Chapter 1

Examples of Control Systems Described by Functional Differential Equations

1.1 Ship Stabilization and Automatic Steering

A ship is rolling in the waves. The differential equation satisfied by the angle of tilt x from the normal upright position is given by

$$m\frac{d^2x(t)}{dt^2} + b\frac{dx(t)}{dt} + kx(t) = 0 \tag{1.1.1}$$

where $m > 0$ is the mass, $b > 0$ is the damping constant, and $k > 0$ is a constant. The damping constant b determines how fast the ship returns to its upright position. It is known that if $b^2 < 4mk$, the ship is "underdamped" and the ship oscillates as it returns to its upright position. If $b^2 > 4mk$, it is "overdamped", there is no oscillation, but it returns to its upright position more slowly the larger b becomes. When $b^2 = 4mk$, it is "critically damped" and upright position is achieved more rapidly. This "roll-quenching" was a very important problem tackled by engineers for ships and destroyers of the second world war. In one such research by Minorsky [5, 6] ballast tanks, partially filled with water, are introduced in each side of the ship in order to obtain the best value of b. A servomechanism is introduced and is designed to reduce the angle of tilt x, and its velocity, $\frac{dx}{dt}$, to 0 as fast as possible. What the contrivance does is to introduce an input to the natural damping of the rolling ship, a term proportional to the velocity at an earlier instant $t-h : q\dot{x}(t-h)$. The delay h is present because the servomechanism cannot respond instantaneously. It takes this time delay h to respond. Also introduced by the servomechanism is a control with components (u_1, u_2), which yields the following equations:

$$\begin{aligned} \dot{x}(t) &= y(t) + u_1(t)\,, \\ \dot{y}(t) &= -\frac{b}{m}y(t) - \frac{q}{m}y(t-h) - -\frac{k}{m}x(t) + u_2(t)\,. \end{aligned} \tag{1.1.2}$$

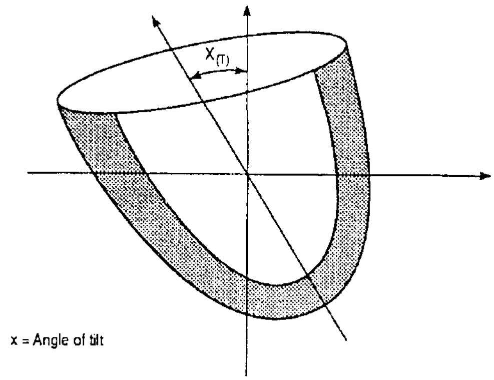

Fig. 1.1.1

We note that (1.1.1) is equivalent to

$$\begin{aligned} \dot{x}(t) &= y(t) \\ \dot{y}(t) &= -\frac{b}{m}y(t) - \frac{k}{m}x(t)\,. \end{aligned} \tag{1.1.3}$$

Equation (1.1.2) can be written in matrix form

$$\dot{\underline{x}}(t) = A_0\underline{x}(t) + A_1\underline{x}(t-h) + Bu(t)\,, \tag{1.1.4}$$

where

$$A_0 = \begin{bmatrix} 0\,, & 1 \\ -\dfrac{k}{m}\,, & -\dfrac{b}{m} \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0 & 0 \\ 0 & -\dfrac{q}{m} \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \underline{x} = \begin{bmatrix} x \\ y \end{bmatrix}.$$

Using matrix notation, (1.1.3) assumes the form

$$\dot{\underline{x}}(t) = A_0\underline{x}(t) + A_1\underline{x}(t-h)\,. \tag{1.1.5}$$

The point (0, 0) can be regarded as an equilibrium position and the aim of the gadget is to steer the angle of tilt and its velocity to the equilibrium position. There are three problems. The first is the problem of stability: Determine necessary and sufficient conditions on A_0, A_1 such that the solution of (1.1.5) satisfies

$$\underline{x}(t) \to 0 \quad \text{as} \quad t \to \infty\,. \tag{1.1.6}$$

But it is undesirable to wait forever for the system to attain the upright position. The second problem is: Is it possible that a control u, introduced by the servomechanism, can drive the system to the equilibrium in finite time? Is the system controllable? The third problem is: Find a control strategy u that will drive the angle of tilt and its velocity to zero in minimum time. Such a control is called a time-optimal control. Our ultimate goal in optimal control theory is to get an optimal control as a function of the appropriate state space, i.e., to obtain the "feedback" or "closed loop" control. The major advantage of such a control, as opposed to an "open loop" one with u as a function of t, is that the system in question becomes self-correcting and automatic.

1.2 Predator-Prey Interactions

Let $x(t)$ be the population at time t of a species of fish called prey, and let $y(t)$ be the population of another species called the predator, which lives off the prey. Under the assumption that without the predator present the prey will increase at a rate proportional to $x(t)$, and that the feeding action of the predator reduces the growth rate of the prey by an amount proportional to the product $x(t)y(t)$, we have

$$\dot{x}(t) = a_1 x(t) - b_1 x(t)y(t).$$

If the predator eats the prey and breeds at a rate proportional to its number and the amount of food available, then

$$\dot{y}(t) = -a_2 y(t) + b_2 x(t)y(t),$$

where a_1, a_2, b_1, b_2 are positive constants. The system of two equations

$$\begin{aligned} \dot{x}(t) &= a_1 x(t) - b_1 x(t)y(t), \\ \dot{y}(t) &= -a_2 y(t) + b_2 x(t)y(t), \end{aligned} \tag{1.2.1}$$

is rather naive. A more realistic model assumes that the birthrate of the prey will diminish as $x(t)$ grows because of overcrowding and shortage of available food. In this model it is assumed that there is a time delay of period h for the predator to respond to changes in the sizes of x and y. Thus

$$\begin{aligned} \dot{x}(t) &= a_1 \left[1 - \frac{x(t)}{p}\right] x(t) - b_1 x(t)y(t), \\ \dot{y}(t) &= -a_2 y(t) + b_2 x(t-h)y(t-h), \end{aligned} \tag{1.2.2}$$

where p is a positive constant.

Volterra [9] studied (1.2.1) and (1.2.2) and various generalizations of (1.2.2). One such system is given by

$$\begin{aligned} \dot{x}(t) &= \left[a_1 - c_1 x(t) - b_1 y(t) - \int_{-h}^{0} g_1(s)y(t+s)ds\right] x(t), \\ \dot{y}(t) &= \left[-a_2 - c_2 y(t) + b_2 x(t) + \int_{-h}^{0} g_2(s)x(t+s)ds\right] y(t), \end{aligned} \tag{1.2.3}$$

where g_1, g_2 are continuous, nonnegative functions, and c_1, c_2 are constants.

In the interaction between the prey and the predator it is important to ask whether there are equilibrium states that may be reached for the systems (1.2.3).

If these states are not zero, then neither the predator nor the prey is extinct and the following is true:

$$a_1 - c_1 x(t) - b_1 y(t) - \int_{-h}^{0} g_1(s) y(t+s) ds = 0\,,$$

$$a_2 - c_2 y(t) + b_2 x(t) + \int_{-h}^{0} g_2(s) x(t+s) ds = 0\,.$$

The function $z^* = (x^*, y^*)$, which solves the above functional equation, is the equilibrium; it may well be the saturation levels of the given species. In this case it may be desirable that every solution $(x(t), y(t)) = z(t)$ of (1.2.3) satisfies $z(t) \to z^*$ as $t \to \infty$. This is an asymptotic stability property. The state z^* is not attained in finite time. A more desirable objective is to "manage" the interaction and to have the population as near as possible to its equilibrium position, and to prevent near periodic outbreaks of predator population y beyond its equilibrium. We aim at coexistence of the two species at their equilibrium states, which are also the saturation levels of the two populations.

One management strategy is fishing. Man is interested in harvesting one or both species at some rates e_i, $i = 1, 2$. For this situation the dynamics of the interaction are

$$\begin{aligned} \dot{x}(t) &= \left[a_1 - c_1 x(t) - b_1 y(t) - \textstyle\int_{-h}^{0} g_1(s) y(t+s) ds\right] x(t) - e_1(t) x(t)\,, \\ \dot{y}(t) &= \left[-a_2 - e_2 y(t) + b_2 x(t) + \textstyle\int_{-h}^{0} g_2(s) x(t+s) ds\right] y(t) - e_2(t) y(t)\,. \end{aligned} \tag{1.2.4}$$

If $u_1(t) = e_1 x(t)$, $u_2(t) = e_2 y(t)$, then $u_i(t)$ is the harvest rate that is proportional to the population density. The effort level $e_i(t)$ is a positive function with dimension 1/time, which measures the total effort at time t made to harvest a given species, and u_i is a piecewise continuous function $[0, T] \to [0, b]$, $T > 0$, $b > 0$. Thus $e_i(t)$ can be considered a control function. The function $u = (u_1, u_2)$ represents the rate the fisherman

(i) selectively kills the prey x,
(ii) selectively kills the predator y, or
(iii) kills both x and y.

There are two problems. By harvesting, what conditions on the systems' parameters ensure that the two species can be driven to the equilibrium z^* in finite time. This is a problem of controllability. If $e_i(t)$ is negative in (1.2.4), u can be said to represent the rate at which laboratory-reared

(i) fishes x, or
(ii) predators y

are released into the system.

One can use a combination of release and harvesting strategies to drive x, y to the saturation level z^* in finite time.

The second question is optimality: What is the best harvesting strategy that will, as quickly as possible, drive the system from any initial population to this equilibrium? Good fish management is interested in the time-optimal control problem for the system (1.2.4).

A more general version of this system is given by

$$\dot{z}(t) = L(t, z_t)z(t) + B(z(t), t)e(t)\,,$$

where $L(t, \phi)$ is a 2×2 matrix function, which is linear in $z(t)$; and $B(z(t), t)$ is the matrix

$$\begin{bmatrix} \beta_1(x(t), t) & 0 \\ 0 & \beta_2(y(t), t) \end{bmatrix}.$$

A more general n-dimensional version of this system is given by

$$\dot{x}(t) = L(t, x_t)x(t) + B(x(t), t)u(t)\,, \tag{1.2.5}$$

where B is an $n \times m$ matrix function, u is in E^m, and

$$L(t, x_t) = \sum_{k=0}^{\infty} A_k(t)x(t - w_k) + \int_{-h}^{0} A(t, s)x(t + s)\,, \qquad 0 \le w_k \le h\,.$$

Here A_k is an $n \times n$ continuous matrix function, $A(t, s)$ is integrable in s for each t, and there is a function $a(t)$ that is integrable such that

$$\left| \int_{-h}^{0} A(t, s)\phi(s)ds \right| \le a(t)\|\phi\|\,.$$

The three questions that we posed for (1.2.4) can also be asked for (1.2.5) or for the general nonlinear equation

$$\dot{x}(t) = f(t, x_t) + B(t, x_t)u(t)\,, \tag{1.2.6}$$

where $f : E \times C \to E^n$, $B : E \times C \to E^{n \times m}$ is an $n \times m$ matrix valued function, and u is a m-vector valued function. Here $E = (-\infty, \infty)$, E^n is the n-dimensional Euclidean space, and $C = C([-h, 0],\ E^n)$ is the space of continuous functions from $[-h, 0] \to E^n$. The symbol $x_t \in C$ is a function defined $x_t(s) = x(t + s)$ $s \in [-h, 0]$.

In the discussion above we required all populations to be driven to the equilibrium, which can be taken as a target. A more general target can be assumed to be

$$T = \{\phi \in C : H\phi = \rho\}\,. \tag{1.2.7}$$

Here H is a linear operator and $\rho \in C$. The target represents the final configuration at which it is desirable for the species to lie after harvesting.

1.3 Fluctuations of Current

Let us consider an electric circuit in which two resistances, a capacitance and inductance, are connected in series. Assume that current is flowing through the loop, and its value at time t is $x(t)$ amperes. We use also the following units: volts for the voltage, ohms for the resistance R, henry for the inductance L, farads for the capacitance c, coulombs for the charge on the capacitance, and seconds for the time t. It is well known that with these systems of units, the voltage drop across the inductance is $L\frac{dx(t)}{dt}$, and that across the resistances it is $(R+R_1)x(t)$. The voltage drop across the capacitance is q/c where q is the charge on the capacitance. It is also known that $x(t) = \frac{dq}{dt}$. A fundamental law of Kirchhoff states that the sum of the voltage drops around the loop must be equal to the applied voltage:

$$L\frac{dx}{dt} + (R+R_1)x(t) + \frac{1}{c}q = 0\,.$$

On differentiating with respect to t we deduce

$$L\frac{d^2x(t)}{dt^2} + (R+R_1)\frac{dx}{dt} + \frac{1}{c}x(t) = 0\,. \tag{1.3.1}$$

In Fig. 1.3.2, the voltage across R_1 is applied to a nonlinear amplifier A. The output is provided with a special phase-shifting network P. This introduces a constant time lag between the input and output P. The voltage drop across R in series with the output P is

$$e(t) = qg(\dot{x}(t-h))\,;$$

q is the gain of the amplifier to R measured through the network. The equation becomes

$$L\frac{d^2x(t)}{dt^2} + R\dot{x}(t) + qg(\dot{x}(t-h)) + \frac{1}{c}x(t) = 0\,.$$

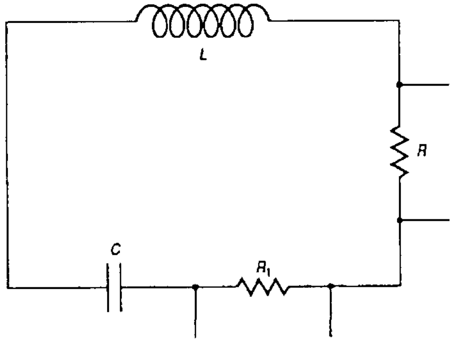

Fig. 1.3.1

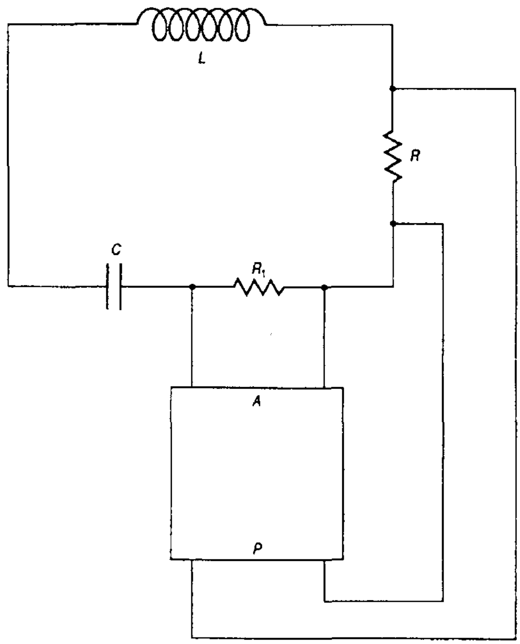

Fig. 1.3.2

Finally, a control device is introduced to help stabilize the fluctuations of the current. If $\dot{x}(t) = y(t)$, the "controlled" system may be described by

$$\begin{aligned} \dot{x}(t) &= y(t) + u_1(t)\,, \\ \dot{y}(t) &= -\frac{R}{L}y(t) - \frac{q}{L}g(y(t-h)) - \frac{1}{cL}x(t) + u_2(t)\,. \end{aligned} \tag{1.3.2}$$

The control $u = (u_1, u_2)$ is "created" and introduced by the stabilizer.

