

Chapter 1

The Breakdown of Classical Mechanics

1.1 Quantum Number of the Earth

Calculate the principle quantum number for the earth in its orbit about the sun. What is the energy difference between two neighbouring energy levels?

Hint: For large n , $E_n \approx E_{classical}$.

Solution

The classical energy of the earth in its orbit about the sun is

$$E = -\frac{GmM}{2R} \quad (1.1.1)$$

where

m is the mass of the earth = 6×10^{27} g ,

M is the mass of the sun = 2×10^{33} g ,

G is the gravitational constant = 6.67×10^{-8} dyn cm²/g² and

R is the sun-earth distance = 1.5×10^{13} cm .

Proceeding as for the hydrogen atom we find that we need only replace e^2 by GmM . Thus, we find

$$E = -\frac{1}{2} \left(\frac{2\pi}{h} GmM \right)^2 m \frac{1}{n^2} . \quad (1.1.2)$$

Solving for n we get

$$n^2 = R \left(\frac{2\pi}{h} \right)^2 GMm^2 = 18 \times 10^{146} . \quad (1.1.3)$$

Therefore,

$$n \approx 4 \times 10^{73} . \quad (1.1.4)$$

Also we see that since

$$\Delta E = \frac{1}{2} \left(\frac{2\pi}{h} GmM \right)^2 m \frac{\Delta n}{n^3} = |E| \frac{\Delta n}{n} \quad (1.1.5)$$

we get $\Delta E = 6.4 \times 10^{-34}$ erg .

1.2 Thermal Wavelength

What is the wavelength associated with gas molecules at a temperature T ? Estimate this wavelength for a typical gas at room temperature and compare it to visible light.

Solution

The energy U of a molecule at temperature T is given by the equipartition principle

$$U = \frac{3}{2} k_B T = \frac{hc}{\lambda} . \quad (1.2.6)$$

Therefore, the wavelength is

$$\lambda = hc/U . \quad (1.2.7)$$

At room temperature $T \approx 300$ K. So,

$$U = 1.5 \times 1.38 \times 10^{-23} \times 300 \text{ J} = 6.21 \times 10^{-21} \text{ J} . \quad (1.2.8)$$

$$\lambda = \frac{6.63 \times 10^{-34} \times 3.00 \times 10^8}{6.21 \times 10^{-21}} = 3.20 \times 10^{-5} \text{ m} . \quad (1.2.9)$$

This is considerably longer than visible light which has a wavelength of about 5×10^{-7} m .

1.3 Photons in a Beam

For a monochromatic beam of electromagnetic radiation of wavelength ($\lambda \approx 5000 \text{ \AA}$), intensity $I = 1 \text{ watt/m}^2$, calculate the number of photons passing an area of $A = 1 \text{ cm}^2$ normal to the beam in one second.

Solution

The energy of a photon is given by

$$E_{\text{photon}} = h\nu = \frac{hc}{\lambda} . \quad (1.3.10)$$

The total energy of the beam is $E = IA = NE_{\text{photon}}$ where N is the number of photons. Therefore,

$$N = \frac{IA\lambda}{hc} = \frac{1 \times 10^{-4} \times 5000 \times 10^{-10}}{6.63 \times 10^{-34} \times 3.00 \times 10^8} = 2.5 \times 10^{14} \text{ photons} . \quad (1.3.11)$$

1.4 Hydrogen Atom and de Broglie

Show that if one assumes that the circumference of a stationary state orbit of an electron in a hydrogen atom is an integral multiple of the de Broglie wavelength, one also obtains the correct energy levels.

