

Rasmussen poses the question how much  $\mathbf{k}'$  may deviate from its exact value without changing the nature of the solution. Put in another form: what is the appropriate value of the bandwidth for application to random waves. To this end, the solution is multiplied by a small variation due to a small perturbation wave number  $\kappa$ ,

$$B(\mathbf{k}, t) = \mathcal{B}e^{i(\alpha t + \kappa \cdot \mathbf{x})} \delta(\mathbf{k} - \mathbf{k}'). \quad (177)$$

By substituting this expression in Eqs. (172) and (175), Rasmussen finds that the solution for the free-surface elevation  $\zeta(\mathbf{x}, t)$  can be written in both cases as:

$$\zeta(\mathbf{x}, t) = a \cos[\mathbf{K} \cdot \mathbf{x} - (1 + (Ka)^2)\omega(\mathbf{K})t], \quad (178)$$

where  $\mathbf{K} = \mathbf{k} + \kappa$  and Eqs. (172) and (175) are fulfilled when  $\kappa/k = \mathcal{O}(\varepsilon)$  and  $\mathcal{O}(\varepsilon^2)$  respectively.

## 7. Generation of Free Long Waves

In coastal areas, long waves with typical periods of minutes can be generated due to several physical phenomena. One of the generation mechanisms is due to the nonlinear effect on modulated wave trains. Generally two types of long waves exist: (1) locked (forced) long waves and (2) free long waves which propagate with their own celerities according to the linear dispersion relation. In coastal areas and for a narrow-banded wave group, the celerities become  $\sqrt{gh}$  where  $h(\mathbf{x})$  is the depth.

For horizontal bottoms, usually only the locked waves are generated. The effect of an uneven bottom is that free long waves are generated and, moreover, that part of the locked wave energy transforms to free waves. Mei and Benmoussa (1984) [see also Liu, 1989] have shown that the free long waves could propagate in a direction different from the wave group and the carrier waves. For shear-current regions, similar phenomenae occur, see Liu, DINGEMANS and Kostense (1990). Other mechanisms of long wave generation have been discussed in Holman and Bowen (1982).

The importance of knowing the locked and free long waves in coastal areas is because they influence the sediment transport rates and especially the amount of free long waves is important for harbour oscillation problems. For harbours with berths for large vessels, often the effect of the long waves exceeds that of the short wind waves not only due to possible resonance of the harbour itself but also due to the mooring systems which do have resonance peaks at much lower frequency closer to the free long waves. So, the knowledge of the amount

of free long waves at the harbour mouth is essential for a good harbour design. The first paper discussing the importance of free and locked waves for harbour design was the one of Bowers (1977). Bowers showed that free long waves were generated because of the difference in the strength of the bound waves across the junction between the two channels. Another cause of the generation of free waves with which we are concerned here is due to the refraction of wave groups, either due to an uneven bottom or due to shear currents, see e.g., Mei and Benmoussa (1984), Liu (1989), Liu *et al.* (1990), Liu *et al.* (1992) and Dingemans *et al.* (1991).

In this section, we primarily consider the second-order formulation for the wave amplitudes. In third-order, the NLS-equation formulation is obtained. For that case, we refer to Dingemans *et al.* (1991).

### 7.1. Formulation of the equations

We consider a train of modulated linear waves propagating over a slowly-varying bottom. In first instance, we also include an ambient current field in the considerations. Because the length scales of the resulting wave groups are much larger than those of the carrier waves, slow variables are introduced by:

$$\mathbf{X} = \beta \mathbf{x} \quad \text{and} \quad T = \beta t, \quad \beta \ll 1. \quad (179)$$

When an ambient current field  $\mathbf{U}(\mathbf{X}, T)$  is also considered, the first-order displacements can be written in the following form (see Liu *et al.*, 1990),

$$\zeta(\mathbf{x}, t; \mathbf{X}, T) = \frac{1}{2} A(\mathbf{X}, T) \exp[i\chi(\mathbf{x}, t)] + CC, \quad (180a)$$

$$\Phi(\mathbf{x}, z, t; \mathbf{X}, z, T) = -\frac{igA(\mathbf{X}, T)}{2\omega_r} \frac{\cosh[k(h+z)]}{\cosh kh} \exp[i\chi(\mathbf{x}, t)] + CC, \quad (180b)$$

with

$$\chi(\mathbf{x}, t) = \int^{\mathbf{x}} d\mathbf{x}' \mathbf{k}(\mathbf{x}') - \omega_0 t, \quad (180c)$$

where the carrier wave frequency  $\omega$  is determined by:

$$\omega = \omega_r + \mathbf{k} \cdot \mathbf{U} \quad \text{and} \quad \omega_r^2 = gk \tanh kh, \quad (180d)$$

and  $\mathbf{k}$  is given by:

$$\mathbf{k} = \nabla \chi(\mathbf{x}, t). \quad (180e)$$

In the same way as derived in Liu and Dingemans (1989) for the case without an ambient current, an equation for the complex amplitude with current becomes,

$$\frac{\partial}{\partial T} \left( \frac{A^2}{\omega_r} \right) + \nabla_X \cdot \left[ (\mathbf{c}_g + \mathbf{U}) \frac{A^2}{\omega_r} \right] = 0, \quad (181)$$

where  $\nabla_X = (\partial/\partial X_1, \partial/\partial X_2)^T$ .

Writing  $\bar{\zeta}$  for  $\zeta^{(2,0)}$  and  $\bar{\phi}$  for  $\phi^{(1,0)}$ , we have the following relation between  $\bar{\zeta}$  and  $\bar{\phi}$ , see Kirby (1983),

$$\bar{\zeta} = -\frac{1}{g} \frac{D\bar{\phi}}{DT} - \frac{\omega_r |A|^2}{4g \sinh^2 kh}, \quad (182a)$$

where  $D/DT \equiv \partial/\partial T + \mathbf{U} \cdot \nabla_X$  and

$$\frac{\partial \bar{\zeta}}{\partial T} + \nabla_X \cdot \left[ \bar{\zeta} \mathbf{U} + h \nabla_X \bar{\phi} + \frac{g|A|^2}{2\omega_r} \mathbf{k} \right] = 0. \quad (182b)$$

As noted by Liu *et al.* (1990), taking the horizontal gradient of Eq. (182a) and eliminating  $\bar{\zeta}$  from the resulting equation and Eq. (182b) yields the long-wave equation for the potential  $\bar{\phi}$ ,

$$\begin{aligned} \frac{D^2 \bar{\phi}}{DT^2} + (\nabla_X \cdot \mathbf{U}) \frac{D\bar{\phi}}{DT} - g \nabla_X \cdot (h \nabla_X \bar{\phi}) = \\ \frac{g^2}{2} \nabla_X \cdot \left( \mathbf{k} \frac{|A|^2}{\omega_r} \right) - \frac{gk}{2 \sinh 2kh} \frac{\partial |A|^2}{\partial T} - \nabla_X \cdot \left[ \frac{gk|A|^2}{2 \sinh 2kh} \mathbf{U} \right], \end{aligned} \quad (183)$$

which was also derived by Kirby (1983) by using an averaged Lagrangian approach.

Alternatively, we can write the long-wave equation in terms of the second-order free-surface elevation  $\bar{\zeta}$  by taking the total derivative of Eq. (182b) and substituting  $D\bar{\phi}/DT$  from Eq. (182a), yielding,

$$\begin{aligned} \frac{D^2 \bar{\zeta}}{DT^2} - \nabla_X \cdot (gh \nabla_X \bar{\zeta}) + \frac{D}{DT} (\bar{\zeta} \nabla_X \cdot \mathbf{U}) = \\ \nabla_X \cdot \left[ h \nabla_X \left( \frac{\omega A^2}{4 \sinh^2 kh} \right) \right] - \frac{D}{DT} \nabla_X \cdot \left( \frac{kgA^2}{2\omega} \right). \end{aligned} \quad (184)$$

In the case of an ambient current and an uneven bottom, the governing equations are the amplitude equation (181) with either Eq. (183) for  $\bar{\phi}$  or Eq. (184) for  $\bar{\zeta}$ .

In absence of an ambient current field and reverting back to the unstretched variables  $\mathbf{x}$  and  $t$ , Eq. (184) simplifies to:

$$\frac{\partial^2 \bar{\zeta}}{\partial t^2} - \nabla \cdot (gh \nabla \bar{\zeta}) = \nabla \cdot \left[ h \nabla \left( \frac{\omega A^2}{4 \sinh^2 kh} \right) \right] - \frac{\partial}{\partial t} \nabla \cdot \left( \frac{\mathbf{k} g A^2}{2\omega} \right), \quad (185a)$$

and in terms of  $\bar{\phi}$ , we then have,

$$\frac{\partial^2 \bar{\phi}}{\partial t^2} - \nabla \cdot (gh \nabla \bar{\phi}) = \frac{g^2}{2} \nabla \cdot \left( \mathbf{k} \frac{|A|^2}{\omega_r} \right) - \frac{gk}{2 \sinh 2kh} \frac{\partial |A|^2}{\partial t}. \quad (185b)$$