The three basic questions are now natural: With $u = 0$, what are the properties of the systems parameter which will ensure that $x^2(t) + y^2(t) \to 0$ as $t \to \infty$? Will "admissible controls" u (say $|u_j| \leq 1,\ j = 1, 2$) bring any wild fluctuations of current (any initial position) to a normal equilibrium in finite time? Can this be done in minimum time?

1.4 Control of Epidemics

1.4.1 *First model*

In this subsection we formulate a theory of epidemics, a problem of time-optimal control theory. It is a slight modification of Bank's presentation of the work of Gupta and Rink [2]. We assume as basic that there are four types of individuals in our population:

(i) susceptibles: $X_1(t)$,
(ii) exposed and infected but not yet infectious (infective) individuals: $X_2(t)$,
(iii) infectives: $X_3(t)$, and
(iv) removed individuals: X_4. We include in (iv) those who have recovered and gained immunity and those who have been removed from the population because of observable symptoms.

We assume a constant rate A of inflow of new susceptible members. The latent period h_1 is the period from the time an individual is exposed and becomes infected to the time he becomes infective. The infectious period h_2 is the period the individual is in the infectious (infective) class. The incubation period will denote the time from exposure and infectedness to the time of removal (i.e., $h_1 + h_2 = \tau$). After normalization by setting $x_i = X_i/N$, where N is the population, the dynamics of interaction is

$$\begin{aligned} \dot{x}_1(t) &= -\beta x_1(t)x_3(t) + A\,, \\ \dot{x}_2(t) &= \beta x_1(t)x_3(t) - \beta x_1(t-h_1)x_3(t-h_1)\,, \\ \dot{x}_3(t) &= \beta x_1(t-h_1)x_3(t-h_1) - \beta x_1(t-h_1-h_2)x_3(t-h_1-h_2)\,, \\ \dot{x}_4(t) &= \beta x_1(t-h_1-h_2)x_3(t-h_1-h_2) - A\,. \end{aligned} \tag{1.4.1}$$

The constant β is the average number of individuals per unit time that any individual will encounter, sufficient to cause infection. To control the epidemic we allow two strategies:

(i) The removal at some rates u_i $i = 2, 3$ of both the infectives and exposed and infected individuals. The removal rate $u_i(t) = e_i(t)b_i(x_i(t))$ is proportional to a function b_i of the population density and the effort level $e_i(t)$.
(ii) Active immunization, the injection of dead or live but attenuated disease microorganisms into members of the population, resulting in antibodies in the population of vaccinated individuals. If $E_1(t)$ represents the reliable rate per day at which members of the population are being actively vaccinated at time t, the normalized rate is

$$e_1(t) = E_1(t)/N\,.$$

We assume that the effective immunization rate is $b_1(x_1(t))e_1(t) = u_1(t)$. If we assume that there is a delay h ($h_i \leq h$, $i = 1, 2$) days before the antibodies become effective, then the rate is $b_1(x_1(t))e_1(t-h) = u_1(t)$. Thus the dynamical system is

$$\begin{aligned} \dot{x}_1(t) &= -\beta x_1(t)x_3(t) - b_1(x_1(t))e_1(t-h) + A\,, \\ \dot{x}_2(t) &= \beta x_1(t)x_3(t) - \beta x_1(t-h_1)x_3(t-h_1) - u_2(t)\,, \\ \dot{x}_3(t) &= \beta x_1(t-h_1)x_3(t-h_1) - \beta x_1(t-h_1-h_2)x_3(t-h_1-h_2) - u_3(t)\,. \end{aligned} \tag{1.4.2}$$

We use control strategies (i) and (ii) to reduce the total number of infected to an "acceptable" size in finite time. This health policy may be used to "prevent an epidemic" in minimum time, where preventing an epidemic means that the solution of (1.4.2) satisfies

(i) $|x_2(t)| + |x_3(t)| \leq A$ for all $t \geq 0$.
(ii) $\max_{s\in[0,T]} |x_3(s)| \leq B_1$ where $A_1 B_1 A$, B are prescribed constants.

The solution of (1.4.2) may be viewed as lying in the space $C([-h, 0], E^3)$ the space of continuous functions mapping $[-h, 0]$ into E^3 with the sup norm. Or it may be viewed as lying in E^3. We define a subset $T \subset E^3$ by

$$T = \{x \in E^3 : |x_2| + |x_3| \leq A_1 \,,\ |x_3| \leq B\}\,.$$

The best health policy may be that solutions of (1.4.2) hit the target, T, in minimum time. Also desirable is to have solutions of (1.4.2) to reach and remain in T, and to do so as fast as possible.

The time-optimal control problem formulated in relation to the epidemic models above can also be formulated for the more general nonlinear system

$$\dot{x}(t) = f(t, x(t),\ x(t-h_1)\cdots x(t-h_n),\ u(t),\ u(t-h))\,, \tag{1.4.3}$$

with target set

$$T = \{x \in E^n : Hx = r,\ r \in E^n\}\,,$$

and H an $n \times n$ matrix. Equation (1.4.3) is a special case of the delay system

$$\dot{x}(t) = f(t, x_t, u_t)\,, \qquad x_\sigma = \phi\,, \tag{1.4.4}$$

which includes the ordinary differential equation

$$\dot{x}(t) = k(t, x(t), u(t))\,. \tag{1.4.5}$$

In (1.4.4) x_t is a function defined by $x_t(s) = x(t+s)$ $s \in [-h, 0]$.
The symbol u_t is defined similarly.

1.4.2 *Control of epidemics: AIDS*

Just as in I, we formulate a theory of acquired immunodeficiency syndrome (AIDS) epidemic as a problem of time-optimal control theory. It is a modification of the recent works of Castillo-Chavez, Cooke, Huang, and Levin [14]. We assume as basic that there are five classes (of sexually active male homosexuals with multiple partners):

x_1: susceptible individuals,
x_2: those infectious individuals who will go on to develop full-blown AIDS,
x_3: infectious individuals who will not develop full-blown AIDS,

x_4: those former x_3 who are no longer sexually active, and
x_5: those former x_2 who have developed full-blown AIDS.

Note that if an individual enters x_5 or x_4, he no longer enters into the dynamics of the disease. In contrast to our earlier treatment of the theory of epidemics, a latent class is excluded, i.e., those exposed individuals who are not yet infectious, since only a very short time is spent in that class. It is our assumption that an individual with full-blown AIDS has no sexual contacts and is therefore not infectious. Also, once infected, an individual is immediately infectious, and sexual inactivity or acquisition of AIDS are at the constant rates of a_3 and a_2, respectively, per unit time. We let Λ denote the total recruitment rate into the susceptible class, defined to be those individuals who are sexually active. Let μ be the natural mortality rate, and d the disease-induced mortality due to AIDS. Suppose p is the fraction of the susceptible individuals that after becoming infectious will enter the AIDS class, and $(1-p)$ the fraction that become infectious but will not develop full-blown AIDS. We use the following diagram:

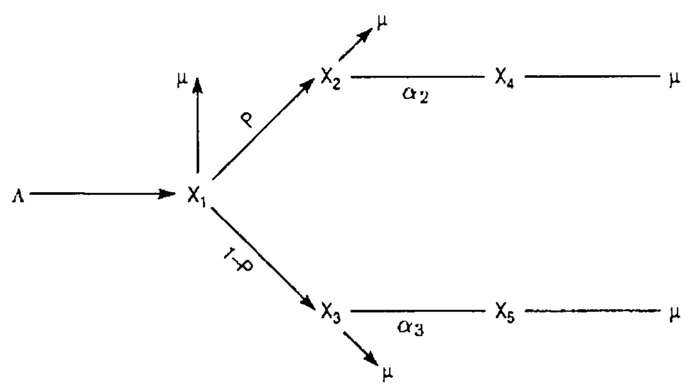

Fig. 1.4.1

to determine the epidemiological dynamics:

$$\frac{dx_1(t)}{dt} = \Lambda - \lambda C(T)(t)x_1(t)\frac{W(t)}{T(t)} - \mu x_1(t)\,, \tag{1.4.6}$$

$$\frac{dx_2(t)}{dt} = \lambda_p C(T)(t)x_1(t)\frac{W(t)}{T(t)} - (\alpha_2+\mu)x_2(t)\,, \tag{1.4.7}$$

$$\frac{dx_3(t)}{dt} = \lambda(1-p)C(T)(t)x_1(t)\frac{W(t)}{T(t)} - (\alpha_3+\mu)x_3(t)\,, \tag{1.4.8}$$

$$\frac{dx_4(t)}{dt} = \alpha_2 x_2(t) - (d+\mu)x_4(t)\,, \tag{1.4.9}$$

$$\frac{dx_5(t)}{dt} = \alpha_3 x_3(t) - \mu x_5(t)\,, \tag{1.4.10}$$

where

$$W = x_2 + x_3 \quad \text{and} \quad T = W + x_1 . \tag{1.4.11}$$

In the above formulation, $C(T)$ represents the mean number of sexual partners an average individual has per unit time when the total population is T, and the constant λ denotes the average sexual risk per partner. Thus the expression $\lambda C(T)SW/T$ denotes the number of newly infected individuals per unit time. Very often the Michaelis–Menten type contact law is accepted for $C(T)$:

$$C(T) = \frac{\beta T}{1 + kT}, \tag{1.4.12}$$

where β, k is constant. Systems (1.4.6–1.4.10) are ordinary differential equations. Further to the assumption above where individuals become immediately infectious and the infectious period is equal to the incubation period, we designate h_2 to be the fixed period of infection for x_2, and h_3 to be that of x_3. We set $x_{20}(t)$, $x_{30}(t)$, to be infectious $x_2(t)$ and $x_3(t)$ at time $t = 0$, and $x_{40}(t), x_{50}(t)$ living $x_4(t), x_5(t)$ at time $t = 0$. Clearly x_{40}, x_{50} have compact support, i.e., $x_{40}(t), x_{50}(t)$ vanish as $t \to \infty$. Note that $x_{20}(t) = x_{30}(t) = 0$ for $t > \max(h_1, h_2)$. If $H(t)$ is the unit step or Heaviside function, i.e., $H(t) = \left\{\begin{smallmatrix} 0, & t<0 \\ 1, & t>1 \end{smallmatrix}\right.$, then the system is modified to be

$$\dot{x}_1(t) = \Lambda - \lambda C(T)(t) x_1(t) \frac{W(t)}{T(t)} - \mu x_1(t), \tag{1.4.13}$$

$$x_2(t) = x_{20}(t) + \lambda p \int_{t-h_2}^{t} C(T)(s) x_1(s) \frac{W(s)}{T(s)} H(s) e^{-\mu(t-s)} ds, \tag{1.4.14a}$$

$$x_3(t) = x_{30}(t) + \lambda(1-p) \int_{t-h_3}^{t} C(T)(s) x_1(s) \frac{W(s)}{T(s)} H(s) e^{-\mu(t-s)} ds, \tag{1.4.14b}$$

$$\begin{aligned} x_4(t) = x_{40}(t) + \lambda p \int_0^{t-h_2} & C(T)(s) x_1(s) \frac{W(s)}{T(s)} H(s) e^{-\mu(t-s)} \\ & - d(t - s - h_2) ds, \end{aligned} \tag{1.4.15a}$$

$$x_5(t) = x_{50}(t) + \lambda(1-p) \int_0^{t-h_3} C(T)(s) \frac{x_1(s) W(s)}{T(s)} H(s) e^{-\mu(t-s)} ds, \tag{1.4.15b}$$

with initial conditions,

$$\begin{aligned} x_1(t) &= x_{10}(t), \qquad t \in [-\max(h_1, h_2), 0] \\ x_2(t) &= x_{20}(t), \\ x_3(t) &= x_{30}(t). \end{aligned} \tag{1.4.16}$$

This is a delay differential equation whose existence and uniqueness of solutions are easily proved. We now restrict our analysis to the case $p = 1$ (i.e., we model AIDS as a progressive disease). The study of the steady states reduces to the following set of equations:

$$\dot{x}_1(t) = \Lambda - \lambda CT(t)x_1(t)\frac{x_2(t)}{T(t)} - \mu x_1(t),$$

$$\dot{x}_2(t) = \lambda|C(T)(t)x_1(t)\frac{x_2(t)}{T(t)} - CT(t-h)x_1(t-h)\frac{x_2(t-h)}{T(t-h)}e^{-\mu h} - \mu x_2(t),$$

with infection free state $(\frac{\Delta}{\mu}, 0)$ as equilibrium. It is possible to allow h_2, h_3 to be distributed. For this we follow the generalizations of [14] and define the survivorship functions $P_2(s), P_3(s)$, which are the proportions of those individuals who become x_2 or x_3 infective at time t and if alive are still infectious at time $t + s$. These are nonnegative nonincreasing with

$$P_2(0) = P_3(0) = 1,$$
$$\int_0^\infty P_2(s)ds < \infty, \qquad \int_0^\infty P_3(s)ds < \infty. \tag{1.4.17}$$

Under these assumptions, $-\dot{P}_2(s)$ and $-\dot{P}_3(s)$ are the removal rates of individuals from x_2 and x_3 into x_4, and x_5 classes s time units after infection. We therefore have

$$\dot{x}_1(t) = \Lambda - \lambda C(T)(t)x_1(t)\frac{W(t)}{T(t)} - \mu x_1(t), \tag{1.4.18}$$

$$x_2(t) = x_0(t) + p\int_0^t \lambda C(T)(s)x_1(s)\frac{W(s)}{T(s)}e^{-\mu(t-s)}P_2(t-s)ds, \tag{1.4.19}$$

$$x_3(t) = x_{30}(t) + (1-p)\int_0^t \lambda C(T)(s)x_1(s)\frac{W(s)}{T(s)}e^{-\mu(t-s)}P_3(t-s)ds, \tag{1.4.20}$$

$$x_4(t) = p\int_0^t \left\{\int_0^\tau \lambda C(T)(s)x_1(s)\frac{W(s)}{T(s)}e^{-\mu(\tau-s)}[-\dot{P}_2(\tau-s)e^{-(\mu+d)(t-\tau)}ds]\right\}d\tau$$
$$+ x_{40}e^{-(\mu+d)t} + x_{40}(t), \tag{1.4.21}$$

$$x_5(t) = (1-p)\int_0^t \left\{\int_0^\tau \lambda C(T(s))x_1(s)\frac{W(s)}{T(s)}e^{-\mu(\tau-s)}[-\dot{P}_3(\tau-s)\right.$$
$$\left.\times e^{-\mu(t-\tau)}]ds\right\}d\tau + x_{50}e^{-\mu t} + x_{50}(t). \tag{1.4.22}$$