Solution

The de Broglie wavelength is given by

$$\lambda = h/p .$$

The circumference is $2\pi r$. Thus, we write

$$2\pi r = n\lambda .$$

The energy is given by

$$E = \frac{p^2}{2m} - \frac{e^2}{r} . \quad (1.4.12)$$

We substitute for p and r in terms of λ and introduce the fine structure constant

$$\alpha = \frac{2\pi e^2}{hc} \quad (1.4.13)$$

to get

$$E = \frac{h^2}{2m\lambda^2} - \frac{hc\alpha}{n\lambda} . \quad (1.4.14)$$

We now equate the Coulomb force with the mass times the centripetal acceleration

$$\frac{mv^2}{r} = \frac{2\pi e^2}{n\lambda} . \quad (1.4.15)$$

This yields

$$\frac{h^2}{m\lambda^2} = \frac{2\pi e^2}{n\lambda} . \quad (1.4.16)$$

We can now solve for λ .

$$\frac{1}{\lambda} = \frac{m\alpha c}{h} \frac{1}{n} . \quad (1.4.17)$$

Combining this with our previous result we find

$$E = \frac{m\alpha^2 c^2}{2n^2} - \frac{m\alpha^2 c^2}{n^2} = -\frac{1}{2} \frac{m\alpha^2 c^2}{n^2} . \quad (1.4.18)$$

1.5 Vibrations in NaCl

The shortest possible wavelength of sound in sodium chloride is twice the lattice spacing, about 5.8×10^{-8} cm. The sound velocity is approximately 1.5×10^5 cm/sec.

- Compute a rough value for the highest sound frequency in the solid.
- Compute the energy of the corresponding phonons, or quanta of vibrational energy.
- Roughly what temperature is required to excite these oscillations appreciably?

Solution

The shortest wavelength is given by

$$\lambda_{min} = 5.8 \times 10^{-8} \text{ cm} = 2 \times \text{lattice spacing} \quad (1.5.19)$$

$$v_0 = 1.5 \times 10^5 \text{ cm/s} . \quad (1.5.20)$$

a)

$$\nu = \frac{v_0}{\lambda} . \quad (1.5.21)$$

Therefore,

$$\nu_{max} = \frac{v_0}{\lambda_{min}} = 2.6 \times 10^{12} \text{ Hz} . \quad (1.5.22)$$

b)

$$E_{max} = h\nu_{max} = 1.7 \times 10^{-21} \text{ J} = 0.011 \text{ eV} . \quad (1.5.23)$$

c)

$$u = \frac{3}{2} k_B T . \quad (1.5.24)$$

Assume $u \approx E_{max}$. Therefore,

$$T \approx \frac{2E_{max}}{3k_B} = 83 \text{ K} . \quad (1.5.25)$$

1.6 Crystal Powder

Estimate the effect on the specific heat of reducing a crystal to a fine powder of dimensions of about 10^{-5} cm.

Hint: Study the Debye model [1.1] of specific heat and realize that the size of the crystal now also imposes an upper limit on the wavelength of the sound waves in the crystal.

Solution

The effect of grinding up the crystal into a powder is to limit the maximum wavelength of a standing wave in the crystal to roughly the size of the crystal particles. This changes the integral in the Debye expression [1.1] for the internal energy from

$$U = \frac{12\pi(k_B T)^4}{h^3 v_0^3} \int_0^{x_{max}} \frac{x^3 dx}{e^x - 1} \quad (1.6.26)$$

to

$$\begin{aligned} U_{\text{powder}} &= \frac{12\pi(k_B T)^4}{h^3 v_0^3} \int_{x_{min}}^{x_{max}} \frac{x^3 dx}{e^x - 1} \\ &= U - \frac{12\pi(k_B T)^4}{h^3 v_0^3} \int_0^{x_{min}} \frac{x^3 dx}{e^x - 1} . \end{aligned} \quad (1.6.27)$$

Here, we have introduced

$$x_{min} = \frac{h}{k_B T} \nu_{min} = \frac{h}{k_B T} \frac{v_0}{\lambda_{max}} \quad (1.6.28)$$

where $\lambda_{max} = 10^{-5}$ cm \approx size of the powder particles. If we now estimate x_{min} at room temperature by using that $v_0 \approx 500$ m/s, we get $x_{min} \approx 8 \times 10^{-4} \ll 1$. Therefore, we can approximate the exponential in the last integral by $e^x \approx 1 + x$ and get

$$\int_0^{x_{min}} \frac{x^3 dx}{e^x - 1} \approx \int_0^{x_{min}} x^2 dx = \frac{1}{3} x_{min}^3 . \quad (1.6.29)$$

