

$$\begin{aligned} i \left(\frac{\partial B}{\partial t} + \frac{\omega}{2k} \frac{\partial B}{\partial x} \right) - \frac{\omega}{8k^2} \frac{\partial^2 B}{\partial x^2} + \frac{\omega}{4k^2} \frac{\partial^2 B}{\partial y^2} - \frac{k^4}{2\omega} |B|^2 B = \\ - \frac{i}{16} \frac{\omega}{k^3} \left(6 \frac{\partial^3 B}{\partial x \partial y^2} - \frac{\partial^3 B}{\partial x^3} \right) + i \frac{k^3}{4\omega} B \left(B \frac{\partial B^*}{\partial x} - 6B^* \frac{\partial B}{\partial x} \right) + kB \frac{\partial \bar{\Phi}}{\partial x} \Big|_{z=0}, \end{aligned} \tag{115}$$

which is also identical to Eq. (10) of Trulsen and Dysthe (1996). As also discussed in Lo and Mei (1985), these equations are derived under the condition that $kh = \mathcal{O}\{(ka)^{-1}\} \ll 1$. Lo and Mei argue that the equations also remain valid for $kh = \mathcal{O}\{(ka)^{-1}\}$ when also the condition $\partial\bar{\Phi}/\partial z = 0$ at $z = -h$ is added.

6.2. Modification due to an ambient current

Modification to Dysthe’s equation in the presence of an ambient current is considered by Stocker and Peregrine (1999). In addition to the wave-induced mean flow, denoted by $\bar{\Phi}(\mathbf{x}, z, t)$ and $\bar{\zeta}$, we now introduce $\Phi_c(\mathbf{x}, t)$ and ζ_c to represent the potential and elevation due to an ambient current. Thus, the expressions for potential and elevation are:

$$\zeta = \bar{\zeta} + \zeta_c + \frac{1}{2}[Ae^{i\vartheta} + A_2e^{i2\vartheta} + \dots + CC], \tag{116a}$$

$$\Phi = \bar{\Phi} + \Phi_c + \frac{1}{2}[Be^{kz}e^{i\vartheta} + B_2e^{2kz}e^{i2\vartheta} + \dots + CC]. \tag{116b}$$

Assuming further that the ambient current \mathbf{U} has a slow variation, the current field may be expressed as a perturbation about a constant (both spatial and temporal) mean $-\mathbf{V}$, i.e., $\mathbf{U} = -\mathbf{V} + (U_1(x, t), V_1(x, t))^T$.

In a frame of reference moving with the mean velocity $-\mathbf{V}$, the current modified higher-order equations read,

$$\begin{aligned} & i \left[\frac{\partial}{\partial t} + c_{g0} \frac{\partial}{\partial x} - \mathbf{V} \cdot \nabla_h \right] B - \frac{(gk_0)^{1/2}}{8k_0^2} (B_{xx} - 2B_{yy}) \\ & - \frac{k_0^4}{2(gk_0)^{1/2}} B|B|^2 - k_0 \Phi_{cx} B = i \frac{k_0^3}{4(gk_0)^{1/2}} B (BB_x^* - 6B_x B^*) \\ & + i \frac{(gk_0)^{1/2}}{16k_0^3} (B_{xxx} - 6B_{yyx}) + k_0 B \bar{\Phi}_x \Big|_{z=0} + P_1, \end{aligned} \tag{117}$$

where the term P_1 contains the higher-order contribution due to the current and is given by:

$$P_1 = ik_0 \left(\frac{1}{2(gk_0)^{1/2}} \left[\frac{\partial}{\partial t} + c_{g0} \frac{\partial}{\partial x} - \mathbf{V} \cdot \nabla \right] \Phi_{cx} - \Phi_{cz} \right) B - i \nabla \Phi \cdot \nabla B. \tag{118}$$

To complete the system, the current potential Φ_c and the wave-induced mean potential $\bar{\Phi}$ need to be defined. The current potential is determined by:

$$\nabla^2 \Phi_c = 0; \quad z \leq 0, \tag{119}$$

$$\left[\frac{\partial}{\partial t} - \mathbf{V} \cdot \nabla \right] \Phi_c + g\zeta_c = 0; \quad z = 0, \quad (120)$$

$$\Phi_{c,z} - \left[\frac{\partial}{\partial t} - \mathbf{V} \cdot \nabla \right] \zeta_c = 0; \quad z = 0, \quad (121)$$

while the wave-induced mean flow valid up to $O(\varepsilon^4)$ is given by:

$$\nabla^2 \bar{\Phi} + \frac{\partial^2 \bar{\Phi}}{\partial z^2} = 0; \quad z \leq 0, \quad (122)$$

$$\left[\frac{\partial}{\partial t} - \mathbf{V} \cdot \nabla \right] \bar{\Phi} + g\bar{\zeta} = 0; \quad z = 0, \quad (123)$$

$$\frac{\partial \bar{\Phi}}{\partial z} - \frac{k_0}{2} \left(\frac{k_0}{g} \right)^{1/2} \frac{\partial |B|^2}{\partial x} = 0; \quad z = 0. \quad (124)$$

As seen from Eq. (117) B , the amplitude of the velocity potential, depends explicitly on the external current. However, the surface elevation terms are related to B in an identical manner as that in Brinch-Nielsen (1988) and Brinch-Nielsen and Jonsson (1986) without any explicit dependence on the current U .

To evaluate the performance of the $O(\varepsilon^4)$ -modulation equation, Stocker and Peregrine (1999) have undertaken numerical solution for a specific case of waves being modulated by a sinusoidally (of much longer wave length) varying current. Computed profiles from both a lower-order ($O(\varepsilon^3)$) and the higher-order models are compared with those from an exactly nonlinear boundary integral model. Comparison shows that starting from the identical initial condition, the evolution predicted from the lower-order NLS equation kept deviating with time. Significant improvement was achieved by adopting the $O(\varepsilon^4)$ -nonlinear equation. Agreement with the fully nonlinear solution was good till about breaking (breaking was said to occur if sharp curvature appeared on the computed surface in the exactly nonlinear model) was initiated.

6.3. *The Zakharov equation*

That the NLS equation is a special case of the Zakharov equation has been proven by Stiassnie (1984) for the case of deep water. Originally, Zakharov (1968) derived a deep-water evolution equation for the amplitude of a wave field. A little later, the equations for restricted depth were given by Zakharov and Kharitanov (1970), still for horizontal bottom.

We first sketch the steps along which the Zakharov equation can be obtained. Here, we follow Stiassnie and Shemer (1984), see also Shemer and Stiassnie (1991), where the bottom has been assumed to be horizontal. A detailed account can also be found in Rasmussen (1998).