Now denote by B the rate of infection — the number of new cases of infection per unit of time:

$$B(t) = C(T(t))x_1(t)\frac{W(t)}{T(t)}\,. \tag{1.4.23}$$

Since μ is the mortality rate, we denote the rate of attrition by $A(t)$: In (1.4.6) this is given by $A(t) = -\mu x_1(t)$. Thus, (1.4.6) becomes

$$\dot{x}_1(t) = \Lambda - B(t) - A(t)\,. \tag{1.4.24a}$$

This can be generalized. If Λ is time varying, then $\Lambda(t)$ is defined as the total rate of recruitment into the susceptible class at time t. Assuming a certain fraction of the total dies in each future time period after recruitment, then $m(\tau)d\tau$ is the natural mortality density, i.e., the fraction that dies in any small interval of length $d\tau$ around the time point τ. Then $\Lambda(t)m(t)d\tau$ will die in a small interval about $t+\tau$, and if we replace t by $t-\tau$ then $\Lambda(t-\tau)m(\tau)d\tau$ of the total recruitment made in a small interval about $t-\tau$ will die. The total natural death at time t of all previous total recruitments is $\int_0^\infty \Lambda(t-\tau)m(\tau)d\tau$. Equation (1.4.24a) becomes

$$\dot{x}_1(t) = \Lambda(t) - B(t) - \int_0^\infty \Lambda(t-\tau)m(\tau)d\tau\,. \tag{1.4.24b}$$

This can be considered an integral equation of renewal theory, where $\dot{x}_1(t)$ and $B(t)$ are known and $\Lambda(t)$ is unknown. If we define the nth convolution, $m^n(\tau)$ of $m(\tau)$ recursively by the relations

$$m^{(1)}(\tau) = m(\tau), \quad m^{n+1}(\tau) = \int_0^\tau m^n(\sigma)m(\tau-\sigma)d\sigma\,, \tag{1.4.25a}$$

and the replacement density, $r(\tau) = \sum_{n=1}^\infty m^{(n)}(\tau)$, then the unique solution of (1.4.24b) is

$$\Lambda(t) = \dot{x}_1(t) + B(t) + \int_0^\infty \dot{x}_1(t-\tau)r(\tau)d\tau\,. \tag{1.4.25b}$$

If the mortality density is exponential, $m(\tau) = \mu e^{-\mu\tau}$ then

$$m^n(\tau) = \frac{\mu^n\tau^{n-1}}{(n-1)!}e^{-\mu\tau} \tag{1.4.26}$$

so that $r(\tau) = \mu$. With this (1.4.25b) becomes

$$\Lambda(t) = \dot{x}_1(t) + \int_0^\infty \dot{x}_1(t-\tau)r(\tau)d\tau + B(t)$$

or

$$\Lambda(t) = \dot{x}_1(t) + \mu x_1(t) + B(t)\,.$$

From this analysis one has

$$\dot{x}_1(t) + \int_0^\infty \dot{x}_1(t-\tau)r(\tau)d\tau = \Lambda(t) - B(t). \tag{1.4.27}$$

Since $r(\tau) \geq 0$ for all τ, there is a number $0 \leq h(t) = h < \infty$ such that

$$\int_0^\infty \dot{x}_1(t-\tau)r(\tau)d\tau = \dot{x}_1(t-h)\int_0^\infty r(\tau)d\tau\,,$$

and we can then postulate that for some finite number a_{-1},

$$\dot{x}_1(t-h(t))a_{-1} = \dot{x}_1(t-h(t))\int_0^\infty r(\tau)d\tau\,. \tag{1.4.28}$$

We have derived the neutral equation

$$\dot{x}_1(t) + a_{-1}\dot{x}_1(t-h(t)) = \Lambda(t) - B(t)\,, \tag{1.4.29}$$

which now replaces (1.4.13). Because of the delays in (1.4.14) and (1.4.15) there is a possibility of oscillation and endemic equilibrium. One can now study the stability of the disease as well as its controllability by using any of the control strategies of Section 1.4.1, i.e., allowing removal and active immunization. The results are available in Chukwu [15].

1.5 Wind Tunnel Model; Mach Number Control

Delay equations of type (1.4.4) are also encountered in the design of an aircraft control facility. For example, in the control of a high-speed closed-air unit wind tunnel known as the National Transonic Facility (NTF), which is at present under development at NASA Langley Research Center, a linearized model of the Mach number dynamics is the system of three state equations with a delay, given by

$$\begin{aligned}\dot{x}_1(t) &= -ax_1(t) + akx_2(t-h) + k_1u_1(t)\,,\\ \dot{x}_2(t) &= x_3(t) + k_2u_2(t)\,,\\ \dot{x}_3(t) &= -w^2x_2(t) - 2\xi wx_3(t) + w^2u_3(t)\,,\end{aligned} \tag{1.5.1}$$

with $a = \frac{1}{1.964}$, $k = 0.117$, $w = 6$, $\xi = 1.6$, and $h = 0.33s$. $|u_i| \leq K$ with K a positive constant. The state variables x_1, x_2, x_3 represent deviations from a chosen operating point (equilibrium point) of the following quantities: x_1 = Mach number, x_2 = actuator position guide vane angle in a driving fan), and x_3 = actuator rate. The delay represents the time of the transport between the fan and the test section. Though in many cases only $k_1 = k_2 = 0$ are considered ([1, 4]), we assume here that there is a control device that creates the control variable with three components that constitute an input to the rate of the Mach number, the actuator velocity, and the actuator servomechanism. The parameter values correspond to one particular operating point. Equation (1.5.1) can be written as

$$\dot{x}(t) = A_0 x(t) + A_1 x(t-h) + Bu(t), \tag{1.5.2}$$

where

$$A_0 = \begin{bmatrix} -a & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -w^2 & -2\xi w \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0 & ak & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} k_1 & 0 & 0 \\ 0 & k_2 & 0 \\ 0 & 0 & w_2 \end{bmatrix}.$$

The following problem can be proposed (Manitius and Tran [4]): Find an optimal control subject to its constraints such that the solution of (1.5.2), with this control ($k_1 = k_2 = 0$) and with an initial configuration

$$(x_1(0) = 0.1\,,\ x_2(\theta) = x_1(\theta)/k\,,\ \theta \in [-h, 0]\,,\ x_3(\theta) = 0)\,,$$

will hit a target

$$G = \{(x_1, x_2, x_3) : |x_1(t)| \leq 0.002\ \ |x_2(t)| \leq 30\ |x_3(t)| \leq 30\}$$

in minimum time T and remain there forever after.

Remark 1.5.1 In (1.3.2) and (1.5.1) the controls (u_1, u_2) and (u_1, u_2, u_3) introduced are such that additional variables have been added in the equations describing the relations between velocity and position. One can question the physical validity of such an effort since velocity is the derivative of position, and no extra control may be allowed in the relation. Such an objection is shallow. Just as a bird flying in the wind with x_1 as its position may have its resultant velocity $\dot{x}_1 = y + u_1$, where u_1 is the velocity of the wind (control), so may the control device bring in an additional u_1 to bear in (1.3.2) and an additional (u_1, u_2) to bear in (1.5.1). These are possibilities. If these components are zero, the Euclidean controllability of these equations can still be studied.

1.6 The Flip-Flop Circuit

In the general system (1.4.4) whose special cases are given in the previous examples, we deduced the equilibrium position by setting $f(t, x_t, u_t) = 0$. The state x_t^* for which this is true is the equilibrium state. The problem of stability is to deduce conditions on f for which every solution x of (1.4.4) satisfies $x_t \to x_t^*$ as $t \to \infty$. It is possible that some dynamical systems possess multiple equilibria and are therefore suited to be used as a memory storage device in the design of a digital computer. The flip-flop circuit has such dynamics [8]. It is the basic element in a digital computer, and a standard model is given in Figs. 1.6.1 and 1.6.2 in [8].

In this model the portion between 0 and 1 is a lossless transmission line with inductance L and capacitance C. The current i flowing through the line and the voltage v across it are both functions of x and t. In [8] the function $g(v)$ is a nonlinear function of v and gives the current in the indicated box in the direction shown. The lossless transmission line can be described by the hyperbolic partial differential equations

$$\frac{\partial v}{\partial x} = -L\frac{\partial i}{\partial t}, \quad \frac{\partial i}{\partial x} = -C\frac{\partial v}{\partial t}, \quad 0 < x < 1, \quad t > 0 \tag{1.6.1}$$

with boundary conditions

$$\begin{aligned} &E - v(0,t) - Ri(0,t) = 0, \\ &C\frac{\partial v(1,t)}{\partial t} = i(1,t) - g(v(1,t)). \end{aligned} \tag{1.6.2}$$

If we let $s = (LC)^{-\frac{1}{2}}$, $z = (L/C)^{\frac{1}{2}}$, $k = (z-R)/(z+R)$, and $\alpha = 2E/(z+R)$, we can convert (1.6.1) with boundary conditions (1.6.2) into a neutral functional differential equation, i.e., a system with delay in the derivative of the state as well as in the state itself. Indeed, the general solution of (1.6.1) is

$$\begin{aligned} v(x,t) &= \phi(x-st) + \psi(x+st), \\ i(x,t) &= \frac{1}{z}[\phi(x-st) - \psi(x+st)] \end{aligned} \tag{1.6.3}$$

which is equivalent to

$$\begin{aligned} 2\phi(x-st) &= v(x,t) + zi(x,t), \\ 2\psi(x+st) &= v(x,t) - zi(x,t). \end{aligned}$$

This implies that

$$\begin{aligned} 2\phi(-st) &= v\left(1, t+\frac{1}{s}\right) + zi\left(1, t+\frac{1}{s}\right), \\ 2\psi(st) &= v\left(1, t-\frac{1}{s}\right) - zi\left(1, t-\frac{1}{s}\right). \end{aligned}$$

We use these expressions in the general solution (1.6.3) and in the first expression of (1.5.2) at the moment $(t - (\frac{1}{s}))$ to deduce

$$i(1,t) - ki\left(1, t-\frac{2}{s}\right) = \alpha - \frac{1}{z}v(1,t) - \frac{k}{z}v\left(1, t-\frac{2}{s}\right).$$

On using the second boundary condition of (1.6.2) and setting $u(t) = v(1,t)$, we obtain the equation

$$\dot{u}(t) - k\dot{u}\left(t-\frac{2}{s}\right) = f\left(u(t), u\left(t-\frac{2}{s}\right)\right), \tag{1.6.4}$$

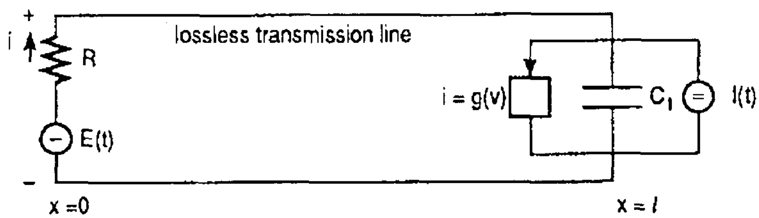

Fig. 1.6.1

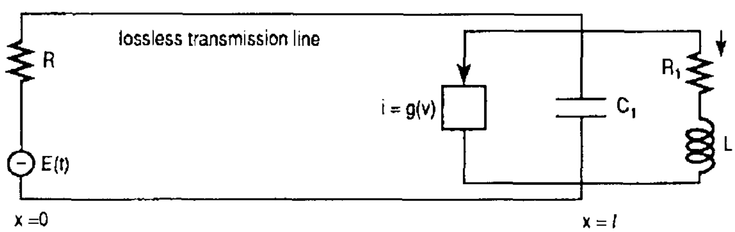

Fig. 1.6.2

where

$$c_1 f(u(t), u(t-h)) = \alpha - \frac{1}{2}u(t) - \frac{k}{z}u(t-h) - g(u(t)) + kgu(t-h))\,.$$

For the flip-flop circuit to operate as a memory storage device, one equilibria it may possess is assumed to be globally asymptotically stable: Criteria on the circuit parameters for a single equilibria point to be globally, asymptotically stable are deduced by a Lyapunov stability theory for a more general system than (1.6.4), namely

$$\frac{d}{dt}[x(t) - A_{-1}x(t-h)] = f(t, x_t)\,, \tag{1.6.5}$$

where A_{-1} is an $n \times n$ matrix and $f : E \times C \to E^n$ is a continuous function.