This means that the internal energy U is reduced by

$$\Delta U = \frac{12\pi(k_B T)^4}{h^3 v_0^3} \times \left(\frac{1}{3} \frac{h v_0}{k_B \lambda_{max}} \right)^3 = \frac{4\pi k_B T}{\lambda_{max}^3} . \quad (1.6.30)$$

Therefore, the specific heat per unit volume at constant volume

$$\frac{1}{V} \frac{\partial U}{\partial T}$$

is decreased by a constant amount, namely

$$\frac{4\pi k_B}{\lambda_{max}^3} = \frac{4\pi \times 1.38 \times 10^{-23} \text{ J/K}}{(10^{-7})^3 \text{ m}^3} \approx 0.173 \text{ J/(K m}^3\text{)} . \quad (1.6.31)$$

1.7 Einstein Coefficients

For a collection of atoms with energies E_n , $n = 1, 2, 3, \dots$ submerged in a background of radiation at a temperature T , the following transitions may occur:

1) spontaneous from $n \rightarrow m$ $E_n > E_m$

- 2) induced from $n \rightarrow m$
 3) induced from $m \rightarrow n$.

At equilibrium, at a temperature T , the emission and absorption probabilities are given by

$$P_{mn} = KN_n [B_{mn}\rho(\nu) + A_{mn}] \quad \text{emission} \quad (1.7.32)$$

$$P_{nm} = KN_m [B_{nm}\rho(\nu)] \quad \text{absorption} . \quad (1.7.33)$$

Here, $h\nu = E_n - E_m$, K is a proportionality constant, and N_n , N_m are respectively the number of atoms in the states n and m . The coefficients A_{mn} , B_{mn} are known respectively as the ‘‘Einstein Coefficients’’ of spontaneous and induced emission, whereas the coefficient B_{nm} is known as the ‘‘Einstein Coefficient’’ of induced absorption. [1.2] Use these equations together with Planck’s radiation law for the radiation density $\rho(\nu)$ at equilibrium to show that

- 1) the Einstein coefficients of induced absorption and emission are the same, that is that $B_{nm} = B_{mn}$ and that
 2) the Einstein coefficients of spontaneous and induced emission are related by

$$A_{nm} = \frac{8\pi}{c^3} h\nu^3 B_{nm} .$$

Solution

In equilibrium, at a temperature T , if the number of atoms in the state n and m is given by N_n and N_m respectively, we have that

$$N_n = N e^{-E_n/k_B T} \quad , \quad N_m = N e^{-E_m/k_B T} \quad (1.7.34)$$

where N is the total number of atoms. Therefore,

$$N_m = N_n e^{(E_n - E_m)/k_B T} = N_n e^{h\nu/k_B T} . \quad (1.7.35)$$

Also at equilibrium the number of transitions from $n \rightarrow m$ equals the number of transitions from $m \rightarrow n$. Thus, we have

$$N_m P_{nm} = N_n P_{mn} \quad (1.7.36)$$

or

$$N_n e^{h\nu/k_B T} B_{nm}\rho(\nu) = N_n [B_{mn}\rho(\nu) + A_{nm}] . \quad (1.7.37)$$

Thus, solving for the radiation density ρ we get

$$\rho(\nu) = \frac{A_{nm}/B_{nm}}{e^{h\nu/k_B T} - B_{nm}/B_{mn}} . \quad (1.7.38)$$

But, at equilibrium the radiation density is given by Planck’s Law

$$\rho(\nu) = \frac{8\pi}{c^3} \frac{h\nu^3}{e^{h\nu/k_B T} - 1} \quad (1.7.39)$$

Therefore, comparing these two equations we see that we have

$$B_{nm} = B_{mn} \quad (1.7.40)$$

and

$$A_{nm} = \frac{8\pi}{c^3} h\nu^3 B_{nm} . \quad (1.7.41)$$

Bibliography

- [1.1] A.Z. Capri, *Nonrelativistic Quantum Mechanics* 3rd edition, World Scientific Publishing Co. Pte. Ltd., section 1.12, (2002) .
- [1.2] F.K. Richtmyer, E.H. Kennard, and J.N. Cooper, *Introduction to Modern Physics*, 6th edition, sec 13.12, McGraw-Hill, New York, (1969).