The kinematic and dynamic free-surface conditions are written in terms of the free-surface potential $\varphi(\mathbf{x}, t) = \Phi\{\mathbf{x}, z = \zeta(\mathbf{x}, t), t\}$, the vertical velocity $w^s = (\partial\Phi/\partial z)|_{z=\zeta}$ and have been given in Eqs. (3). Together with the Laplace equation $\nabla^2\Phi = 0$ and the kinematic bottom condition $\partial\Phi/\partial z = 0$ at $z = -h$, these equations constitute the description of the physical problem. We stress the fact that in this derivation, the bottom is taken to be horizontal. The derivation of the Zakharov equation proceeds in the following steps,

- (1) The horizontal Fourier transform of Eqs. (3) yields two integro-differential equations for the Fourier transforms $\hat{\zeta}$ and $\hat{\varphi}$ where the Fourier transform of a function $f(\mathbf{x})$ is defined by:

$$\hat{f}(\mathbf{k}) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\mathbf{x} f(\mathbf{x}) e^{-i\mathbf{k}\cdot\mathbf{x}}, \tag{125a}$$

and the delta function is defined as:

$$\delta(\mathbf{k}) = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} d\mathbf{x} e^{i\mathbf{k}\cdot\mathbf{x}}. \tag{125b}$$

In addition to the variables $\hat{\zeta}$ and $\hat{\varphi}$, \hat{w}^s also features in the transformed form of Eqs. (3). An expression for \hat{w}^s has to be found. This is achieved in the following steps.

- (2) Taking the horizontal Fourier transform of the Laplace equation and satisfying subsequently the bottom condition yields a separation of the vertical structure in the following way,

$$\hat{\Phi}(\mathbf{k}, z, t) = \hat{\varphi}(\mathbf{k}, t) \cosh[|\mathbf{k}|(z + h)]; \tag{126}$$

this makes it possible to express the free-surface variables φ and w^s in terms of $\hat{\varphi}(\mathbf{k}, t)$ and $\zeta(\mathbf{x}, t)$ in the following way,

$$\begin{aligned} \varphi(\mathbf{x}, t) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} d\mathbf{k} [\cosh[|\mathbf{k}|h] \cosh[|\mathbf{k}|\zeta(\mathbf{x}, t)] \\ &\quad + \sinh[|\mathbf{k}|h] \sinh[|\mathbf{k}|\zeta(\mathbf{x}, t)]] e^{i\mathbf{k}\cdot\mathbf{x}}, \end{aligned} \tag{127a}$$

$$w^s(\mathbf{x}, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\mathbf{k} [|\mathbf{k}| \hat{\phi} \cosh[|\mathbf{k}|h] \sinh[|\mathbf{k}|\zeta(\mathbf{x}, t)] + \sinh[|\mathbf{k}|h] \cosh[|\mathbf{k}|\zeta(\mathbf{x}, t)]] e^{i\mathbf{k}\cdot\mathbf{x}}. \tag{127b}$$

(3) The next step is to express w^s in terms of $\hat{\zeta}$ and $\hat{\phi}$. It is here that the first approximations have to be made. The expressions $\sinh(|\mathbf{k}|\zeta)$ and $\cosh(|\mathbf{k}|\zeta)$ are replaced by their Taylor expansions up to $\mathcal{O}\{(|\mathbf{k}|\zeta)^3\}$; ζ is expressed by its Fourier transform $\hat{\zeta}$. Finally the Fourier transform of Eq. (127) is considered. This yields two equations in which $\hat{\phi}$ and \hat{w}^s are expressed in terms of $\hat{\phi}$ and $\hat{\zeta}$. An iterative solution of the equation for $\hat{\phi}$ is applied to obtain $\hat{\phi}$ as a function of $\hat{\phi}$ and the subsequent use of $\hat{\phi}(\hat{\phi})$ in the equation for \hat{w}^s yields an expression for \hat{w}^s in terms of $\hat{\zeta}$ and $\hat{\phi}$. This expression for \hat{w}^s is now used in the Fourier transform of the free-surface equations (3). Multiplying the equation for $\hat{\zeta}$ by $\sqrt{g/(2\omega(\mathbf{k}))}$ and multiplying the equation for $\hat{\phi}$ by $\sqrt{\omega(\mathbf{k})/(2g)}$ and adding these two equations together, the result is an evolution equation for the complex variable,

$$b(\mathbf{k}, t) = \left[\frac{g}{2\omega(\mathbf{k})} \right]^{1/2} \hat{\zeta}(\mathbf{k}, t) + i \left[\frac{\omega(\mathbf{k})}{2g} \right]^{1/2} \hat{\phi}(\mathbf{k}, t), \tag{128}$$

where the dispersion relation is:

$$\omega = [g|\mathbf{k}| \tanh(|\mathbf{k}|h)]^{1/2}. \tag{129}$$

The evolution equation for $b(\mathbf{k}, t)$ then is:

$$\begin{aligned} \frac{\partial b}{\partial t}(\mathbf{k}, t) + i\omega(\mathbf{k})b(\mathbf{k}, t) + i \sum_{n=1}^3 \iint_{-\infty}^{\infty} d\mathbf{k}_1 d\mathbf{k}_2 V^{(n)}(\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2) C_{2n} \\ + i \sum_{n=1}^4 \iiint_{-\infty}^{\infty} d\mathbf{k}_1 d\mathbf{k}_2 d\mathbf{k}_3 W^{(n)}(\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) C_{3n} \\ + \sum_{n=1}^5 \iiiii_{-\infty}^{\infty} d\mathbf{k}_1 d\mathbf{k}_2 d\mathbf{k}_3 d\mathbf{k}_4 X^{(n)}(\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \mathbf{k}_4) C_{4n} = 0, \end{aligned} \tag{130}$$

with the $C_{\ell n}$ given by:

$$C_{\ell n} = \left(\prod_{m=1}^{n-1} b^*(\mathbf{k}_m, t) \right) \left(\prod_{m=n}^{\ell} b(\mathbf{k}_m, t) \right) \cdot \delta \left(\mathbf{k} + \sum_{m=1}^{n-1} \mathbf{k}_m - \sum_{m=n}^{\ell} \mathbf{k}_m \right), \tag{131}$$

with $*$ denoting the complex conjugate and $\sum_{m=n}^{\ell}(\cdot) = 0$ and $\prod_{m=n}^{\ell}(\cdot) = 1$ whenever $\ell < n$.