Equation (1.6.5) can be obtained from a lossless transmission line in which conditions (1.6.2) are more complicated. See Ref. [7]. In Fig. 1.6.3, for example, (1.6.1) is satisfied and the following boundary conditions hold:

$$v(t,0) = u_0(t) - R_0 i(t,0) + E_0(t)\,,$$

$$v(t,1) = u_1(t) + R_1 i(t,1)\,, \tag{1.6.6}$$

$$u_0(t) = -L_0 i_0(t)\,, \qquad u_1(t) = L_1 i_1(t)\,, \tag{1.6.7}$$

$$i(t,0) - i_0(t) = -c_0 \dot{u}_0(t)\,, \qquad i(t,1) - i_1(t) = c_1 \dot{u}_1(t)\,. \tag{1.6.8}$$

If we integrate (1.6.1) along its characteristics, then we have

$$x_1(t) = \sqrt{c}v(t,0) + \sqrt{L}i(t,0) = \sqrt{c}v(t+h,1) + \sqrt{L}i(t+h,1)\,,$$

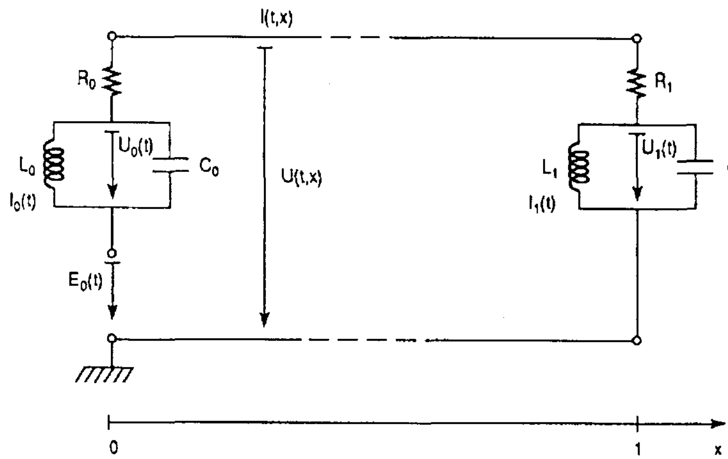

Fig. 1.6.3

$$x_2(t) = \sqrt{c}v(t,1) - \sqrt{L}i(t,1) = \sqrt{c}v(t+h,0) - \sqrt{L}i(t+h,0)\,,$$

where $h = \sqrt{cL}$. Setting $x_3(t) = 2\sqrt{L}i_0(t)$, $x_4(t) = 2\sqrt{L}i_1(t)$, and $u(t) = 2\sqrt{c}E_0(t)$, and using the boundary conditions (1.6.6)–(1.6.8), we deduce an equation of the form

$$\frac{d}{dt}[x(t) - A_{-1}x(t-h) - B_{-0}u(t)] = A_0x(t) + A_1x(t-h) + B_0u(t)\,, \tag{1.6.9}$$

where

$$A_0 = \begin{bmatrix} -\alpha_0 & 0 & \alpha_0 & 0 \\ 0 & -\alpha_1 & 0 & -\alpha_1 \\ -\alpha_2 & 0 & 0 & 0 \\ 0 & \alpha_3 & 0 & 0 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0 & \alpha_0 & 0 & 0 \\ \alpha_1 & 0 & 0 & 0 \\ 0 & \alpha_2\alpha_4 & 0 & 0 \\ -\alpha_3\alpha_5 & 0 & 0 & 0 \end{bmatrix},$$

$$B_0 = \begin{bmatrix} 0 \\ 0 \\ \alpha_2\beta_0 \\ 0 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0 & \alpha_4 & 0 & 0 \\ \alpha_5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad B_{-0} = \begin{bmatrix} \beta_0 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

and where

$$\alpha_0 = \frac{1}{c_0} \frac{\sqrt{c}}{R_0\sqrt{c} + \sqrt{L}}, \qquad \alpha_2 = \frac{1}{L_0} \frac{R_0\sqrt{c} + \sqrt{L}}{\sqrt{c}},$$

$$\alpha_1 = \frac{1}{c_1} \frac{\sqrt{c}}{R_1\sqrt{c} + \sqrt{L}}, \qquad \alpha_3 = \frac{1}{L_1} \frac{R_1\sqrt{c_1} + \sqrt{L}}{\sqrt{c}},$$

$$\alpha_4 = \frac{R_0\sqrt{c} - \sqrt{L}}{R_0\sqrt{c} + \sqrt{L}}, \qquad \alpha_5 = \frac{R_1\sqrt{c} - \sqrt{L}}{R_1\sqrt{c} + \sqrt{L}}.$$

Equation (1.6.9) is a neutral functional differential equation with control u that is related to the initial power source $E_0(t)$.

Equations (1.6.4), (1.6.5), and (1.6.9) are special cases of the system

$$\frac{d}{dt}[D(t, x_t)] = f(t, x_t, u(t)), \tag{1.6.10}$$

where $D : E \times C \to E^n$, $f : E \times C \times E^m \to E^n$ are continuous functions with

$$D(t, x_t) = x(t) - g(t, x_t), \tag{1.6.11}$$

and $g : E \times C \to E^n$ is continuous.

1.7 Hyperbolic Partial Differential Equations with Boundary Controls

A final example involves the derivation of (1.6.10) from linear hyperbolic partial differential equations with boundary control. The derivation is treated in [3]. Let the wave equation for $w(t, x)$ be given by

$$w_{tt} - c^2 w_{xx} = 0, \qquad t \in [0, T], \qquad x \in [0, 1], \tag{1.7.1}$$

with boundary conditions

$$\begin{aligned} A_0(t)w_{tt}(t,0) + B_0(t)w_{tx}(t,0) &= G_0(t, w(t,0),\ w_t(t,0),\ w_x(t,0)), \\ A_1(t)w_{tt}(t,1) + B_1(t)w_{tx}(t,1) &= G_1(t, w(t,1),\ w_t(w,1),\ w_x(t,1)), \end{aligned} \tag{1.7.2}$$

and initial conditions

$$w(0, x) = f_0(x), \quad w_t(0, x) = f_1(x) \quad \text{on} \quad [0, 1], \tag{1.7.3}$$

where the initial functions satisfy the boundary conditions at $t = 0$. Here the subscript t or x denotes partial derivative. The prime in the sequel denotes total derivative. The terms G_0, G_1 contain the controls if

$$G_0(t, \rho, \sigma, \tau) = F_0(t, u(t), \rho, \sigma, \tau),$$

$$G_1(t, \rho, \sigma, \tau) = F_1(t, v(t), \rho, \sigma, \tau),$$

and $\binom{u}{v}$ is the control to be chosen from a prescribed class. We assume that A_i, B_i are continuously differentiable, G_i are absolutely continuous in t, continuously differentiable in other arguments with G_{it} dominated by a square integrable function. We assume further that

$$A_0(t) - B_0(t)/c \neq 0\,, \qquad A_1(t) + B_1(t)/c \neq 0\,, \tag{1.7.4}$$

for $t \in [0, T]$.

We assume a D'Alamberto solution of the form

$$w(t,x) = \phi(t + x/c) + \psi(t - x/c)\,, \tag{1.7.5}$$

and substitute this in (1.7.2). On setting

$$\alpha_i(s) = \left[A_i(s) + \frac{1}{c}B_i(s)\right], \qquad \beta_i(s) = \left[A_i(s) - \frac{1}{c}B_i(s)\right], \qquad i = 0, 1\,,$$

the substitution yields

$$\begin{aligned}\alpha_0(t)\phi''(t) + \beta_0(t)\psi''(t) &= G_0\left(t, f_0(0) + \int_0^t \phi'(s)ds + \int_0^t \psi'(s)ds,\right.\\ &\qquad \left.\times\ \phi'(t) + \psi'(t)\frac{1}{c}\phi'(t) - \frac{1}{c}\psi'(t)\right)\\ &\equiv \tilde{G}_0(\phi'(\cdot),\ \psi'(\cdot), t),\ t \geq 0\,,\end{aligned}$$

and

$$\begin{aligned}&\alpha_1\left(t - \frac{1}{c}\right)\phi''(t) + \beta_1\left(t - \frac{1}{c}\right)\psi''\left(t - \frac{2}{c}\right)\\ &= G_1\left(t - \frac{1}{c}, f_0(0) + \int_0^t \phi'(s)ds + \int_0^{t-\frac{2}{c}} \psi'(s)ds, \phi'(t)\right.\\ &\quad \left. + \psi'\left(t - \frac{2}{c}\right),\ \ \frac{1}{c}\phi'(t) - \frac{1}{c}\psi'\left(t - \frac{2}{c}\right)\right)\\ &\equiv \tilde{G}_1(\phi'(\cdot),\ \psi'(\cdot), t), \quad t \geq \frac{1}{c}\,.\end{aligned}$$

It follows from condition (1.7.4) that we can multiply the last pair of equations by the matrix

$$\begin{bmatrix} 0 & \dfrac{1}{\alpha_1(t - \frac{1}{c})} \\ \dfrac{1}{\beta_0(t)} & \dfrac{\alpha_0(t)}{\alpha_1(t - \frac{1}{c})\beta_0(t)} \end{bmatrix}$$

to deduce the equations

$$\phi''(t) + \frac{\beta_1(t-\frac{1}{c})\psi''(t-\frac{2}{c})}{a_1(t-\frac{1}{c})} = \frac{1}{\alpha_1(t-\frac{1}{c})}\tilde{G}_1(\phi'(\cdot),\psi'(\cdot),t)\,,$$

$$\begin{aligned}\psi''(t) &= \frac{\alpha_0(t)\beta_1(t-\frac{1}{c})}{\alpha_1(t-\frac{1}{c})\beta_0(t)}\psi''\left(t-\frac{2}{c}\right)\\ &= \frac{1}{\beta_0(t)}\tilde{G}_0(\phi'(\cdot),\psi'(\cdot),t) - \frac{\alpha_0(t)G_1(\phi'(\cdot),\psi'(\cdot),t)}{\alpha_1(t-\frac{1}{c})\beta_0(t)}\,,\end{aligned}$$

$$\frac{1}{c} \le t\,.$$

Denoting (ϕ',ψ') by (y,z), this equation is a neutral equation of the form

$$\begin{aligned}\dot{y}(t) + h_1(t)\dot{z}\left(t-\frac{2}{c}\right) &= H_1(t,y(\cdot),z(\cdot))\,,\\ \dot{z}(t) + h_2(t)\dot{z}\left(t-\frac{2}{c}\right) &= H_2(t,y(\cdot),z(\cdot))\,.\end{aligned} \tag{1.7.6}$$

The data of (1.7.3) can now be used to produce initial and terminal functions of (y,z) for (1.7.6). Since there are controls on G_i and therefore on H_i, the system (1.7.6) is a control system for $t \in [\frac{1}{c},T]$ with initial and terminal values for y given on $[0,\ \frac{1}{c}]$ and at $t=T$, and the corresponding values of z given on $[-\frac{1}{c},\frac{1}{c}]$ and $[T-\frac{2}{c},T]$. Because of the smoothness conditions on G_i and therefore on H_i, (1.7.6) may be argued from the fundamental existence theorem below to have a solution $(y,z) = (\phi',\psi')$, which is absolutely continuous with square integrable derivatives. One argues that this (ϕ,ψ) utilized in (1.7.5) yields a solution of (1.7.1) in the generalized sense or in the sense of almost everywhere, i.e., $w(t,x) = \phi(t+\frac{x}{c}) + \psi(t-\frac{x}{c})$ is continuously differentiable with w_t, w_x being absolutely continuous and possessing square integrable partials that satisfy (1.7.1) a.e.

It is interesting to note that the boundary conditions of (1.7.2) cover the usual ones for transverse vibrations of a string or longitudinal vibrations in an elastic rod with elastically supported ends. Just as in earlier examples, it is there at the boundary that controls are introduced. It sometimes requires some ingenuity to introduce in practice the type of control that is required for the system $\frac{d}{dt}[D(t)x_t] = f(t,x_t) + B(t)u(t)$ (or of (1.6.10)) to have a desired effect on $B(t)$. One can then relate this to the control device. An appropriate time-optimal control problem can now be formulated for such systems.

1.8 Control of Global Economic Growth

Let $x(t)$ denote the value of capital stock at time t. Let $u(t)$, $0 \le u(t) \le 1$ be the fraction of the stock that is allocated to investment. This investment is used to increase productive capacity. We can assume the value of this investment $I(t)$ is given by

$$I(t) = ku(t)x(t)$$

where k is the constant of proportionality. There is depreciation, $D(t)$, of capital stock, and it is proportional to capital stock

$$D(t) = -\delta x(t).$$

The net investment is

$$N(t) = I(t) - \delta x(t),$$

and it is used to produce new capital. Thus net capital formation $\dot{x}(t)$ is given by

$$\dot{x}(t) = ku(t)x(t) - \delta x(t). \tag{1.8.1}$$

If $-1 \leq u(t) \leq 0$, we can interpret $u(t)$ to be the fraction of the value of stock that is used for payment of taxes, etc. Thus, in general, $u(t)$ satisfies $-1 \leq u(t) \leq 1$. Such models are very naive and unrealistic. As pointed out by Takayama [10, p. 705], this implies that adjustment of desired stock of capital is instantaneous and frictionless, an implication that has no valid empirical foundation. There is a time lag that needs to be incorporated to the control procedure of the firm. Investment in new capital equipment does not yield new productive capacity until the equipment is delivered, installed, and tested. There is a time delay $h > 0$. It is more realistic to express $I(t)$ as a function of the present and past values of $u(t)$ and $x(t)$ and of time:

$$I(t) = g(t, x(t), u(t), x(t-h), u(t-h)).$$

In general,

$$I(t) = B(t, x_t, u_t)u_t \equiv \int_{-h}^{0} d_s H(t, x(t+s), u(t+s))u(t+s).$$

Here the past history of x and u over an entire interval of length h enters equations through the Riemann–Stieltjes integral. Realism dictates that depreciation be incorporated into the model. We assume that the value of the capital stock that has depreciated is not just proportional to $x(t)$ as in (1.8.1), but is a function g of $(x(t), u(t))$ and is given by $g(x(t), u(t))x(t)$ at time t. Suppose this value decreases by a factor $P(a)$ at time a; ($P(0) = 1$, $P(L) = 0$ where L is the lifetime of the asset). If $p(a) = \frac{dP(a)}{da}$, the depreciation density or mortality density, $p(\tau)d\tau$, represents the fraction lost to productive use in any small time interval of length $d\tau$ around the time τ. Therefore, $g(x(\tau), u(\tau))x(\tau)p(\tau)d\tau$ will disappear in a small interval about $t + \tau$. If we replace t by $t - \tau$, we can say that at time t the amount

$$g(x(\tau - h), u(\tau - h))x(\tau - h)(\tau)p(\tau)d\tau$$

in a small time interval about $t - \tau$ will disappear. The total evaporation at time t is

$$-\int_0^L g(x(\tau - h), u(\tau, -h))x(\tau - h)p(\tau)d\tau\,,$$

or

$$-\int_0^L g(x(\tau - h), u(\tau - h))x(\tau - h)dp(\tau)$$

if we use Riemann–Stieltjes integral. A general representative expression is

$$L(t, x_t, u_t)x_t \equiv \int_{-h}^0 d_s\eta(t, s, x(t+s), u(t+s)), x(t+s)$$

where we use Riemann–Stieltjes integral. We can therefore state that net capital formation $\dot{x}(t)$ is given by

$$\dot{x}(t) = L(t, x_t, u_t)x_t + B(t, x_t, u_t)u_t\,. \tag{1.8.2}$$

We postulate initial capital endowment function as $x_s = \phi$, and initial fraction of stock allocated to investment as $u_\sigma = v$. The key elements in (1.8.2) are the time patterns of capital $x(t)$ and the capital policy described by the function $u(t)$. The problem of optimal capital policy is to describe $u(t)$ subject to its constraints $-1 \leq u(t) \leq 1$, such that the system with initial endowment function ϕ will hit a prescribed target x_1 while minimizing the firms' power or effort defined by some function $E(u)$, and perhaps to do it in minimum time. Possible definitions of effort or power are given by

$$E_1(u(t)) = \int_\sigma^{t_1} |u(t)|^2 dt\,, \tag{1.8.3a}$$

$$E_2(u(t)) = \sup_{0 \leq t \leq t_t} |u(t)|\,, \tag{1.8.3b}$$

where

$$u \in U = \{u \in E,\ u \text{ piecewise continuous } |u(t)| \leq 1\ 0 \leq t \leq t_1\}\,.$$

This represents maximum investment thrust available to the firms.

$$E_3(u(t)) = \left(\int_0^{t_1} |u(t)|^p dt\right)^{\frac{1}{P}}\,, \tag{1.8.3c}$$

where

$$u \in U = \{u \in E,\ u \text{ piecewise continuous } |u(t)| \leq 1\ 0 \leq t \leq t_1\}\,.$$

If $p = 2$, $E(u(t))$ represents the investment energy or power that must be minimized.

$$E_4(u(t)) = \int_0^{t_1} |u(t)| dt \equiv \|u\|_{L_1}\,, \tag{1.8.3d}$$

where $u \in U = \{u : \|u\|_{L_1} \leq 1\}$. The expression $E_4(u(t))$ can be called investment; the problem is to minimize it when we achieve our growth target from the initial endowment ϕ to our prescribed target x_1. We may wish to maximize some attainable value of the social welfare criterion

$$J = \int_0^T E(u)\exp(-\gamma t)dt\,,$$

where E is a specified "well-behaved" utility function and γ is a fixed discount rate. E is usually a strictly concave monotone increasing in u with second derivative defined everywhere and such that

$$\lim_{u \to 0} E(u) = \infty\,.$$

The fixed time T is the term of years in which the above objectives will be fulfilled. To solve this problem one must first solve the problem of controllability. The controllability question is investigated in Chaps. 8 and 12. Problems of optimality are reported in Chap. 7. Our research here deals with $-1 \leq u(t) \leq 1$, the situation $0 \leq u(t) \leq 1$ will be explored.