It is to be noted that both the amplitude and the long-wave equations are of second order and the validity of the formulations is restricted to spatial scale  $|\mathbf{X}| \equiv |\beta \mathbf{x}| = \mathcal{O}(1)$  and similarly for the time. Notice that these long-wave equations have already been encountered in the derivation of NLS-type equations, see, e.g., Eq. (82b). The main attention is directed in this section towards the forced long-wave equation. The forcing itself is described by the amplitude equation. When proceeding to third order, a NLS-type equation is obtained for the amplitude. Using this NLS-type equation instead of the second-order equation (181) means that the forcing in the right-hand side of the long-wave equation can be determined more accurately than is possible with the second-order amplitude equation. In the present section, the accuracy of the forcing itself is not so important but the fact that a forcing is present matters.

Notice that the coupling between the amplitude equation (181) and the long-wave equation (183) occurs in one way only. No coupling from long wave to amplitude equation exists. In the case of the NLS equation for restricted depth, two-way coupling exists. Therefore, we will only consider the case of a NLS equation with its companion long-wave equation.

That bound waves become free when the wave group is progressing into a region with variable depth seen as follows. Suppose we have a 1D situation with shelf-like region consisting of a horizontal part with  $h = h_0$  for  $x < X_0$ , a variable part  $h = h(x)$  for  $X_0 \leq x \leq X_1$  and again a horizontal part for  $x > X_1$ . Suppose that we have a permanent wave group with accompanying bound long wave in the region  $x < X_0$ . Upon progressing of this wave group in the region with variable depth, the carrier waves forming the wave group experience shoaling and the group itself changes and is not permanent anymore. This has as a consequence that the long wave also changes; during this change also free long waves are formed. This situation is not very much different from

which is encountered by propagation of solitary and cnoidal waves in regions with uneven bottoms. Also in these situations long waves are formed, see the discussion and the references mentioned in Dingemans (1997, section 6.6.3).

## 7.2. 1D situation, no ambient currents

We consider the long-wave equation (185b) in this section. Because  $A$  is a function of  $\xi \equiv x - c_g t$ , as becomes clear from the amplitude equation (181) a particular solution  $\bar{\phi}_\ell$  to Eq. (185b) is also a function of  $x - c_g t$ . As done in Dingemans *et al.* (1991), we can integrate the long wave equation once and we obtain, for the case of a horizontal bottom,<sup>§</sup>

$$(c_g^2 - gh) \frac{\partial \bar{\phi}_\ell}{\partial \xi} = \left[ \frac{1}{2} \frac{g^2 k}{\omega} + \frac{g k c_g}{2 \sinh 2kh} \right] |A|^2, \quad (186)$$

where  $\bar{\phi}_\ell = 0$  is chosen for  $A = 0$ .

For the amplitude of the wave envelope, we now use the NLS equation which reads (Dingemans *et al.* (1991, Eq. (8)),

$$\left( i \frac{\partial A}{\partial t} + c_g \frac{\partial A}{\partial x} \right) + \frac{1}{2} \frac{\partial c_g}{\partial k} \frac{\partial^2 A}{\partial x^2} + A \mathcal{G} \bar{\phi} - k^2 \omega \kappa |A|^2 A = \mathcal{H} A, \quad (187a)$$

where the operators  $\mathcal{G}$  and  $\mathcal{H}$  are given by:

$$\mathcal{G} \bar{\phi} = \left( \frac{\omega k_\infty}{2g} (\sigma^2 - 1) \frac{\partial}{\partial t} - k \frac{\partial}{\partial x} \right) \bar{\phi}, \quad (187b)$$

$$\mathcal{H} A = \left\{ \frac{\partial}{\partial x} \left( \frac{\partial c_g}{\partial k} \right) - \mu \right\} \frac{\partial A}{\partial x} - \left( \frac{i}{2} \frac{\partial c_g}{\partial k} - \frac{1}{2} \nu \right) A, \quad (187c)$$

and

$$\kappa = \frac{\cosh 4kh + 8 - 2 \tanh^2 kh}{16 \sinh^4 kh}, \quad k_\infty = \frac{\omega^2}{g} \quad \text{and} \quad \sigma = \coth kh, \quad (187d)$$

with  $\mu$  and  $\nu$  coefficients depending amongst others on the bottom slope which are given in Eqs. (B1) and (B2) of Liu and Dingemans (1989).