Notice that $\hat{\zeta}$ and $\hat{\varphi}$ can be expressed in terms of b as:

$$\hat{\zeta}(\mathbf{k}, t) = \left| \frac{\omega(\mathbf{k})}{2g} \right|^{1/2} [b(\mathbf{k}, t) + b^*(-\mathbf{k}, t)], \tag{132a}$$

$$\hat{\varphi}(\mathbf{k}, t) = -i \left| \frac{g}{2\omega(\mathbf{k})} \right|^{1/2} [b(\mathbf{k}, t) - b^*(-\mathbf{k}, t)]. \tag{132b}$$

(4) We now use the transformation,

$$B(\mathbf{k}, t) = b(\mathbf{k}, t)e^{-i\omega(\mathbf{k})t}. \tag{133}$$

The term $i\omega b$ then disappears from Eq. (130) while for b can be read B in Eq. (130). The next assumption now is that we suppose that B is composed of a (in time) slowly-varying part \tilde{B} and faster varying parts B' , B'' and B''' ,

$$B(\mathbf{k}, t) = \varepsilon \tilde{B}(\mathbf{k}, t_2, t_3) + \varepsilon^2 B'(\mathbf{k}, t, t_2, t_3) + \varepsilon^3 B''(\mathbf{k}, t, t_2, t_3) + \varepsilon^4 B'''(\mathbf{k}, t, t_2, t_3), \tag{134}$$

where $t_j = \varepsilon^j t$, $j = 2, 3$. The slow time t_1 is omitted because no exact resonance between three waves is possible. When surface tension is included, exact resonance is possible for three-wave interaction, but, these waves are very short and not of interest for us here. It is assumed also that most of the wave energy is contained in \tilde{B} . The representation of Eq. (134) is substituted into the evolution equation (130); separating terms of equal power in ε leads to evolution equations for the \tilde{B} and B' , B'' and B''' . For ε^1 , no information is obtained. Terms with ε^2 yield an evolution equation for $\partial B'/\partial t$ which depends on \tilde{B} and therefore can be integrated to t while keeping t_2 and t_3 fixed.

At $\mathcal{O}(\varepsilon^3)$ is obtained an equation for $i\partial_{t_2}\tilde{B} + i\partial_t B''$ in which the right-hand member depends on terms with \tilde{B} where both slow and fast-varying terms are present. Separating this equation into a slow and fast-varying part and also writing $B = \varepsilon \tilde{B}$, we obtain the evolution equation,

$$i \frac{\partial B}{\partial t} = \iiint_{-\infty}^{\infty} d\mathbf{k}_1 d\mathbf{k}_2 d\mathbf{k}_3 T_{0,1,2,3}^{(2)} B_1^* B_2 B_3 \delta(\mathbf{k} + \mathbf{k}_1 - \mathbf{k}_2 - \mathbf{k}_3) \times \exp[i(\omega + \omega_1 - \omega_2 - \omega_3)t], \tag{135}$$

where B_j stands for $B(\mathbf{k}_j, t)$. For B'' , an evolution equation is obtained in which the right-hand side does not depend on the slow time-scale so that an integration to t is possible.

Equation (135) is the so-called *Zakharov equation* which is also valid for restricted depth when the dispersion relation $\omega(\mathbf{k})$ and the definition of T^2 are adapted for finite depth.

Once B has been determined, the free-surface elevation $\zeta(\mathbf{x}, t)$ follows by:

$$\zeta(\mathbf{x}, t) = \int_{-\infty}^{\infty} d\mathbf{k} \left(\frac{\omega(\mathbf{k})}{2g} \right)^{1/2} [B(\mathbf{k}, t)e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)} + CC]. \tag{136}$$

A problem with the Zakharov equation is that many different forms for the interaction coefficient $T^{(2)}$ exist. The reason for this is that there is some freedom in the definition of $T^{(2)}$ without changing the value of the integral in Eq. (135). The interaction coefficients can be symmetrised as noted by Stiassnie and Shemer (1984).

The Zakharov equation has been reconsidered by Krasitskii (1994) who showed that previously used forms did not give a truly Hamiltonian system of equations; this had to do with the definition of the interaction coefficients. It appears that in the older form of the Zakharov equation, the coefficients were not sufficiently symmetric. For an extensive discussion of these matters is referred to Krasitskii (1994) and Badulin *et al.* (1995). As put forward by Krasitskii (1994), the symmetry conditions are not clear without considering the Hamiltonian formulation.

Notice that Rasmussen (1998, Eq. (2.61)) writes the Zakharov equation (135) in the form,

$$\frac{\partial B}{\partial t}(\mathbf{k}, t) = -i \iiint d\mathbf{k}_1 d\mathbf{k}_2 d\mathbf{k}_3 X_{0,1,2,3}^{(2)} C'_{3,2}, \tag{137a}$$

with for C' the expression,

$$\begin{aligned} C'_{\ell n} &= \left(\prod_{m=1}^{n-1} b^*(\mathbf{k}_m, t) \right) \left(\prod_{m=n}^{\ell} b(\mathbf{k}_m, t) \right) \cdot \delta \left(\mathbf{k} + \sum_{m=1}^{n-1} \mathbf{k}_m - \sum_{m=n}^{\ell} \mathbf{k}_m \right) \\ &\times \exp \left[i \left(\sum_{m=1}^{n-1} \omega_m - \sum_{m=n}^{\ell} \omega_m \right) t \right]. \end{aligned} \tag{137b}$$

Taking $T^{(2)} = -X^{(2)}$, the same equation as given in Eq. (135) is obtained. For $X_{0,1,2,3}^{(2)} = -T_{0,1,2,3}^{(2)}$, Rasmussen (1998, Eq. (2.60)) gives the expression,

$$X_{0,1,2,3}^{(2)} = \frac{1}{2}(Y_{0,1,2,3}^{(2)} + Y_{0,1,3,2}^{(2)}), \tag{138}$$

whenever both $\mathbf{k} + \mathbf{k}_1 - \mathbf{k}_2 - \mathbf{k}_3 = 0$ and $|\omega + \omega(\mathbf{k}_1) - \omega(\mathbf{k}_2) - \omega(\mathbf{k}_3)| \leq \mathcal{O}(\varepsilon^2)$; otherwise, $X^{(2)} = 0$. The coefficient $Y_{0,1,2,3}^{(2)}$ has been given in Rasmussen (1998, Eq. (A.18)).