In the previous discussion $x(t)$ is a real number, the value of one stock. We can denote $x(t) = (x_1(t) \cdots x_n(t))$ to be the value of n capital stocks at time t, with investment and tax strategy $u = (u_1 \cdots u_n)$, where $-1 \leq u_j(t) \leq 1$. We therefore consider (1.8.2) as the equation of the net capital function for n stocks in a region that is isolated. These stocks are linked to ℓ other such systems in the globe, and the "interconnection" or "solidarity function" is given by

$$g_i(x_{1t}, \ldots, x_{\ell t},\ u_{1t}, \ldots, u_{\ell t})\,.$$

This function describes the effects of other subsystems on the ith subsystem as measured locally at the ith location. Thus,

$$\dot{x}_i(t) = L_i(t, x_{it}, u_{it})x_{it} + B_i(t, x_{it}, u_{it})u_{it} + g_i(t, x_{1t}, \ldots, x_{\ell t}, u_{1t} \cdots u_{\ell t})\,,\ i = 1, \ldots, \ell \tag{1.8.4}$$

is the decomposed interconnected system whose free subsystem is

$$\dot{x}_i(t) = L_i(t, x_{it}, u_{it})x_{it} + B_i(t, x_{it}, u_{it})u_{it}\,. \tag{S_i}$$

We now introduce the following notation: Let $\sum_{i=1}^{\ell} n_i n$, $\sum_{i=1}^{\ell} m_i = m$.

$$\phi = [\phi_1, \ldots, \phi_\ell] \in E^n\,, \quad x = [x_1, \ldots, x_\ell] \in E^n\,, \quad u = [u_1, \ldots, u_\ell] \in E^n\,,$$

$$L(t, x_t, u_t) = [L_1(t, x_{1t}, u_1) \cdots L_\ell(t, x_{\ell t}, u_{\ell t})]\,,$$

$$g(t, x_t, u_t) = [g_1(t, x_t, u_t) \cdots g_\ell(t, x_t, u_t)]\,,$$

$$B(t, x_t, u_t) = \text{diag}[B_1(t, x_{1t}, u_{1t}) \cdots B_\ell(t, x_{\ell t} u_{(t)})]\,.$$

Then (1.8.3) is given as

$$\dot{x}(t) = L(t, x_t, u_t) + B(t, x_t, u_t)u_t + g(t, x_t, u_t)\,. \tag{S}$$

To clarify our terminology we introduce the following notations and definitions. Let $E = (-\infty, \infty)$, and E^r be the r-dimensional Euclidean space with norm $|\cdot|$. The symbol C denotes the space of continuous functions mapping the interval $[-h, 0]$, $h > 0$, $h \in E$ into E^n with the sup norm $\|\cdot\|$ defined by $\|\phi\| = \sup_{-h \le s \le 0} |\phi(s)|$, $\phi \in C$. If $t \in [\sigma, t_1]$, we let $x_t \in C$ be defined by $x_t(s) = x(t+s)$, $s \in [-h, 0]$. The symbol $L_\infty([\sigma, t_1], E^m) = L_\infty$ denotes the space of essentially bounded measurable functions on $[\sigma, t_1]$ with the norm $\max_{1 \le j \le m} \sup_{t \in [\sigma, t_1]} |u_j(t)| = \|u\|_\infty$.

Definition 1.8.1 The system (S) is Euclidean controllable on $[\sigma, t_1]$ if for any function $\phi \in C$ and any vector $x_1 \in E$, there is a control $u \in L_\infty([\sigma, t_1], E^m)$ such that the solution $x(t) = x(t, \sigma, \phi, u)$ of (S) satisfies $x_\sigma(\cdot, \sigma, \phi, u) = \phi$, $x(t_1, \sigma, \phi, u) = x_1$. It is Euclidean null controllable on $[\sigma, t_1]$ if $x_1 = 0$ in the above definition.

The system (S) is controllable on $[\sigma, t_1]$, $t_1 > \sigma + h$, if for each $\phi, \psi \in C$ there is a control function $u \in L_\infty([\sigma, t_1], E^m)$ such that the solution $x(t) = x(t, \sigma, \phi, u)$ satisfies $x_\sigma(\cdot) = \phi$, $x_{t_1} = \psi$. It is null controllable on $[\sigma, t_1]$ if $\psi \equiv 0$ in the preceding definition. In C we drop the qualifying phrase "on the interval $[\sigma, t_1]$", if controllability obtains on every interval $[\sigma, t_1]$, with $t_1 > \sigma + h$. In E^n we drop "on the interval $[\sigma, t_1]$" if Euclidean controllability holds on every interval $[\sigma, t_1]$, $t_1 \ge \sigma$.

The problem of controllability of (S) will be explored in Chaps. 8 and 9. Conditions are stated for the controllability of the isolated system S_i. Assuming that the subsystem (S_i) is controllable we shall deduce conditions for (S) to be controllable when the solidarity function is "nice".

In (1.8.2) we postulated that capital growth is described by a functional differential equation with delay. The crucial assumption is that the net capital formation $\dot{x}(t)$ is given by the nonlinear function, $I(t)$, the gross investment, on the right. A more general situation can be obtained. Arrow [11, p. 184] showed that indeed it is realistic to have

$$I(t) = \dot{x}(t) + \int \dot{x}(t-s) r(s) ds\,,$$

where $r(t) \ge 0$ is the replacement density and $\dot{x}(t)$ the net capital formation. Since $I(t)$ is finite valued and $r(t) \ge 0$, the mean value theorem for integrals allows one to write

$$\int_0^\infty \dot{x}(t-s) r(s) ds = \dot{x}(t - h(t)) \int_0^\infty r(s) ds\,,$$

where $0 \le h \le \infty$. It is therefore reasonable to postulate that

$$I(t) = \dot{x}(t) + \dot{x}(t - h(t))a\,,$$

where $0 \leq h < \infty$ and a is determined from the replacement density. As a consequence of this hypothesis we can reasonably postulate that

$$\dot{x}(t) + a\dot{x}(t-h) = L(t, x_t, u_t)x_t + B(t, xt, u_t)u_t\,, \tag{1.8.4}$$

in place of (1.8.2). This is a functional differential equation of neutral type. Analogous systems (1.8.4) (Si) and (S) can be formulated.

1.9 The General Time-Optimal Control Problem and the Stability Problem

The general form of equation studied is

$$\frac{d}{dt}[D(t)x_t] = f(t, x_t, u(t))\,, \qquad x_\sigma = \varphi\,, \tag{1.9.1}$$

where

$$D(t)x_t = x(t) - g(t, x_t)\,,$$

$g(t, \varphi)$ is linear in φ, and $f(t, \varphi, u)$ may be nonlinear. It includes the delay system

$$\dot{x}(t) = f(t, x_t, u(t))\,, \qquad x_\sigma = \varphi\,, \tag{1.9.2}$$

and the ordinary differential system

$$\dot{x}(t) = k(t, x(t), u(t))\,, \qquad x(0) = x_0\,. \tag{1.9.3}$$

The initial state φ can appropriately be given as the space C of continuous functions from $[-h, 0]$ into E^n with the uniform norm. In this case, finding a solution of (1.9.2) is equivalent to finding a solution of the integral equation

$$x(t) = \varphi(0) + \int_\sigma^t f(s, x_s, u(s))ds\,, \qquad t \geq \sigma\,,$$
$$x(\sigma + \theta) = \varphi(\theta)\,, \qquad -h \leq \theta \leq 0\,.$$

Other spaces with this property of equivalence of solution may be taken, and will be considered in subsequent discussions. If spaces other than C are used, the final point and the solution lie in C for $t \geq \sigma + h$. Often the final point we reach may be taken to be in E^n, or in C. The space W_p^1, consisting of all absolutely continuous functions from $[-h, 0] \to E^n$ that have p-integrable derivatives where $1 \leq p \leq \infty$, will play a significant role in our investigations. Another space of initial functions that is found to be very useful in application is the space $L_p([-a, 0], E^n)$, i.e., the space of p-integrable functions with the usual norm. If we appropriate this as the state of initial conditions, the solutions of (1.9.1) can be considered in the product space

$$M^p = E^n \times L_p([-h, 0], E^n) \times L_p([0, T], E^m)\,,$$

$1 \le p \le \infty$, endowed with the norm

$$\|\varphi\| = [\|\varphi^0\|^p + \|\varphi^1\|_p^p + \|\varphi^2\|^p]^{\frac{1}{p}}, \qquad \varphi = (\varphi^0, \varphi^1, \varphi^2) \in M^p.$$

With the state space selected, the target is either a point or a subset of the appropriately selected state space.

The admissible controls are measurable vector valued functions with values constrained to lie in a compact convex set $\mathcal{U}$ of E^m. Often $\mathcal{U}$ is specialized to be the unit cube,

$$C^m = \{u \in E^m : |u_j| \le 1, \ h = 1, 2, \ldots, m\}.$$

The time-optimal control problem is now formulated as follows: Determine an admissible control u^* such that the solution $x(\varphi, \sigma, u^*)$ of (1.9.1) hits a continuously moving target point or set in the appropriate space, in minimum time, $t^* \ge 0$. Such controls u^* are called time-optimal controls. Our ultimate goal is to get an optimal control u^* as a function of the appropriate state space, i.e., to obtain the "feedback" or "closed loop" control. The major advantage of such a feedback-optimal control as against an "open-loop" one with u as a function of t is that the system in question becomes self-correcting and automatic.

Thus if C is the state space, we hunt for a measurable function $m : C \to E^m$ such that

(i) $m(x_t) \equiv u(t), \ u(t) \in \mathcal{U}$.
(ii) m is optimal feedback control for (1.9.1) in the following sense. In addition to (1.9.1), consider the differential equation

$$\frac{d}{dt}[D(t)y_t] = f(t, y_t, m(y_t)). \tag{1.9.4}$$

Then each optimal control of (1.9.1) is a solution of (1.9.4), and conversely each solution of (1.9.4) is an optimal control solution of (1.9.1). Once found, the time-optimal control problem for the system (1.9.1) is completely solved.

In (1.2.4) for instance, the optimal feedback control is the fishing strategy that would drive the system to the target (equilibrium) as quickly as possible.

Sometimes the theory and techniques of the solution of the time-optimal problem can be appropriated to tackle the problem of minimizing a cost function which, for example, is given in (1.8.3). The general situation will be treated in Chaps. 7 and 9.

1.9.1 *The stability problem*

The complete solution of the time-optimal problem is dependent on the solution of the corresponding stability problem, which is formulated as follows: Let x^* be a solution of

$$f(t, x_t, 0) \equiv g(t, x_t) = 0 . \tag{1.9.5}$$

Find necessary and sufficient conditions on D and g such that every solution $x(\phi)$ of

$$\frac{d}{dt}[D(t, x_t] = g(t, x_t) , \quad t \geq \sigma , \qquad x_\sigma = \phi , \tag{1.9.6}$$

is uniformly, globally, asymptotically stable in the following sense: For each ϕ, the solution $x(\phi)$ of (1.9.6) satisfies

$$x_t(\phi) \to x^* \quad \text{as} \quad t \to \infty . \tag{1.9.7}$$

These two problems of stability and optimal control are the primary questions we attempt to address in this monograph.

Remark 1.9.1 In the time-optimal or minimum-effort problem treated here, the controls of essential interest are small and bounded in components: $|u_j(t)| \leq 1$, $j = 1, \ldots, m$. Because of this constraint, the global asymptotic stability of the system without control are needed and will be studied to ensure global constrained controllability of our dynamics, on which rest the existence of an optimal control. This contrasts with the situation in which the constraint $|u_j(t)| \leq 1$, $j = 1, \ldots, m$ is removed, and we require admissible controls to be square integrable. For such "big" controls one could study feedback stabilization and optimal control with quadratic cost functional. Though such a study is important, we shall not pursue it in depth because the main applications we have in mind for this introductory text have essentially bounded controls.