In order to find an initial condition on the horizontal-bottom part of the shelf-like geometry, Dingemans *et al.* (1991) substituted the locked wave solution  $\bar{\phi}_\ell$  in the expression for  $\mathcal{G} \bar{\phi}$  in Eq. (187a). This leads to the NLS equation

<sup>§</sup>Notice that, because  $gk/\sinh 2kh = \omega^2/4 \sinh^2 kh$ , the long-wave equation as given in Dingemans *et al.*, 1991, Eq. (9), is equal to the formulation (185b).

for horizontal bottom,

$$\left( i \frac{\partial A}{\partial t} + c_g \frac{\partial A}{\partial x} \right) + \frac{1}{2} \frac{\partial c_g}{\partial k} \frac{\partial^2 A}{\partial x^2} - \nu_2 |A|^2 A = 0, \quad (188a)$$

with

$$\nu_2 = \left( \frac{\omega k_\infty c_g}{2g \sinh^2 kh} + k \right) \frac{\frac{kg^2}{2\omega} + \frac{\omega^2 c_g}{4 \sinh^2 kh}}{c_g^2 - gh} + k^2 \omega \kappa. \quad (188b)$$

As shown in section 5, an initial condition for solution of the wave envelope follows from the horizontal-bottom equation (188a) as:

$$A(x, t) = a \operatorname{sech} \left( \sqrt{\frac{-\nu_2}{\partial c_g / \partial k}} a(x - c_g t) \right). \quad (189)$$

The corresponding bound long-wave is given by the differential equation (186).

The solution of  $\bar{\phi}$  and  $A$  is achieved by a simultaneously numerical solution of Eqs. (187a) and (185b). The solution for  $\bar{\phi}$  is the combination of free and locked long waves. The omission of the reflection in the derivation of the evolution equation for the wave envelope (see Liu and Dingemans, 1989) leads to the consequence that there is no reflected component of the locked-wave. For the free wave, we have both a forward ( $\bar{\phi}_f^+$ ) and a backward ( $\bar{\phi}_f^-$ ) propagating component. We have,

$$\bar{\phi} = \bar{\phi}_\ell + \bar{\phi}_f^+ + \bar{\phi}_f^-. \quad (190)$$

The solution of the long-wave equation is considered now. Because Dingemans *et al.* (1991) wanted a solution for  $\bar{\phi}$  itself and not only for its derivatives, the following method was used. Divergence-like and curl-like differential operators  $\mathcal{D}$  and  $\mathcal{R}$  were introduced by:

$$\mathcal{D}\mathbf{u} \equiv \mathcal{D} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \frac{\partial u_1}{\partial t} + \frac{\partial u_2}{\partial x} \quad \text{and} \quad \mathcal{R}(p) = \begin{pmatrix} \frac{\partial p}{\partial x} \\ -\frac{\partial p}{\partial t} \end{pmatrix}. \quad (191)$$

It can be proven that any twice differentiable vector field  $\mathbf{u}$  with  $\mathcal{D}\mathbf{u} = 0$  has the property  $\mathbf{u} = \mathcal{R}(p)$  for some scalar function  $p$  determined uniquely up to a constant, in the same way as the usual stream function for incompressible flow is determined.<sup>h</sup> The long-wave equation (185b) can also be written in the

<sup>h</sup>The proof follows simply by substitution of  $\mathbf{u} = \mathcal{R}(p)$  into  $\mathcal{D}\mathbf{u} = 0$ .

form of  $\mathcal{D}\mathbf{u}$ . With the special choice  $p = -gh\psi$ , with  $\psi$  an auxiliary function, Eq. (185b) then is equivalent to the system,

$$\frac{\partial \bar{\phi}}{\partial t} + \frac{\partial}{\partial x}(gh\psi) = - \left( \frac{\omega_0}{2 \sinh q} \right)^2 |A|^2, \quad (192a)$$

$$\frac{\partial \psi}{\partial t} + \frac{\partial \bar{\phi}}{\partial x} = - \frac{gk_0}{2\omega_0 h} |A|^2. \quad (192b)$$

The set (192) now replaces the long-wave equation (185b). It has to be stressed that the variable  $\psi$  is only an auxiliary variable, it has no physical significance and  $\psi$  even does not fulfill the wave equation (185b).