Taking the symmetric form of Krasitskii (1994) (in his notation, $T^{(2)}$ is called $\tilde{V}^{(2)}$), the underlying system is a Hamiltonian system and we have the property that (see also Badulin *et al.*, 1995):

$$\begin{aligned} T^{(2)}(\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) &= T^{(2)}(\mathbf{k}_1, \mathbf{k}, \mathbf{k}_2, \mathbf{k}_3) = T^{(2)}(\mathbf{k}_1, \mathbf{k}, \mathbf{k}_3, \mathbf{k}_2) \\ &= T^{(2)}(\mathbf{k}_2, \mathbf{k}_3, \mathbf{k}, \mathbf{k}_1). \end{aligned} \tag{139}$$

6.4. Reduction of Zakharov equation to NLS-type equation

6.4.1. Narrow-band approximation in both dispersion and nonlinearity

Stiassnie (1984) showed that the NLS equation can be derived from the Zakharov equation by restricting the waves to have narrow spectra only. To this end, it is supposed that the energy is concentrated around the wave number $\mathbf{k} = \mathbf{k}_0 = (k_0, 0)^T$ which is in accordance with the usual assumption for NLS equations that the waves have one predominant direction, taken here as the $x = x_1$ direction. We then write,

$$\mathbf{k} = \mathbf{k}_0 + \boldsymbol{\kappa}, \quad \boldsymbol{\kappa} = (\kappa_1, \kappa_2)^T \quad \text{with} \quad |\boldsymbol{\kappa}|/k_0 = o(1). \tag{140}$$

To facilitate the expansion for narrow spectral width, a new amplitude variable \mathcal{B} is introduced by:

$$\mathcal{B}(\boldsymbol{\kappa}, t) = B(\boldsymbol{\kappa}, t) \exp\{-i[\omega(\mathbf{k}) - \omega(\mathbf{k}_0)]\}. \tag{141}$$

The Zakharov equation (135) becomes in terms of \mathcal{B} ,

$$\begin{aligned} i \frac{\partial \mathcal{B}}{\partial t}(\boldsymbol{\kappa}, t) - [\omega(\mathbf{k}) - \omega(\mathbf{k}_0)]\mathcal{B}(\boldsymbol{\kappa}, t) &= \\ \iint\int_{-\infty}^{\infty} d\boldsymbol{\kappa}_1 d\boldsymbol{\kappa}_2 d\boldsymbol{\kappa}_3 T_{0,1,2,3}^{(2)}(\mathbf{k}_0 + \boldsymbol{\kappa}, \mathbf{k}_0 + \boldsymbol{\kappa}_1, \mathbf{k}_0 + \boldsymbol{\kappa}_2, \mathbf{k}_0 + \boldsymbol{\kappa}_3) & \\ \times \mathcal{B}^*(\boldsymbol{\kappa}_1)\mathcal{B}(\boldsymbol{\kappa}_2)\mathcal{B}(\boldsymbol{\kappa}_3)\delta(\boldsymbol{\kappa} + \boldsymbol{\kappa}_1 - \boldsymbol{\kappa}_2 - \boldsymbol{\kappa}_3). & \end{aligned} \tag{142}$$

Substitution of Eq. (141) in Eq. (136) yields the following expression for ζ ,

$$\zeta(\mathbf{x}, t) = \frac{1}{2\pi} e^{i[k_0\mathbf{x} - \omega(k_0)t]} \int_{-\infty}^{\infty} d\boldsymbol{\kappa} \left(\frac{\omega(\mathbf{k}_0 + \boldsymbol{\kappa})}{2g} \right)^{1/2} [\mathcal{B}(\boldsymbol{\kappa}, t)^{i\boldsymbol{\kappa} \cdot \mathbf{x}} + CC]. \quad (143)$$

We also write ζ as:

$$\zeta(\mathbf{x}, t) = \text{Re}\{a(\mathbf{x}, t)e^{i[k_0\mathbf{x} - \omega(\mathbf{k}_0)t]}\}, \quad (144)$$

Expansion of $\sqrt{\omega(\mathbf{k}_0 + \boldsymbol{\kappa})}$ to first order in $\boldsymbol{\kappa}$ for the case of deep-water waves^f (the case considered by Stiassnie, 1984) permits us to approximate the complex amplitude a to:

$$a(\mathbf{x}, t) = \left(\frac{2\omega(\mathbf{k}_0)}{g} \right)^{1/2} \frac{1}{2\pi} \int_{-\infty}^{\infty} d\boldsymbol{\kappa} \left(\frac{1 + \kappa_1}{4k_0} \right) \mathcal{B}(\boldsymbol{\kappa}, t) e^{i\boldsymbol{\kappa} \cdot \mathbf{x}}. \quad (145)$$

In the Zakharov equation (142), we now expand the term $\omega(\mathbf{k}) - \omega(\mathbf{k}_0)$,

$$\begin{aligned} \omega(|\mathbf{k}_0 + \boldsymbol{\kappa}|) - \omega(k_0) &= \frac{1}{2} \sqrt{\frac{g}{k_0}} \left[\kappa_1 - \frac{\kappa_1^2}{4k_0} + \frac{\kappa_2^2}{2k_0} \right. \\ &\quad \left. + \frac{\kappa_1^3}{8k_0^2} - \frac{3\kappa_1\kappa_2^2}{4k_0^2} + \mathcal{O}\left(\frac{|\boldsymbol{\kappa}|^4}{k_0^3}\right) \right]. \end{aligned} \quad (146)$$

The Zakharov equation (142) has to be expressed in terms of the complex amplitude a instead of A . Therefore (Stiassnie, 1984), Eq. (142) is multiplied by $\sqrt{2\omega(k_0)/g} \cdot (1 + \kappa/(4k_0))$ and subsequently the inverse Fourier transform is taken. This results in (Stiassnie, 1984):

$$\begin{aligned} i \frac{\partial a}{\partial t} + \frac{1}{2} \sqrt{\frac{g}{k_0}} \left[i \frac{\partial a}{\partial x} - \frac{1}{4k_0} \frac{\partial^2 a}{\partial x^2} + \frac{1}{2k_0^2} \frac{\partial^2 a}{\partial y^2} + \frac{i}{8k_0^3} \frac{\partial^3 a}{\partial x \partial y^2} \right] = \\ \left(\frac{2\omega(k_0)}{g} \right)^{1/2} \frac{1}{2\pi} \iiint_{-\infty}^{\infty} d\boldsymbol{\kappa}_1 d\boldsymbol{\kappa}_2 d\boldsymbol{\kappa}_3 \left(1 + \frac{\kappa_2 + \kappa_3 - \kappa_1}{4k_0} \right) \\ \cdot T_{0,1,2,3}^{(2)}(k_0 + \boldsymbol{\kappa}_2 + \boldsymbol{\kappa}_3 - \boldsymbol{\kappa}_1, k_0 + \boldsymbol{\kappa}_1, k_0 + \boldsymbol{\kappa}_2, k_0 + \boldsymbol{\kappa}_3) \\ \cdot \mathcal{B}^*(\boldsymbol{\kappa}_1) \mathcal{B}(\boldsymbol{\kappa}_2) \mathcal{B}(\boldsymbol{\kappa}_3) e^{i(\boldsymbol{\kappa}_2 + \boldsymbol{\kappa}_3 - \boldsymbol{\kappa}_1) \cdot \mathbf{x}}. \end{aligned} \quad (147)$$

^fIn deep water, one has $\omega = \sqrt{g|\mathbf{k}_0 + \boldsymbol{\kappa}|} = \sqrt{gk_0(1 + 2\kappa_1/k_0 + |\boldsymbol{\kappa}|^2/k_0^2)}^{1/4}$ and thus, to first order, $\omega(\mathbf{k}_0 + \boldsymbol{\kappa}) = \omega(k_0)(1 + \kappa_1/(2k_0))$.