1.10 Economic Models with Delay

In the construction of dynamic economic models, the kind of "time", "continuous" or "discrete", dictates whether a differential equation or a difference equation is the model. It has been argued strongly and persuasively by G. Gandolfo that mixed differential-difference equations, functional differential equations are much more suitable than differential equations alone or difference equations alone for an adequate treatment of dynamic economic phenomena [20]. Functional differential equations were used by Kalecki to model capital stock. More recent studies include Gandolfo [20], Chukwu [19, 21] and [26]. Kalecki [27] argued that the growth of capital stock $x(t)$ of a single firm is given by

$$\dot{x}(t) = I(t) = a_0 x(t) + a_1 x(t-h) , \tag{1.10.1}$$

where a_i are constants and the delay h represents the time lag between the decision to invest and the deliveries of capital equipment. The crucial assumption for (1.10.1) is that the net capital formation $\dot{x}(t)$ is given by $I(t)$. To obtain (1.10.1), Kalecki assumes that the decision to invest B is given by

$$B(t) = a(1-c)y(t) - kx(t) + \varepsilon ,$$

a, c, k are constants, ε may be time varying, y is income (output), $x(t)$ denotes the stock capital assets at time t. Later as an exercise, Kalecki suggests that the decision to invest should be

$$B(t) = a(1-c)y(t) - kx(t) + \varepsilon + v\frac{dy}{dt}\,.$$

The outcome of this analysis is the functional differential equation of neutral type,

$$\frac{d}{dt}(x(t) - a_{-1}x(t-h)) = a_0x(t) - a_1x(t-h) + a_2\,.$$

This is a functional differential equation of neutral type which describes the growth of capital stock of a single firm. We can introduce $b(t)u(t)$ at the right-hand side of this system to obtain

$$\dot{x}(t) + a_{-1}\dot{x}(t-h) = a_0x(t) + a_1x(t-h) + b(t)u(t)\,.$$

We interpret $b(t)u(t)$ as follows. If $0 \le u(t) \le 1$, then $u(t)$ is the fraction of "available capital assets, $b(t)$" at time t that is allocated to investment. If $-1 \le u(t) \le 0$, then $u(t)$ is the fraction allocated to consumption or for payment of taxes. Thus $-1 \le u(t) \le 1$ and $u(t)$ is an investment consumption strategy which is appropriated as a control to drive an initial capital endowment ϕ to a target while minimizing a cost function.

There are at least two ways that time delays emerge in the dynamics of economic variables: there is some time lag between the time economic decisions are made and the time the decisions bear fruit. See [35] and Chukwu [28, 29]. There is a "hidden" way, the way rational expectation (see Ray C. Fair [30], Luigi Amoroso [31], J. B. Taylor [32, Chap. 3]).

For dynamic economic systems we appropriate the argument of Fair that the "rational expectations hypothesis" can better be approximated by assuming that the expected values of the economic model are a function of the current and past values. Indeed we assume that aggregate demand z is a sum of investment I, consumption C, net export X, and government outlay G:

$$z = I + C + X + G\,, \tag{1.10.2}$$

where

$$\begin{aligned} C = C_0 &+ C_1(y(t) - T(t)) + C_2(y(t-h) - T(t-h)) + C_3(\dot{y}(t) - \dot{T}(t)) \\ &+ C_4(\dot{y}(t-h) \\ &- \dot{T}(t-h)) + C_5R(t) + C_6R(t-h) + C_7(ML)\,. \end{aligned} \tag{1.10.3}$$

Here,

$$\begin{aligned} C &= \text{private consumption}\,, & M &= \text{money supply}\,, \\ \tilde{L} &= \text{money demand} = Md\,, & ML &= \tilde{L} - M\,, \end{aligned} \tag{1.10.4}$$

$$\tilde{L} = M_0 + M_1 y(t) + M_1 y(t-h) + M_3 R(t) + M_4 R(t-h) + M_5 \dot{R}(t-h) + M_6 p(t)\,, \tag{1.10.5}$$

$$I = I_0 + I_1 y(t) + I_2 y(t-h) - I_3 \dot{y}(t) + I_4 \dot{y}(t-h) + I_5 R(t) + I_6 R(t-h) + I_8 L(t) + I_9 L(t-h) - I_{11} K(t) - I_{13}(M - \tilde{L})\,, \tag{1.10.6}$$

where

K = value of capital stock, L = employment,

M = money supply = $M1$, $\tilde{L}$ = Liquidity = money demand = Md.

Also

$$G = g_0 + g_1 y(t) + g_2 y(t-h) + g_4 \dot{y}(t-h) + g_5 R(t) + g_8 L(t)\,, \tag{1.10.7}$$

$$(g_0 = \text{federal budget net expenditure}) \tag{1.10.8}$$

where

$$X = x - m = \text{Export} - \text{Import}$$

$$x = \text{export function} = x(p, e, d, \tau, y, L)$$

where p = price, y = income, e = exchange rate, τ = tariffs, L = employment. We know that $\partial x/\partial e < 0, \partial x/\partial p < 0$, if the elasticity of substitution among importables (ESM) is greater than 1 and if the elasticity of transformation between exportables (ETE) is greater than ETDF, the elasticity of transformation between production for domestic market and for foreign markets. We can therefore assume a linear model

$$x = x_0 + x_1 y(t) + x_2 y(t-h) + x_3 \dot{y}(t) + x_4 \dot{y}(t-h) + x_5 R(t) + x_6 L(t) + x_{10} \dot{L}(t-h) + x_{16}\tau(t) + x_{15} e(t) - x_{12} p(t) + z_{17} d(t)\,, \quad (x_{15} > 0, x_{12} > 0)\,. \tag{1.10.9}$$

(It can be assumed that

$$M = M_0 + M_1 y + M_2 p + M_3 e + M_4 L + M_5 \tau \tag{1.10.10}$$

where obviously, $M_1 > 0$, $M_2 > 0$, $M_3 > 0$.) Since income y has a predictable positive effect on trade flows and increase in imports we have $M_1 > 0$. Thus net export

$$X = X_0 + X_1 y(t) + X_2 y(t-h) + X_3 \dot{y}(t) + X_4 \dot{y}(t-h) + X_5 R(t) + X_8 L(t) + X_{10} \dot{L}(t-h) + X_{12} p(t) + X_{16}\tau(t) + X_{15} e(t) + X_{17} d(t)\,, \tag{1.10.11}$$

where

$$X_0 = X_0 - M_0\,, \qquad X_1 = X_1 - M_1\,, \text{ etc}\,.$$

Therefore, on gathering results we have that aggregate demand

$$\begin{aligned} z &= C + I + X + G \\ &= z_0 + z_1 y(t) + z_2 y(t-h) + z_3 \dot{y}(t) + z_4 \dot{y}(t-h) + z_5 R(t) + z_6 R(t-h) \\ &\quad + z_8 L(t) + z_9 L(t-h) + z_{10} \dot{L}(t-h) + z_{11} K(t) + z_{13} \dot{R}(t-h) + z_{18} p(t) \\ &\quad - z_{14} T(t) - z_{15} e(t) + z_{16} \tau(t) + z_{17} d(t) - z_{19} T(t-h) \\ &\quad - z_{20} \dot{T}(t) - z_{21} \dot{T}(t-h)\,, \end{aligned} \tag{1.10.12}$$

where

$$\begin{aligned} z_0 &= g_0 + I_0 - M_0(C_7 + I_{13}) + C_0 + X_0\,, \\ z_1 &= g_1 + I_1 - M_1(I_{13} + C_7) + C_1 + X_1\,, \\ z_2 &= g_2 + I_2 + C_2 + X_2 + M_2(I_{13} + C_7)\,, \\ z_3 &= g_3 - I_3 + C_3 + x_3\,, \\ z_4 &= g_4 + I_4 + C_4 + X_4 + M_4(C_7 + I_{13})\,, \\ z_5 &= g_5 + I_5 + C_5 + X_5 - C_7 M_7\,, \\ z_6 &= I_6 + C_6 - C_7 M_6\,, \\ z_8 &= g_8 + I_8 + X_8\,, \\ z_9 &= I_9\,, \\ z_{10} &= X_{10}\,, \\ z_{11} &= -I_{11}\,, \\ z_{13} &= (I_{13} + C_7) M_6\,, \\ z_{14} &= -C_1\,, \\ z_{15} &= X_{15}\,, \\ z_{16} &= X_{16}\,, \\ z_{17} &= X_{17}\,, \\ z_{18} &= (C_7 + z_{13}) M_6\,, \\ z_{19} &= C_2\,, \end{aligned}$$

$$z_{20} = C_3 ,$$

$$z_{12} = C_4 .$$

By the market principle of supply and demand,

$$\frac{dy(t)}{dt} = \lambda_1(z(t) - y(t)) , \qquad (1.10.14)$$

where λ_1 is the speed of response of supply to demand, the speed of adjustment. The reciprocal of the speed of adjustment $(1/\lambda_1)$ is the mean time lag, i.e. the time necessary for about 63% of the discrepancy between y and z (or between the actual and desired value of the variable) to be eliminated [33, p. 94]. From (1.10.14) the following equation emerges:

$$\begin{aligned}\frac{dy}{dt} - a_{-11}\dot{y}(t-h) - a_{-13}\dot{L}(t-h) = \lambda_1\sigma_1^{-1}[&(z_1 - 1 - z_{13}M_1)y(t) \\ &+ z_2 y(t-h) + (z_5 - z_{13}M_2)R(t) \\ &+ z_6 R(t-h) + z_8 L(t) \\ &+ z_9 L(t-h) + z_{11}k(t) - z_{13}M_3 p(t) \\ &- z_{14}T(t) - z_{15}e(t) \\ &+ z_{16}\tau(t) + z_{19}T(t-h) - z_{20}\dot{T}(t) \\ &+ z_{21}\dot{T}(t-h) + z_{17}d(t)] + z_0\lambda_1\sigma_1^{-1} ,\end{aligned}$$

where

$$\sigma_1 = 1 - \lambda_1 z_3 , \qquad a_{-11} = \lambda_1\sigma_1^{-1}z_4 , \qquad a_{-13} = \lambda_1\sigma_1^{-1}z_{10} .$$

Set

$$\begin{aligned} a_{01} &= \lambda_1\sigma_1^{-1}(z_1 - 1 - z_{13}M_1) , \\ a_{11} &= z_2\lambda_1\sigma_1^{1} , \\ a_{12} &= (z_5 - z_{13}M_2)\lambda_1\sigma_1^{-1} , \\ a_{13} &= z_6\lambda_1\sigma_1^{-1} , \\ a_{14} &= z_8\lambda_1\sigma_1^{-1} , \\ a_{15} &= z_9\lambda_1\sigma_1^{-1} , \\ a_{16} &= z_{11}\lambda_1\sigma_1^{-1} , \\ a_{17} &= z_{19}\lambda_1\sigma_1^{-1} , \\ a_{18} &= \lambda_1\sigma_1^{-1}z_{13}M_3 , \end{aligned}$$

and set

$$q_1(t) = \lambda_1 \sigma_1^{-1} [g_0 - z_{14} T(t) + z_{19} T(t-h) - z_{20} \dot{T}(t) - z_{21} \dot{T}(t-h) - z_{15} e(t) + z_{16} \tau(t) + z_{17} d(t)] , \quad (1.10.17)$$

$$r_1(t) = \lambda_1 \sigma_1^{-1} [(C_0 + I_0 + X_0) - M_0 (I_{13} + C_7)] . \quad (1.10.18)$$

Then the dynamics of GNP in our economic system is

$$\frac{dy}{dt} - a_{-11} \dot{y}(t-h) - a_{13} \dot{L}(t-h) = a_{01} y(t) + a_{11} y(t-h) + a_{12} R(t) + a_{13} R(t-h) + a_{14} L(t) + a_{15} L(t-h) + a_{16} k(t) - a_{18} p(t) + q_1(t) + r_1(t) . \quad (1.10.19)$$

The rate of interest is determined by the typically Kenesian dynamics,

$$\frac{dR}{dt} = \lambda_2 (L - M) . \quad (1.10.20)$$

This yields

$$\dot{R}(t) = \lambda_2 M_1 y(t) + \lambda_2 M_2 y(t-h) + \lambda_2 M_3 R(t) + \lambda_2 M_4 R(t-h) + \lambda M_5 \dot{R}(t-h) + \lambda_2 M_6 p(t) + \lambda M_0 - \lambda_2 M1 . \quad (1.10.21)$$

If we set

$$\begin{aligned} q_2(t) &\equiv -\lambda_2 M1 , & \sigma_2(t) &= -\lambda_2 M_0 , \\ a_{-22} &= \lambda_2 M_5 , & a_{21} &= \lambda_2 M_1 , \\ a_{22} &= \lambda_2 M_2 , & a_{23} &= \lambda_2 M_3 , \\ a_{25} &= \lambda_2 M_6 , & a_{24} &= \lambda_2 M_4 , \end{aligned} \quad (1.10.22)$$

then

$$\dot{R}(t) - a_{-22} \dot{R}(t-h) = a_{21} y(t) + a_{22} y(t-h) + a_{33} R(t) + a_{24} R(t-h) + a_{25} p(t) - \sigma_2(t) + q_2(t) . \quad (1.10.23)$$

1.10.1 *Prices*

To obtain the dynamics of domestic prices we appropriate as our background the treatment by Gandolfo and Padoan [33]. Price of output is determined as

$$D \log p = \alpha_{12} \log(\hat{p}/p) + \alpha_{13} DM + \alpha_{14} \log(M/M_d) \quad (1.10.24)$$

where

$$\hat{p} = \gamma_7(p^f \cdot e)^{\beta_{12}} w^{\beta_{13}} \text{ prod} - \beta_{14}, \tag{1.10.25}$$

$$\begin{aligned} DM &= \text{proportional rate of change of } M1 \\ &\quad \text{or money supply} \end{aligned} \tag{1.10.26}$$

$$\begin{aligned} M1 &= \text{Nominal stock of money} \\ p^f &= \text{import price level (in foreign currency)} \\ e &= \text{exchange rate (country–dollar spot exchange rate)} \\ M_d &= \tilde{L} = \text{money demand} \\ W &= \text{money wage rate} \\ &= \text{wage/labor} \\ \text{prod} &= \text{labor productivity} = n\,. \end{aligned} \tag{1.10.27}$$

We observe that

$$D \log p = \frac{\dot{p}(t)}{p(t)}, \tag{1.10.28}$$

and if we linearize we obtain

$$\begin{aligned} \frac{\dot{p}}{p(t)} &= p_0 + p_1 p^f(t) \cdot e(t) + p_2 W(t) - p_3 n(t) \\ &\quad - p_4 p(t) + p_5 \dot{M}1 + p_6(M1 - L)\,, \end{aligned} \tag{1.10.29}$$

where L is the liquidity function,

$$\begin{aligned} P_0 = \log \gamma_7 &= \beta_{12}\gamma_7(a_1 p_0^f e_0 + a_0) + \beta_{13}(C_0\omega_0 - C_1\omega_0) \\ &\quad + \beta_{14}(p_0 - p_1 n_0)\,. \end{aligned} \tag{1.10.30}$$

Thus

$$\begin{aligned} \dot{p}(t) &= p(t)[p_0 + p_1 p^f(t) \cdot e(t) + p_2 w(t) - p_3 n(t) \\ &\quad - p_4 p(t) + p_5 \dot{M}_1 + p_6(M - L)]\,. \end{aligned} \tag{1.10.31}$$