The computations are performed in the interval  $0 \leq x \leq L$ . At both the inflow and the outflow boundaries, we take the bottom to be horizontal in order to facilitate the application of radiating boundary conditions. For the envelope  $A$ , we give the value at  $x = 0$  and at  $x = L$ , we give the weakly reflection condition  $(\partial_t + c_g \partial_x)A = 0$ . For the boundary condition of the long-wave potential  $\bar{\phi}$ , distinction has to be made between the three long-wave components: the locked-wave component  $\bar{\phi}_\ell^+$  and the free waves  $\bar{\phi}_f^+$  and  $\bar{\phi}_f^-$ . For a horizontal bottom, these components satisfy,

$$(\partial_t + c_g \partial_x) \bar{\phi}_\ell^+ = 0, \quad (\partial_t + \sqrt{gh} \partial_x) \bar{\phi}_f^+ = 0, \quad (\partial_t - \sqrt{gh} \partial_x) \bar{\phi}_f^- = 0. \quad (193)$$

The inflow condition for  $\bar{\phi}$  is found as the sum of these equations. Using also the solution of Eq. (186), we obtain as weakly reflective condition at the inflow boundary,

$$\frac{\partial \bar{\phi}}{\partial t} - \sqrt{gh} \frac{\partial \bar{\phi}}{\partial x} = \frac{g^2 k}{2\omega} + c_g \left( \frac{\omega}{2 \sinh kh} \right)^2 |A|^2 - 2\sqrt{gh} \frac{\partial \bar{\phi}_f^+}{\partial x}. \quad (194a)$$

The inflow condition for the auxiliary function  $\psi$  can be obtained from the system of Eq. (192) with  $\bar{\phi}$  being substituted into the boundary condition, Eq. (194a), resulting in:

$$\frac{\partial \psi}{\partial t} - \sqrt{gh} \frac{\partial \psi}{\partial x} = \frac{c_g gk}{2\omega h} + \left( \frac{\omega}{2 \sinh kh} \right)^2 |A|^2 - 2 \frac{\partial \bar{\phi}_f^+}{\partial x}. \quad (194b)$$

Similar conditions can be derived at the outflow side, see Dingemans *et al.* (1991). It has to be remarked that the weak point in the analysis of Dingemans

*et al.* (1991) in this respect is that for the locked wave, the assumption of a single wave group has been used. However, after progressing over an uneven bottom part, usually the wave group is split in two or more groups with their own velocity. At the inflow boundary, one can make sure that a single wave group is entering the computational domain. This restriction can also be rephrased in the following way: the coupling between the wave envelope and the long wave should be weak or the coupling term  $\mathcal{G}\bar{\phi}A$  should be weak.

From the computed results, the mean free-surface elevation  $\bar{\zeta}$  follows from Eq. (182a).

The determination of the free and bound components can be performed only for a horizontal bottom. Because both  $\partial\bar{\phi}/\partial t$  and  $\partial\bar{\phi}/\partial x$  are available from the numerical computation, we also have,

$$\frac{\partial\bar{\phi}}{\partial t} = \frac{\partial\bar{\phi}_\ell^+}{\partial t} + \frac{\partial\bar{\phi}_f^+}{\partial t} + \frac{\partial\bar{\phi}_f^-}{\partial t} \quad \text{and} \quad \frac{\partial\bar{\phi}}{\partial x} = \frac{\partial\bar{\phi}_\ell^+}{\partial x} + \frac{\partial\bar{\phi}_f^+}{\partial x} + \frac{\partial\bar{\phi}_f^-}{\partial x}. \quad (195)$$

The sets (193) and (195) yield five equations for six unknowns. Together with Eq. (186), the system is closed and can be solved. The splitting in free and bound wave components is thus achieved for a horizontal bottom only. This procedure is usually also used for uneven bottom with the understanding that the procedure then is not very accurate but with the hope of sufficient accuracy. However, this is still to be proven. The solution of these six equations leads to:

$$\partial_x\bar{\phi}_f^+ = \partial_x\bar{\phi} - C_1 - C|A|^2, \quad \partial_x\bar{\phi}_f^- = C_1, \quad \partial_x\bar{\phi}_\ell^+ = C|A|^2, \quad (196a)$$

with

$$C = \frac{1}{c_g^2 - gh} \left[ \frac{g^2k}{2\omega} + c_g \left( \frac{\omega}{2 \sinh kh} \right)^2 \right], \quad (196b)$$

and

$$C_1 = \frac{1}{2c} [\partial_t\bar{\phi} + c\partial_x\bar{\phi} - (c - c_g)C|A|^2], \quad (196c)$$

and  $c = \sqrt{gh}$ . The solutions for  $\partial_t\bar{\phi}_f^+$ ,  $\partial_t\bar{\phi}_f^-$  and  $\partial_t\bar{\phi}_\ell^+$  follow immediately from Eqs. (193). An example is given in Dingemans *et al.* (1991).

## 8. Observations of Wave Modulations

### 8.1. Theoretical aspects of modulational instability

Basic ideas of Benjamin–Feir instability mechanism have been presented in the introductory section 2. The first mathematical treatment of instability of water