For the case of deep-water waves, it is possible to show that a first-order Taylor expansion of the interaction coefficient $T_{0,1,2,3}^{(2)}$ in the spectral width becomes,

$$T_{0,1,2,3}^{(2)}(\mathbf{k}_0 + \boldsymbol{\kappa}_2 + \boldsymbol{\kappa}_3 - \boldsymbol{\kappa}_1, \mathbf{k}_0 + \boldsymbol{\kappa}_1, \mathbf{k}_0 + \boldsymbol{\kappa}_2, \mathbf{k}_0 + \boldsymbol{\kappa}_3) = \frac{k_0^3}{4\pi^2} \left[1 + \frac{3}{2k_0}(\kappa_2 + \kappa_3) - \frac{(\kappa_1 - \kappa_2)^2}{2k_0|\boldsymbol{\kappa}_1 - \boldsymbol{\kappa}_2|} - \frac{(\kappa_1 - \kappa_3)^2}{2k_0|\boldsymbol{\kappa}_1 - \boldsymbol{\kappa}_3|} + \mathcal{O}\left(\frac{|\boldsymbol{\kappa}|^2}{k_0^2}\right) \right]. \quad (148)$$

Using Eq. (148) in Eq. (147), the following equation is found,

$$\begin{aligned} & i \frac{\partial a}{\partial t} + \frac{1}{2} \left(\frac{g}{k_0} \right)^{1/2} \left[i \frac{\partial a}{\partial x} - \frac{1}{4k_0} \frac{\partial^2 a}{\partial x^2} + \frac{1}{2k_0} \frac{\partial a}{\partial y} - \frac{i}{8k_0^2} \frac{\partial^3 a}{\partial x^3} + \frac{3i}{4k_0^2} \frac{\partial^3 a}{\partial x \partial y^2} \right] \\ & = \frac{g}{2\omega(\mathbf{k}_0)} \left[k_0^3 |a|^2 a - \frac{ik_0^2}{2} a^2 \frac{\partial a^*}{\partial x} - 3ik_0^2 |a|^2 \frac{\partial a}{\partial x} \right] - \frac{k_0^6 a I}{4\pi^2}, \end{aligned} \quad (149)$$

where the sign of the second term in square brackets in the right-hand side of the equation is negative instead of positive as in Stiassnie’s equation (10). Notice that this correction is the same as the correction of Janssen (1983) of Dysthe’s equation. See also Hogan (1985, p. 371). In Eq. (149), I is an integral which can be related to the derivative of the wave potential Φ at $z = 0$ (Stiassnie, 1984) as:

$$I = \frac{4g\pi^2}{\omega^2(\mathbf{k}_0)} \frac{\partial \bar{\Phi}}{\partial x} \Big|_{z=0}. \quad (150)$$

Substituting Eq. (150) in Eq. (149) and rewriting results in:

$$\begin{aligned} & i \left(\frac{\partial a}{\partial x} + \frac{2k_0}{\omega_0} \frac{\partial a}{\partial t} \right) - \frac{1}{4k_0} \frac{\partial^2 a}{\partial x^2} + \frac{1}{2k_0} \frac{\partial^2 a}{\partial y^2} - \frac{i}{8k_0^2} \frac{\partial^3 a}{\partial x^3} + \frac{3i}{4k_0^2} \frac{\partial^3 a}{\partial x \partial y^2} = \\ & k_0^3 |a|^2 a - \frac{ik_0^2}{2} a^2 \frac{\partial a^*}{\partial x} - 3ik_0^2 |a|^2 \frac{\partial a}{\partial x} + \frac{2k_0^2}{\omega(\mathbf{k}_0)} a \frac{\partial \bar{\Phi}}{\partial z} \Big|_{z=0}. \end{aligned} \quad (151)$$

6.4.2. Modification due to surface tension

Hogan (1985) extended the analysis of Dysthe (1979) by also taking surface-tension effects into account and derived a fourth-order equation valid for deep-water waves with surface tension effects included. The dispersion relation used is $\omega^2(k) = (1 + \bar{s})gk$ and $\bar{s} = \gamma k^2/(\rho g)$ with γ the surface tension which is given as force per unit length (N/m). He starts with the Zakharov equation

with surface-tension effects included and then follows Stiassnie's (1984) method to obtain the fourth-order evolution equation. Writing,

$$\zeta(\mathbf{x}, t) = \text{Re}\{a(\mathbf{x}, t)e^{i(k_0x - \omega_0t)}\}, \tag{152}$$

new scaled (primed) variables are introduced by $t' = \omega_0t$, $\mathbf{x}' = k_0\mathbf{x}$, $a' = k_0a$, $\bar{\Phi}' = (2k_0^2/\omega_0)\bar{\Phi}$ and $c_g' = (k_0/\omega_0)c_g$. The dimensionless higher-order NLS equation then is, dropping the primes,

$$2i \left(\frac{\partial a}{\partial t} + c_g \frac{\partial a}{\partial x} \right) + p \frac{\partial^2 a}{\partial x^2} + q \frac{\partial a}{\partial y} - \chi_1 |a|^2 a = -is \frac{\partial^3 a}{\partial x \partial y^2} - ir \frac{\partial^3 a}{\partial x^3} - iua^2 \frac{\partial a^*}{\partial x} + iv|a|^2 \frac{\partial a}{\partial x} + a \frac{\partial \bar{\Phi}}{\partial x} \Big|_{z=0}, \tag{153}$$

where the coefficients are given by:

$$p = \frac{3\tilde{s}^2 + 6\tilde{s} - 1}{4(1 + \tilde{s}^2)}, \quad q = \frac{1 + 3\tilde{s}}{2(1 + \tilde{s})}, \tag{154a}$$