Observe that

$$\begin{aligned} p_6(M - L) &= p_6(M - M_0 - M_1 y(t) - M_2 y(t-h) - M_3 R(t) \\ &\quad - M_4 R(t-h) - M_6 p(t) - M_7 \dot{R}(t-h))\,, \end{aligned} \tag{1.10.32}$$

where

$$L = M_0 + M_1 y(t) + M_2 y(t-h) + M_3 R(t) + M_4 R(t-h) + M_6 p(t) + M_5 \dot{R}(t-h)\,. \tag{1.10.33}$$

Set

$$\begin{aligned} q_5(t) &= p_1 p^f(t) \cdot e(t) + p_5 \dot{M}_1(t) + p_6 M_1(t)\,, \\ \sigma_5(t) &= p_0 - p_3 n(t) + p_2 w(t) - p_6 M_0(t)\,. \end{aligned} \tag{1.10.34}$$

Then

$$\begin{aligned} \dot{p}(t) &= p(t)[(M_6 + p_4)p(t)] - p_6 M_1 y(t) \\ &\quad - p_6 y(t-h) - p_6 M_3 R(t) - M_4 p_6 R(t-h) \\ &\quad - p_6 M_7 \dot{R}(t-h) + [q_5(t) + \sigma_5(t)]p(t)\,. \end{aligned} \tag{1.10.35}$$

1.10.2 *Balance of payment*

The Balance of Payment is given by $B = X - F - T$ where X is defined in (1.10.11) and

$$\begin{aligned} F + T &= f_0 + f_1 R(t) + f_2 R(t-h) + f_4 \dot{R}(t-h) \\ &\quad + f_5 y(t) + f_6 y(t-h) + f_8 L(t) + f_9 L(t-h) \\ &\quad + f_{11} \dot{L}(t-h) + f_{12} E(t-h) + f_{13} \dot{E}(t-h)\,. \end{aligned} \tag{1.10.36}$$

Hence

$$\begin{aligned} B &= b_0 + b_1 y(t) + b_2 y(t-h) + b_3 p(t) \\ &\quad + b_4 \dot{y}(t-h) + b_5 R(t) + b_6 R(t-h) \\ &\quad + b_7 e(t) + b_8 \dot{R}(t-h) + b_9 L(t) \\ &\quad + b_{10} L(t-h) + b_{12} \dot{L}(t-h) \\ &\quad + b_{15} d(t) + b_{13} \tau(t) + b_{17} B(t-h)\,, \end{aligned} \tag{1.10.37}$$

where

$p, e, d =$ have the usual meaning, and where E is the cumulative balance of payment,

$$E(t) = \int_0^t B(s)ds\,. \tag{1.10.38}$$

Thus the differential equation for the cumulative balance of payment is

$$\frac{dE}{dt} = B(t); \tag{1.10.39}$$

or

$$\begin{aligned}\frac{dE(t)}{dt} &= b_0 + b_1 y(t) + b_2 y(t-h) + b_4 \dot{y}(t-h) + b_5 R(t) \\ &\quad + b_6 R(t-h) + b_8 \dot{R}(t-h) + b_9 L(t) + b_{10} L(t-h) \\ &\quad + b_{12} \dot{L}(t-h) + b_3 p(t) + b_7 e(t) + b_{15} d(t) \\ &\quad + b_{13} \tau(t) + b_{17} B(t-h).\end{aligned} \tag{1.10.40}$$

On setting

$$X_0 = x_0 - M_0, \qquad b_0 = X_0 - f_0,$$

and

$$-r_6(t) = x_0 - m_0 = X_0. \tag{1.10.41}$$

Import quotas and taxes (tariff, which is privately firm generated and partially controlled); and

$$q_6(t) = b_7 e(t) + b_8 \tau(t) + b_{15} d(t) - f_0, \tag{1.10.42}$$

government control instruments: exchange rate, e, tariffs, foreign credit, interest equalization tax, f_0, preferential arrangement (which reduce trade barriers and enhance trade flows between nations) d, transportation and distance between partners. The dynamics of cumulative balance of payment becomes

$$\begin{aligned}\dot{E}(t) &= b_1 y(t) + b_2 y(t-h) + b_4 \dot{y}(t-h) + b_5 R(t) \\ &\quad + b_6 R(t-h) + b_8 \dot{R}(t-h) + b_9 L(t) + b_{10} L(t-h) \\ &\quad + b_{12} \dot{L}(t-h) + b_{17} B(t-h) - r_6(t) + q_6(t).\end{aligned} \tag{1.10.43}$$

1.10.3 *Employment and capital stock*

We now derive the equation of Capital Stock and Employment. The definition of national income from the expenditure side is

$$y = \tilde{C} + \tilde{I} + \tilde{X} + \tilde{G}, \tag{1.10.44}$$

where

$$\begin{aligned}\tilde{C} &= y_{10} + y_{11} y(t) + y_{12} \dot{y}(t) + y_{13} R(t) + y_{15}(M - \tilde{L}) + y_{18}(y - T), \\ \tilde{I} &= I_0 + I_1,\end{aligned} \tag{1.10.45}$$

and where

$$I_1 = \frac{1}{h}\int_{t-h}^{t} D(s)ds\,, \tag{1.10.46}$$

and

$$\frac{dk(t)}{dt} = D(t-h)\,, \tag{1.10.47}$$

which is the rate of deliveries of new equipment. We postulate that

$$D(t) = a(1-c)y(t) - k_0 k(t) + k_{13}\dot{k}(t) + L_5 L(t) + L_6 p(t) + v\dot{y}(t)\,. \tag{1.10.48}$$

Now,

$$\begin{aligned} I_1(t) &= \frac{1}{h}\int_{t-h}^{t} \frac{dk(s+h)ds}{ds} = \frac{1}{h}\int_{t}^{t+h} \frac{dk(\tau)d\tau}{d\tau} \\ &= \frac{1}{h}k(t)\Big|_{t}^{t+h} = \frac{1}{h}[k(t+h) - k(t)]\,. \end{aligned} \tag{1.10.49}$$

Also

$$\begin{aligned} \tilde{X} &= x_0 + x_1 y(t) + x_2 y(t-h) + x_5 R(t) + x_8 L(t) \\ &\quad + X_{10}\dot{L}(t) + x_{11}e(t) + x_{12}\tau(t) + x_{13}d(t)\,, \\ \tilde{G} &= g_{s0} + g_{s1}y(t) + g_{s4}\dot{y}(t) + g_{s5}R(t) + g_{s8}L(t)\,. \end{aligned} \tag{1.10.50}$$

Thus

$$\begin{aligned} y(t) &= z_{s0} + z_{s1}y(t) + z_{s2}y(t-h) + z_{s4}\dot{y}(t) + z_{s5}R(t) \\ &\quad + z_{s8}L(t) + z_{s10}\dot{L}(t) + z_{s13}M_1 - z_{s14}T(t) + z_{s15}e(t) \\ &\quad + z_{s16}\tau(t) + z_{s17}d(t) + \frac{1}{h}[k(t+h) - k(t)]\,. \end{aligned} \tag{1.10.51}$$

It follows that

$$\begin{aligned} y(t) &= (1-z_{s1})^{-1}[z_{s0} + z_{s2}y(t-h) + z_{s4}\dot{y}(t) \\ &\quad + z_{s5}R(t) + z_{s8}L(t) + z_{s10}\dot{L}(t) + z_{s13}M_1 \\ &\quad - z_{s14}T(t) + z_{s15}e(t) + z_{s16}\tau(t) + z_{s17}d(t)] \\ &\quad + \frac{1}{(1-z_{s1})h}[k(t+h) - k(t)]\,. \end{aligned} \tag{1.10.52}$$

Substitute $D(t) = \dot{k}(t+h)$ and (1.10.52) into (1.10.48) then

$$\dot{k}(t+h) = \frac{a(1-c)}{(1-z_{s1})h}[k(t+h) - k(t)]$$

$$
\begin{aligned}
&+ \frac{a(1-c)}{1-z_{s1}}[z_{s0} + z_{s13}M1 - z_{s14}T(t) + z_{s15}e(t) \\
&+ z_{s16}\tau(t) + z_{s17}d(t)] \\
&+ \frac{a(1-c)}{1-z_{s1}}[z_{s1}y(t) + z_{s4}\dot{y}(t) + z_{s5}R(t) + z_{s8}L(t) \\
&+ z_{s10}\dot{L}(t) + z_{s11}k(t) + k_{13}\dot{k}(t)] \\
&+ L_4R(t) + L_5L(t) + L_6p(t)\,. \qquad (1.10.53)
\end{aligned}
$$

Now to switch t to $t-h$ to deduce that

$$
\begin{aligned}
\dot{k}(t) + a_{-1}\dot{k}(t-h) = a_0k(t) &+ a_1k(t-h) + a_2y(t-h) \\
&+ a_3\dot{y}(t-h) + a_4R(t-h) + a_5L(t-h) \\
&+ a_6\dot{L}(t-h) + a_8p(t-h) + b_0q_0 + b_1p_1\,, \qquad (1.10.54)
\end{aligned}
$$

where

$$
\begin{aligned}
a_{-1} &= -k_{13}\left(\frac{a(1-c)}{1-z_{s1}}\right), \\
a_0 &= \frac{a(1-c)}{(1-z_{s1})h}, \\
a_1 &= \frac{-a(1-c)}{1-z_{s1}}(1/h)z_{s11}\,, \\
a_2 &= \frac{a(1-c)}{(1-z_{s1})}(z_{s1})\,, \\
a_3 &= \frac{a(1-c)}{1-z_{s1}}z_{s4}\,, \\
a_4 &= \frac{a(1-c)}{(1-z_{s1})}z_{s5} + L_4\,, \qquad (1.10.55) \\
a_5 &= \frac{a(1-c)}{(1-z_{s1})}\cdot z_{s8} + L_5\,, \\
a_6 &= \frac{a(1-c)}{(1-z_{s1})}\cdot z_{s10}\,, \\
a_8 &= L_6\,,
\end{aligned}
$$

$$
\begin{aligned}
b_0q_0 + b_1p_1 &= \frac{a(1-c)}{(1-z_{s1})}[z_{s0} + z_{s13}M1 - z_{s14}T(t) + z_{s15}e(t) \\
&+ z_{s16}\tau(t) + z_{s17}d(t)]\,,
\end{aligned}
$$

$$z_{s0} = g_0 + x_0 + y_{10} + I_0 \,.$$

Let

$$q_4 = g_0 + z_{s13}M1 - z_{s14}T(t) + z_{s15}e(t) + z_{s16}\tau(t) + z_{s17}d(t)\,; \tag{1.10.56}$$

and

$$\sigma_4(t) = x_0 + y_{10} + I_0\,; \tag{1.10.57}$$

then the dynamics of capital stock emerges as

$$\begin{aligned} &\dot{k}(t) + a_{-1}\dot{k}(t-h) - a_3\dot{y}(t-h) - a_6\dot{L}(t-h) \\ &\quad = a_0k(t) - a_1k(t-h) + a_2y(t-h) + a_4R(t-h) \\ &\qquad + a_5L(t-h) + a_8p(t) + \sigma_4(t) + q_4(t)\,. \end{aligned} \tag{1.10.58}$$

Remark We note that the coefficients of the dynamics in (1.10.54) and therefore of (1.10.58) are identified in (1.10.55). These are obtained by a simple MATLAB linear regression using the arx command on (1.10.47), (1.10.48), (1.10.49), (1.10.50), (1.10.51) and (1.10.52). To obtain the functional differential equation satisfied by employment we recall the Cobb–Douglas equation

$$y = f(k, L) = k^{\alpha}L^{1-\alpha}\,; \tag{1.10.59}$$

and the relation

$$L = m(\omega)k\,, \qquad \dot{L}(t) = m(\omega)\dot{k}(t)\,, \tag{1.10.60}$$

where

$$m(\omega) = \left[(1-\alpha)\frac{1}{\omega}\right]^{1/\alpha}\,. \tag{1.10.61}$$

Because

$$\dot{k}(t) = \frac{\dot{L}(t)}{m(\omega)}\,. \tag{1.10.62}$$

The capital stock equation becomes

$$\begin{aligned} \frac{\dot{L}(t)}{m(\omega)} + \left(\frac{a_{-1}}{m(\omega)} - a_6\right)\dot{L}(t-h) - a_3\dot{y}(t-h) &= \frac{a_0}{m(\omega)}L(t) - \frac{a_1}{m(\omega)}L(t-h) \\ &\quad + a_2y(t-h) + a_4R(t-h) \\ &\quad + a_5L(t-h) + a_8p(t) \\ &\quad + \sigma_4(t) + q_4(t)\,. \end{aligned} \tag{1.10.63}$$

Multiply both sides of this equation by $m(\omega)$:

$$\dot{L}(t) + (a_1 - m(\omega)a_6)\dot{L}(t-h) - a_3 m(\omega)\dot{y}(t-h)$$
$$= a_0 L(t) - a_1 L(t-h) + m(\omega)a_2 y(t-h)$$
$$+ a_4 m(\omega)R(t-h) + a_5 m(\omega)L(t-h)$$
$$+ m(\omega)a_8 p(t) + m(\omega)\sigma_4(t) + m(\omega)q_4(t)\,. \qquad (1.10.64)$$

Recall that profit $P = y - \omega L - rK$ where ω is the wage of labor per unit time, r is the rent per unit time of the use of capital. We built into our model the maximization of profit so that

$$\frac{\partial P}{\partial L} = 0\,, \qquad \frac{\partial P}{\partial k} = 0\,. \qquad (1.10.65)$$

As a consequence $L = m(\omega)k$ where

$$m(\omega) = \left[(1-\alpha)\frac{1}{\omega}\right]^{1/\alpha}.$$

Let

$$\begin{aligned} a_{-1} - m(\omega)a_6 &= l_{-01}\,, & a_3 m(\omega) &= l_{-03}\,, \\ a_0 &= l_0\,, & -a_1 &= l_1\,, \\ m(\omega)a_2 &= l_2\,, & m(\omega)a_4 &= l_4\,, \\ m(\omega)a_5 &= l_5\,, & m(\omega)a_8 &= l_8\,, \\ m(\omega)\sigma_4(t) &= -\sigma_3(t)\,, & m(\omega)q_4(t) &= q_3\,. \end{aligned} \qquad (1.10.66)$$