$$r = -\frac{(1 - \tilde{s})(1 + 6\tilde{s} + \tilde{s}^2)}{8(1 + \tilde{s})^3}, \quad s = \frac{3 + 2\tilde{s} + 3\tilde{s}^2}{4(1 + \tilde{s})^2}, \tag{154b}$$

$$u = \frac{(1 - \tilde{s})(8 + \tilde{s} + 2\tilde{s}^2)}{16(1 - 2\tilde{s})(1 + \tilde{s})^2}, \quad \chi_1 = \frac{8 + \tilde{s} + 2\tilde{s}^2}{8(1 - 2\tilde{s})(1 + \tilde{s})}, \tag{154c}$$

$$v = \frac{3(4\tilde{s}^4 + 4\tilde{s}^3 - 9\tilde{s}^2 + \tilde{s} - 8)}{8(1 + \tilde{s})^2(1 - 2\tilde{s})^2}, \quad c_g = \frac{\omega_0}{2k_0} \frac{1 + 3\tilde{s}}{1 + \tilde{s}}, \tag{154d}$$

with

$$\tilde{s} = \frac{\gamma k_0^2}{\rho g}. \tag{154e}$$

In variables with dimension, one obtains from Eq. (153),

$$2i \left(\frac{\partial a}{\partial t} + c_g \frac{\partial a}{\partial x} \right) + p \frac{\omega_0}{k_0^2} \frac{\partial^2 a}{\partial x^2} + q \frac{\omega_0}{k_0^2} \frac{\partial^2 a}{\partial y^2} - \chi_1 \omega_0 k_0^2 |a|^2 a = -is \frac{\omega_0}{k_0^3} \frac{\partial^3 a}{\partial x \partial y^2} - ir \frac{\omega}{k_0^3} \frac{\partial^3 a}{\partial x^3} - iuk_0\omega_0 a^2 \frac{\partial a^*}{\partial x} + ivk_0\omega_0 |a|^2 \frac{\partial a}{\partial x} + 2k_0 a \frac{\partial \bar{\Phi}}{\partial x} \Big|_{z=0}, \tag{155}$$

where the coefficients of Eq. (154) have been used.

To be able to compare this equation with other higher-order equations, we notice that in absence of surface tension ($\tilde{s} = 0$), Hogan’s result reduces to:

$$i \left(\frac{\partial a}{\partial t} + c_g \frac{\partial a}{\partial x} \right) - \frac{\omega_0}{8k_0^2} \frac{\partial^2 a}{\partial x^2} + \frac{\omega_0}{4k_0^2} \frac{\partial^2 a}{\partial y^2} - \frac{1}{2} \omega_0 |a|^2 a = \frac{1}{16} i \frac{\omega_0}{k_0^3} \frac{\partial^3 a}{\partial x^3} - \frac{3}{8} i \frac{\omega_0}{k_0^3} \frac{\partial^3 a}{\partial x \partial y^2} - \frac{1}{4} i k_0 \omega_0 a^2 \frac{\partial a^*}{\partial x} - \frac{3}{2} i k_0 \omega_0 |a|^2 \frac{\partial a}{\partial x} + k_0 a \frac{\partial \bar{\Phi}}{\partial x} \Big|_{z=0}. \quad (156)$$

Furthermore, as in Dysthe (1979), the same condition for $\bar{\Phi}_x$ follows from the kinematic condition,

$$\frac{\partial \bar{\Phi}}{\partial z} = \frac{1}{2} \omega_0 \frac{\partial (a^2)}{\partial x} \quad \text{at } z = 0. \quad (157)$$

6.4.3. Narrow-band approximation in nonlinearity only

The limited bandwidth for which the NLS and the modified equation (mNLS) as given by Dysthe (1979) are derived hampers the application to real water-wave problems as noted amongst others by Trulsen and Dysthe (1996). These authors enhanced the extent of the bandwidth. Both the NLS and the mNLS equations are derived under the conditions that:

$$\frac{|\Delta \mathbf{k}|}{k} = \mathcal{O}(\varepsilon) \quad \text{and} \quad kh = \mathcal{O}(\varepsilon^{-1}) \quad \text{with} \quad \varepsilon = ka. \quad (158)$$

The resulting equation, valid up to $\mathcal{O}(\varepsilon^4)$ has been given in Eq. (115). We rewrite that equation in the following form,

$$\mathcal{S}(B) = \mathcal{L}_2(B) + \mathcal{N}_2(B), \quad (159a)$$

where $\mathcal{S}(B) = 0$ stands for the NLS equation with $\mathcal{S} \equiv \mathcal{L}_1 + \mathcal{N}_1$, $\mathcal{L}_1 B$ and $\mathcal{N}_1 B$ being the linear part and the nonlinear part of the nonlinear Schrödinger equation respectively. Furthermore, we still have Eqs. (114b) and (114c). The extensions of the NLS equation needed for the mNLS equation are given by:

$$\mathcal{S}(B) = i \left(\frac{\partial B}{\partial t} + \frac{\omega}{2k} \frac{\partial B}{\partial x} \right) - \frac{\omega}{4k^2} \left(\frac{1}{2} \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} \right) B - \frac{k^4}{2\omega} |B|^2 B, \quad (159b)$$

$$\mathcal{L}_2(B) = -\frac{i\omega}{16k^3} \left(6 \frac{\partial^3}{\partial x \partial y^2} - \frac{\partial^3}{\partial x^3} \right) B, \quad (159c)$$

$$\mathcal{N}_2(B, \bar{\Phi}) = \frac{ik^3}{4\omega} \left(B \frac{\partial B^*}{\partial x} - 6B^* \frac{\partial B}{\partial x} \right) B + kB \frac{\partial \bar{\Phi}}{\partial x} \Big|_{z=0}. \quad (159d)$$

Trulsen and Dysthe (1996) now assumed a wider bandwidth to exist,

$$\frac{|\Delta \mathbf{k}|}{k} = \mathcal{O}(\varepsilon^{1/2}) \quad \text{and} \quad kh = \mathcal{O}(\varepsilon^{-1/2}). \tag{160}$$

With $\varepsilon^{1/2}$ being the new expansion parameter, the new slow spatial and time variables are $\varepsilon^{1/2}\mathbf{x}$, $\varepsilon^{1/2}t$ and also the vertical coordinate is somewhat faster than before: $\varepsilon^{1/2}z$. With a similar expansion procedure as before, now up to fifth order in the expansion parameter $\varepsilon^{1/2}$, the resulting equation is, in variables with dimension,

$$S(B) = \mathcal{L}_2(B) + \mathcal{N}_2(B, \bar{\Phi}) + \mathcal{L}_3(B), \tag{161a}$$

the difference being only in the linear dispersive terms. For the nonlinearity, the narrow-band assumption is retained. We have,

$$\begin{aligned} \mathcal{L}_3(B) = & \frac{\omega}{32k^4} \left(-\frac{5}{4} \frac{\partial^4}{\partial x^4} + 15 \frac{\partial^4}{\partial x^2 \partial y^2} + 3 \frac{\partial^4}{\partial y^4} \right) B \\ & - \frac{7i}{64} \left(\frac{1}{4} \frac{\partial^5}{\partial x^5} - 5 \frac{\partial^5}{\partial x^3 \partial y^2} + \frac{3}{2} \frac{\partial^5}{\partial x \partial y^4} \right) B. \end{aligned} \tag{161b}$$

While Eq. (161) is derived for wider bandwidth, it is still an approximation for a somewhat narrow bandwidth, albeit wider than before. The effect is seen in a further shrinking of the instability region for Stokes waves. Further progress is given in the paper of Trulsen *et al.* (2000) where no conditions on the bandwidth are set. Using pseudo-differential operators which capture the full dispersive behaviour, further progress is achieved.

Trulsen *et al.* (2000) start by expressing the surface displacement as:

$$\zeta(\mathbf{x}, t) = \frac{1}{2} \int_{-\infty}^{\infty} a(\mathbf{k}) e^{i(\mathbf{k} \cdot \mathbf{x} - \omega(\mathbf{k})t)} d\mathbf{k} + CC, \tag{162a}$$

with the dispersion relation given by the deep-water form $\omega = \sqrt{g|\mathbf{k}|}$. Supposing that the free-surface elevation may be described by a modulation of a carrier wave with wave number $\mathbf{k}_0 = (k_0, 0)^T$ and frequency $\omega_0 = \omega(\mathbf{k}_0)$, ζ can also be expressed as:

$$\zeta(\mathbf{x}, t) = \frac{1}{2} A(\mathbf{x}, t) e^{i(\mathbf{k}_0 \cdot \mathbf{x} - \omega_0 t)} + CC. \tag{162b}$$

We now use $\mathbf{k} = \mathbf{k}_0 + \boldsymbol{\kappa}$ as the modulation vector. Equating the two expressions for ζ in Eqs. (162), one obtains,

$$A(\mathbf{x}, t) = \int_{-\infty}^{\infty} \hat{A}(\boldsymbol{\kappa}, t) e^{i\boldsymbol{\kappa} \cdot \mathbf{x}} d\boldsymbol{\kappa}, \tag{163a}$$

with

$$\hat{A}(\boldsymbol{\kappa}, t) = a(\mathbf{k}_0 + \boldsymbol{\kappa}) e^{i[\boldsymbol{\kappa} \cdot \mathbf{x} - (\omega(k_0 + \boldsymbol{\kappa}) - \omega(k_0))]} . \tag{163b}$$

Differentiation of Eq. (163b) with respect to t shows that \hat{A} satisfies,

$$\frac{\partial \hat{A}}{\partial t} + i[\omega(k_0 + \boldsymbol{\kappa}) - \omega_0] \hat{A} = 0 . \tag{164}$$

To return to physical space, we use operator correspondence where we have to account for the fact that $\boldsymbol{\kappa}$ is the change of wave number relative to \mathbf{k}_0 . Writing $\boldsymbol{\kappa} = (\kappa_1, \kappa_2)^T$, we thus have,

$$\frac{\kappa_1}{k_0} \rightarrow i\partial_x \quad \text{and} \quad \frac{\kappa_2}{k_0} \rightarrow i\partial_y . \tag{165}$$

In physical space, we then have,

$$\frac{\partial A}{\partial t} + L(\partial_x, \partial_y)A = 0 , \tag{166a}$$

with

$$L(\partial_x, \partial_y) = i\{[(1 - i\partial_x)^2 - \partial_y^2]^{1/4} - 1\} , \tag{166b}$$

and thus,

$$\frac{\partial A}{\partial t} + \frac{1}{4\pi^2} \int_{-\infty}^{\infty} d\boldsymbol{\kappa} i[\omega(\mathbf{k}_0 + \boldsymbol{\kappa}) - \omega_0] \int_{-\infty}^{\infty} d\mathbf{x}' e^{i\boldsymbol{\kappa} \cdot (\mathbf{x} - \mathbf{x}')} A(\mathbf{x}', t) = 0 . \tag{166c}$$

By expanding Eq. (166b) in the derivatives, linear evolution equations to all orders can be obtained. Expansion up to fifth-order derivatives yields the linear part of Eq. (159). Trulsen *et al.* (2000) use the exact linear dispersive equation (166b) for the linear part of the higher-order NLS-type equation instead of some power-series approximation such as Eq. (159). Keeping the usual nonlinear term, the equation considered then is:

$$\frac{\partial A}{\partial t} + L(\partial_x, \partial_y)A + \frac{1}{2}i\omega_0 k_0^2 |A|^2 A = 0 . \tag{167}$$

Trulsen *et al.* (2000) also consider the equation with the nonlinear terms as introduced by Dysthe (1979) and then get the system,

$$\begin{aligned} \frac{\partial A}{\partial t} + L(\partial_x, \partial_y)A + \frac{1}{2}i\omega_0 k_0^2 |A|^2 A + \frac{3}{2}\omega_0 k_0 |A|^2 \frac{\partial A}{\partial x} \\ + \frac{1}{4}\omega_0 k_0 A^2 \frac{\partial A^*}{\partial x} + ik_0 A \frac{\partial \bar{\Phi}}{\partial x} \Big|_{z=0} = 0, \end{aligned} \tag{168a}$$

with

$$\frac{\partial \bar{\Phi}}{\partial z} \Big|_{z=0} = \frac{1}{2}\omega_0 \frac{\partial}{\partial x} |A|^2, \tag{168b}$$

$$\nabla^2 \bar{\Phi} = 0 \quad \text{for } -\infty < z < 0, \tag{168c}$$

$$\frac{\partial \bar{\Phi}}{\partial z} = 0 \quad \text{for } z \rightarrow -\infty. \tag{168d}$$

Equation (167) can also be derived from the Zakharov equation (142) with ζ given by Eqs. (143) and (144) in essentially the same way as done by Stiassnie (1984). Expansion of representation in Eq. (143) to leading order in the spectral width yields (cf. Trulsen *et al.*, 2000) expression (144) with the amplitude a given by:

$$a(\mathbf{x}, t) = \left(\frac{\omega_0}{2g}\right)^{1/2} \int_{-\infty}^{\infty} d\boldsymbol{\kappa} \mathcal{B}(\boldsymbol{\kappa}, t) e^{i\boldsymbol{\kappa} \cdot \mathbf{x}}. \tag{169}$$