Then the dynamics which we are seeking is

$$\dot{L}(t) - l_{-01}\dot{L}(t-h) - l_{-03}\dot{y}(t-h) = l_0 L(t) - l_1 L(t-h) + l_2 y(t-h) + l_4 R(t-h)$$
$$+ l_5 L(t-h) + \sigma_3(t) + q_3(t)\,. \qquad (1.10.67)$$

The equations we have displayed can be put in matrix form as follows

$$\dot{x}(t) - A_{-1}\dot{x}(t-h) = A_0 x(t) + A_1 x(t-h) - \sigma_{(t)} + q(t)\,, \qquad (1.10.68)$$

where

$$x = \begin{bmatrix} y \\ R \\ L \\ k \\ p \\ E \end{bmatrix}, \qquad (1.10.69)$$

$$A_{-1} = \begin{bmatrix} a_{-11} & 0 & a_{-13} & 0 & 0 & 0 \\ 0 & a_{-22} & 0 & 0 & 0 & 0 \\ I_{-03} & 0 & -l_{01} & 0 & 0 & 0 \\ a_3 & 0 & a_6 & -a_{-1} & 0 & 0 \\ 0 & -M_7 p_6 p(t) & 0 & 0 & 0 & 0 \\ b_4 & b_8 & b_{12} & 0 & 0 & 0 \end{bmatrix}, \tag{1.10.70}$$

$$A_0 = \begin{bmatrix} a_{01} & a_{12} & a_{14} & a_{16} & -a_{18} & 0 \\ a_{21} & a_{23} & 0 & 0 & q_{25} & 0 \\ 0 & 0 & a_{33} & 0 & a_{35} & 0 \\ 0 & 0 & 0 & a_{44} & a_{45} & 0 \\ a_{51} & a_{52} & 0 & 0 & a_{55} & 0 \\ a_{61} & a_{62} & a_{63} & 0 & 0 & 0 \end{bmatrix}, \quad \text{or} \tag{1.10.71}$$

$$A_0 = \begin{bmatrix} a_{01} & a_{12} & a_{14} & a_{16} & -a_{18} & 0 \\ a_{21} & a_{23} & 0 & 0 & a_{25} & 0 \\ 0 & 0 & l_0 & 0 & M a_8 & 0 \\ 0 & 0 & 0 & a_0 & a_8 & 0 \\ -p_6 M_1 p & -p_6 M_3 p & 0 & 0 & (M_6 + p_4)p & 0 \\ b_1 & b_5 & b_9 & 0 & 0 & 0 \end{bmatrix}.$$

$$a_{01} = \gamma_1 \sigma_1^{-1}(z_1 - 1 - z_{13} M_1)\,,$$

$$a_{12} = (z_5 - z_{13} M_2)\gamma_1 \sigma^{-1}\,,$$

$$a_{14} = \gamma_1 \sigma_1^{-1} z_8 = \gamma_1 \sigma_1^{-1}(g_8 + I_8 + x_8)\,,$$

$$a_{16} = \gamma_1 \sigma_1^{-1} z_{11} = \gamma_1 \sigma_1^{-1}(-I_{11})\,,$$

$$a_{18} = \gamma_1 \sigma_1^{-1} z_{13} M_3 = -\gamma_1 \sigma_1^{-1} M_3 M_5 (I_{13} + C_7)\,,$$

$$a_{21} = \gamma_2 M_1\,,$$

$$a_{23} = \gamma_2 M_3\,,$$

$$a_{25} = \gamma_2 M_6\,,$$

$$a_{33} = a_0 = l_0 = \frac{a(1-c)}{(1-z_{s1})h}\,,$$

$$a_{35} = l_8 = m(\omega) a_8 = \left[(l - \alpha)\frac{1}{\omega}\right]^{1/\alpha} L_6\,, \tag{1.10.72}$$

$$a_{44} = a_0 = \frac{a(1-c)}{(1-z_{s1})h},$$

$$a_{45} = a_8 = L_6,$$

$$a_{51} = -p_6 M_1 p(t),$$

$$a_{52} = -p_6 M_3 p(t),$$

$$a_{55} = -(M_6 + p_4)p(t),$$

$$a_{61} = b_1,$$

$$a_{62} = b_5,$$

$$a_{63} = b_9.$$

$$A_1 = \begin{bmatrix} a_{111} & a_{112} & a_{113} & 0 & 0 & 0 \\ a_{121} & a_{122} & 0 & 0 & 0 & 0 \\ a_{131} & a_{132} & a_{133} & 0 & 0 & 0 \\ a_{141} & a_{142} & a_{143} & -\tilde{a}_1 & 0 & 0 \\ a_{151} & a_{152} & 0 & 0 & 0 & 0 \\ a_{161} & a_{162} & a_{163} & a_{164} & a_{165} & a_{166} \end{bmatrix} \tag{1.10.73}$$

or
$$A_1 = \begin{bmatrix} a_{11} & a_{13} & 1_{15} & 0 & 0 & 0 \\ a_{22} & a_{24} & 0 & 0 & 0 & 0 \\ l_2 & l_4 & l_5 - l_1 & 0 & 0 & 0 \\ a_2 & a_4 & a_5 & -a_1 & 0 & 0 \\ -p_6 p(t) & -M_4 p_6 p(t) & 0 & 0 & 0 & 0 \\ b_2 & b_6 & b_{10} & 0 & 0 & b_{17} \end{bmatrix}$$

where

$$a_{111} = a_{11} = z_2\gamma_1\sigma_1^{-1} = \gamma_1\sigma_1^{-1}[g_2 + I_2 + C_2 + x_2 + M_2(I_{13} + C_7)],$$

$$a_{112} = a_{13} = \gamma_1\sigma_1^{-1}[I_6 + C_6 - C_7 M_6],$$

$$a_{113} = a_{15} = \gamma_1\sigma_1^{-1}z_9 = \gamma_1\sigma_1^{-1}I_9,$$

$$a_{114} = 0,$$

$$a_{115} = 0,$$

$$a_{116} = 0,$$

$$a_{121} = a_{22} = \gamma_2 M_2,$$

$$a_{122} = a_{24} = \gamma_2 M_4 \,,$$

$$a_{123} = 0 \,,$$

$$a_{124} = 0 \,,$$

$$a_{125} = 0 \,,$$

$$a_{126} = 0 \,,$$

$$a_{131} = l_2 = m(\omega)\frac{a(1-c)}{(1-z_{s1})}z_{s2} \,,$$

$$= \frac{m(\omega)a(1-c)x_2}{1-(x_1+g_{s1}+y_{11})} \,,$$

$$a_{132} = l_4 = m(\omega)a_4 = \frac{a(1-c)}{(1-z_{s1})}z_{s5} + L_4$$

$$a_{133} = l_5 - l_1 = m(\omega)a_5 - l_1 = \frac{m(\omega)a(1-c)z_{s8}}{1-z_{s1}} + L_5 - l_1 \,,$$

$$a_{134} = 0 \,,$$

$$a_{135} = 0 \,,$$

$$a_{136} = 0 \,,$$

$$a_{141} = a_2 = \frac{a(1-c)}{(1-z_{s1})}z_{s2} \,,$$

$$a_{142} = a_4 = \frac{a(1-c)}{(1-z_{s1})}z_{s5} + L_4 \,,$$

$$a_{143} = a_5 = \frac{a(1-c)}{(1-z_{s1})} \cdot z_{s8} + L_5$$

$$a_{144} = 0 = a_{145} = a_{146} \,,$$

$$a_{151} = -p_6 p(t) \,,$$

$$a_{152} = -M_4 p_6 p(t) \,,$$

$$a_{153} = 0 \,,$$

$$a_{154} = 0 \,,$$

$$a_{155} = 0 \,,$$

$$a_{156} = 0 \,,$$

$$a_{161} = b_2 \,,$$

$$a_{162} = b_6 \,,$$

$$a_{163} = b_{10} \,,$$

$$a_{164} = 0\,,$$
$$a_{165} = 0\,,$$
$$a_{166} = b_{17}\,. \tag{1.10.74}$$

Let

$$q = \begin{bmatrix} T_1 \\ g_0 \\ e \\ \tau \\ d \\ M_1 \\ \dot{M}_1 \\ f_0 \end{bmatrix}, \tag{1.10.75}$$

where $T_1 = -z_{14}T(t) + z_{19}T(t-h) - z_{20}\dot{T}(t) - z_{21}\dot{T}(t-h)$.

$$\sigma = \begin{bmatrix} C_0 \\ I_0 \\ X_0 \\ M_0 \\ n \\ w \\ x_0 \\ y_{10} \\ p_0 \end{bmatrix}, \tag{1.10.76}$$

$$\begin{aligned}
\xi_1 &= \gamma_1 \sigma_1^{-1}\,,\\
q_1 &= \gamma_1 \sigma_1^{-1}[g_0 + z_{13}M1 + T_1 - z_{15}e(t) + z_{16}\tau(t) + z_{17}d(t)]\,,\\
\sigma_1(t) &= \gamma_1 \sigma_1^{-1}[C_0 + I_0 + X_0 - z_{13}M_0]\,,\\
q_2(t) &= -\gamma_2 M1\,,\\
-\sigma_2(t) &= \gamma_2 M_0\,,\\
q_5(t) &= p_1 p^f(t)\cdot e(t) + p_5\dot{M}1 + p_6 M1\,, \qquad (1.10.77)\\
\sigma_5(t) &= P_0 - p_3 n(t) + p_2\omega(t) - p_6 M_0\,,\\
q_4(t) &= g_0 + z_{s13}M - z_{s14}T(t) + z_{s15}e(t) + z_{s16}\tau(t) + z_{s17}d(t)\,,
\end{aligned}$$

$$
\begin{aligned}
\sigma_4(t) &= x_0 + y_{10} + I_0\,, \\
q_3(t) &= m(\omega) q_4(t)\,, \\
\sigma_3(t) &= m(\omega)\sigma_4(t)\,. \\
q_6 &= b_7 e(t) + b_8 \tau(t) + b_{15} d(t) - f_0\,, \\
-\sigma_6(t) &= x_0 - M_0 = X_0\,.
\end{aligned} \tag{1.10.78}
$$

$$
B_1 = \begin{bmatrix}
-\xi_1 & \xi_1 & -z_{15}\xi_1 & -\xi_1 z_{16} & -\xi_1 z_{17} & \xi_1 z_{18} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -\gamma_2 & 0 & 0 \\
-m(\omega) z_{s14} & m(\omega) & m(\omega) z_{s15} & m z_{s16} & m z_{s17} & m z_{s13} & 0 & 0 \\
-z_{s14} & 1 & z_{s15} & z_{s16} & z_{s17} & z_{s13} & 0 & 0 \\
0 & 0 & p_1 p f(t) & 0 & 0 & p_6 & p_5 & 0 \\
0 & 0 & b_7 & b_8 & b_{15} & 0 & 0 & -1
\end{bmatrix}, \tag{1.10.79}
$$

$$
B_2 = \begin{bmatrix}
-\xi_1 & \xi_1 & \xi_1 & \xi_1 (I_{13} + C_7) & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -\gamma_2 & 0 & 0 & 0 & 0 & 0 \\
0 & m(\omega) & 0 & 0 & 0 & 0 & m & m & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & -p_6 & -p_3 & p_2 & 0 & 0 & 1 \\
0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0
\end{bmatrix}. \tag{1.10.80}
$$

Then

$$
\dot{x}(t) + A_{-1}\dot{x}(t-h) = A_0 x(t) + A_1 x(t-h) + B_1 q + B_2 \sigma\,. \tag{1.10.81}
$$

Compared with (1.10.81):

$$
\begin{aligned}
D(t, x_t) &= x(t) + A_{-1} x(t-h)\,, \\
\frac{d}{dt}(D(t, x_t)) &= \frac{d}{dt}(x(t) - A_{-1} x(t-h))\,, \\
f(t, x_t, u(t)) &= A_0 x(t) + A_1 x(t-h) + u(t)\,,
\end{aligned}
$$

where

$$
u(t) = B_1 q(t) + B_2 \sigma(t)\,.
$$

We consider the general nonlinear functional differential equation

$$
\dot{x}(t) - A_{-1}\dot{x}(t-h) = f(t, x_t, u(t)) + B(t, x_t) u(t)\,, \tag{1.10.82}
$$

where $f : E \times C^0 \times E^m \to E^n$ is a nonlinear function. Here E^r is the r-dimensional Euclidean space with norm $|\cdot|$. The symbol C^0 denotes the space on continuous functions mapping the interval $[-h, 0]$, $h > 0$, into E^n, with the sup norm $\|\cdot\|$, defined by $\|\Phi\| = \sup_{-h \leq s \leq 0} |\Phi(s)|$, $\Phi \in C$. The control matrix function $B : E \times C^0 \to E^{n \times m}$ is possibly nonlinear. The controls are square integrable functions $u \in L_2([\sigma, t_1], E^m)$, σ, $t_1 \in E$, $t_1 > \sigma$, and L_2 is the space of measurable functions u defined on intervals $[\sigma, t_1]$ for which $|u|^2$ is summable. If $t \in [\sigma, t_1]$, we let $x_t \in C^0$ be defined by $x_t(s) = x(t + s)$, $-h \leq s \leq 0$. With L_2 as the set of admissible controls the state space is either E^n, or $W_2^{(1)}$, the Sobolev space of absolutely continuous functions $x : [-h, 0]$ with the property that $t \to \dot{x}(t) \in L_2([-h, 0], E^n)$. The targets are points in E^n or function in $W_2^{(1)}$.

The concept of function space controllability is very appropriate. Once a target is hit, it remains (hopefully) on a certain growing function for some time T. For example, once employment is brought to a desired level, it is important to keep it there at that level for some T.

The control variable of our economic system are of two kinds and of the form

$$u = B_1 p - B_2 g ;$$

where g is the control instrument of government (taxes, money, supply, public consumption, exchange rate, subsidy, preferential trade arrangement, tariff), and where p is the control instrument of private initiative (autonomous consumption, investment, net export, money holding wages productivity) B_i, $i = 1, 2$ are the respective control matrices. It can be proved that any function target $x_T = (y_T, R_T, L_T, K_T, p_T, E_T)$, can be reached from any position if and only if the number of effective control instruments is equal to the number (S) of target functions. This is a resurgence of "Tinbegen's static controllability condition". There exist a set of policy instruments which is capable of moving the initial function state into some other desired position in a finite time. As a result levels of national income, interest rate, employment, prices, value of capital stock and cumulative balance of payment can be controlled simultaneously. As a result, inflation and employment can be controlled at the same time, provided all the control instruments are in force. This seems to be a basic insight and argument of Robert Eisner in his recent book [34, Chap. 8]. See the Theorem reported in Chukwu [29, pp. 81–89].

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