The narrow-band approximation of the kernel $T_{0,1,2,3}^{(2)}$ yields,

$$T_{0,1,2,3}^{(2)}(\mathbf{k}_0, \mathbf{k}_0, \mathbf{k}_0, \mathbf{k}_0) = \frac{k_0^3}{4\pi^2}. \tag{170}$$

Multiplying Eq. (142) by $(2\omega_0/g)^{1/2}$, taking its inverse Fourier transform and applying the narrow-band approximation, Eq. (170), for the nonlinear term, yields the following NLS-type equation with fully dispersive behaviour and narrow-band nonlinearity,

$$\frac{\partial a}{\partial t} + \frac{1}{4\pi^2} \int_{-\infty}^{\infty} d\mathbf{x}' d\boldsymbol{\kappa} i[\omega(\mathbf{k}_0 + \boldsymbol{\kappa}) - \omega_0] a(\mathbf{x}', t) + \frac{1}{2}i\omega_0 k_0^2 |a|^2 a = 0. \tag{171}$$

6.5. Extensions of the Zakharov equation

By allowing for spatial variations, Rasmussen (1998) derives for the deep-water case a so called *Zakharov-Nonlinear Schrödinger equation* (his equation (3.50)) which reads,

$$\begin{aligned} \frac{\partial B}{\partial t} + \mathbf{c}_g \cdot \nabla B + \frac{ig}{8\omega k} \left(\frac{k_x^2 - 2k_y^2}{k^2} \frac{\partial^2 B}{\partial x^2} + \frac{k_y^2 - 2k_x^2}{k^2} \frac{\partial^2 B}{\partial y^2} + \frac{6k_x k_y}{k^2} \frac{\partial^2 B}{\partial x \partial y} \right) \\ = -i \iiint d\mathbf{k}_1 d\mathbf{k}_2 d\mathbf{k}_3 C'_{3,2}, \end{aligned} \tag{172}$$

where $C'_{3,2}$ is already defined in Eq. (137b). As Rasmussen noticed, the left-hand side is similar to the linear part of the nonlinear Schrödinger equation, apart from the mixed derivative term which can be transformed away by changing the x -direction into the main propagation direction. We notice that the extra terms (compared to the original Zakharov equation) are due to the partial variation of the wave amplitudes. Because the case of deep water is considered and no current is present, the modulation is solely due to the frequency modulation, introduced from outside the computational domain. In this respect, the situation is similar as in Stiassnie (1984) in which a NLS equation is derived from the Zakharov equation for deep water.

It can be shown (see Rasmussen, 1998) that for the case that the wave field consists of only one dominant wave component, $B(\mathbf{k}, t) = \mathcal{B}(\mathbf{k}', t)\delta(\mathbf{k} - \mathbf{k}')$ and also choosing $\mathbf{k}' = (k, 0)^T$, the deep-water NLS equation results,

$$\frac{\partial \mathcal{B}}{\partial t} + \frac{\omega}{2k} \frac{\partial \mathcal{B}}{\partial x} + \frac{ig}{8\omega k} \left(\frac{\partial \mathcal{B}}{\partial x} - 2 \frac{\partial \mathcal{B}}{\partial y} \right) = -\frac{ik^3}{4\pi^2} |\mathcal{B}|^2 \mathcal{B}. \tag{173}$$

Here is used the fact that:

$$X_{0,1,2,3}^{(2)} = \frac{k_0^3}{4\pi^2}, \tag{174}$$

as can be found in Stiassnie (1984, p. 432). Neglecting the variation on the slow spatial scales, Eq. (172) reduces to:

$$\frac{\partial B}{\partial t} + \mathbf{c}_g \cdot \nabla B = -i \iiint d\mathbf{k}_1 d\mathbf{k}_2 d\mathbf{k}_3 C'_{3,2}, \tag{175}$$

which is called the *inhomogeneous Zakharov equation* by Rasmussen.

By considering a Stokes wave with constant amplitude,

$$B(\mathbf{k}, t) = \mathcal{B}e^{i\alpha t} \delta(\mathbf{k} - \mathbf{k}'), \tag{176}$$

Rasmussen poses the question how much \mathbf{k}' may deviate from its exact value without changing the nature of the solution. Put in another form: what is the appropriate value of the bandwidth for application to random waves. To this end, the solution is multiplied by a small variation due to a small perturbation wave number κ ,

$$B(\mathbf{k}, t) = \mathcal{B}e^{i(\alpha t + \kappa \cdot \mathbf{x})} \delta(\mathbf{k} - \mathbf{k}'). \quad (177)$$

By substituting this expression in Eqs. (172) and (175), Rasmussen finds that the solution for the free-surface elevation $\zeta(\mathbf{x}, t)$ can be written in both cases as:

$$\zeta(\mathbf{x}, t) = a \cos[\mathbf{K} \cdot \mathbf{x} - (1 + (Ka)^2)\omega(\mathbf{K})t], \quad (178)$$

where $\mathbf{K} = \mathbf{k} + \kappa$ and Eqs. (172) and (175) are fulfilled when $\kappa/k = \mathcal{O}(\varepsilon)$ and $\mathcal{O}(\varepsilon^2)$ respectively.

7. Generation of Free Long Waves

In coastal areas, long waves with typical periods of minutes can be generated due to several physical phenomena. One of the generation mechanisms is due to the nonlinear effect on modulated wave trains. Generally two types of long waves exist: (1) locked (forced) long waves and (2) free long waves which propagate with their own celerities according to the linear dispersion relation. In coastal areas and for a narrow-banded wave group, the celerities become \sqrt{gh} where $h(\mathbf{x})$ is the depth.

For horizontal bottoms, usually only the locked waves are generated. The effect of an uneven bottom is that free long waves are generated and, moreover, that part of the locked wave energy transforms to free waves. Mei and Benmoussa (1984) [see also Liu, 1989] have shown that the free long waves could propagate in a direction different from the wave group and the carrier waves. For shear-current regions, similar phenomena occur, see Liu, DINGEMANS and KOSTENSE (1990). Other mechanisms of long wave generation have been discussed in Holman and Bowen (1982).

The importance of knowing the locked and free long waves in coastal areas is because they influence the sediment transport rates and especially the amount of free long waves is important for harbour oscillation problems. For harbours with berths for large vessels, often the effect of the long waves exceeds that of the short wind waves not only due to possible resonance of the harbour itself but also due to the mooring systems which do have resonance peaks at much lower frequency closer to the free long waves. So, the knowledge of the amount