Chapter 1

First Concepts

1.1 Fundamentals of the complex field

In this section we give a rapid introduction to the complex numbers themselves. Here we shall assume the reader is largely familiar with this material, or at most needs to have some jogging of the memory.

We denote the complex numbers by $\mathbb{C}$ which is identified with the Euclidean plane $\mathbb{R}^2$ by writing the pair (a, b) as $a + bi$. In $\mathbb{C}$ we add by $(a + bi) + (c + di) = a + c + (b + d)i$ and multiply by $(a + bi) \cdot (c + di) = ac - bd + (ad + bc)i$. Thus $i^2 = -1$. It is easy to check, and this we leave to the reader, that $(\mathbb{C}, +, \cdot)$ is a commutative ring with unit, where $0 + 0i = 0$ is the zero and $1 + 0i = 1$ is the unit. $\mathbb{C}$ is actually a field, that is, every non-zero element has a multiplicative inverse.

To see this let $z = x + iy \in \mathbb{C}$. We define two quantities the norm of a complex number z given by

$$|z| = \sqrt{x^2 + y^2}\,.$$

(here $|z| \geq 0$ and $|z| = 0$ only if $z = 0$) and the conjugate $\bar{z}$ of $z = x + iy \in \mathbb{C}$ by

$$\bar{z} = x - iy\,.$$

A direct calculation shows that $z\bar{z} = |z|^2$. Hence if $z \neq 0$, $z\frac{\bar{z}}{|z|^2} = 1$. Thus $z^{-1} = \frac{\bar{z}}{|z|^2}$ so $\mathbb{C}$ is a commutative field. We denote its multiplicative group by $\mathbb{C}^\times$. Another direct calculation shows $\overline{(z+w)} = \bar{z} + \bar{w}$ and $\overline{(z \cdot w)} = \bar{z} \cdot \bar{w}$. Thus conjugation is an automorphism of the complex field. Using this last remark it follows readily that

$$|zw| = |z||w|,$$

and for $w \neq 0$

$$|\frac{z}{w}| = \frac{|z|}{|w|}.$$

This in turn tells us that if $z \neq 0$ then $\frac{z}{|z|}$ has modulus 1. This process is called *normalizing* z.

The fact that conjugation preserves multiplication also shows that the unit circle

$$\mathbb{T} = \{z : |z| = 1\}$$

forms a subgroup of $\mathbb{C}^\times$ whose elements are characterized by $z^{-1} = \bar{z}$. So in particular $i^{-1} = -i$ and $-i^{-1} = i$. The group $\mathbb{T}$ is compact. It is the group on which Fourier analysis is done. Actually, under the metric defined by

$$d(z, w) = |z - w|$$

the field $\mathbb{C}$ forms a complete metric space. An important ingredient is the so-called triangle inequality

$$|z + w| \leq |z| + |w|.$$

Finally, we mention the polar decomposition of a non-zero complex number. If $z \in \mathbb{T}$, then there is a unique angle θ (actually unique up to an integer multiple of 2π) such that $z = \cos\theta + i\sin\theta$. Of course, this means that more generally, if $z \neq 0$, then $z = |z|(\cos\theta + i\sin\theta)$. This is called the *polar decomposition* of $z \neq 0$. The angle θ is called the argument, or arg of z. (Obviously we must have $z \neq 0$ to have a well defined argument). This decomposition shows $\mathbb{C}^\times$ is a product space, $\mathbb{C}^\times = {\mathbb{R}_+}^\times \times \mathbb{T}$, where the first component of z is its modulus $|z|$ and the second its argument θ. We ask the reader to verify all these facts.

Exercise 1.1 *Let $p(z)$ be a polynomial with real coefficients. Show that the non-real roots occur in conjugate pairs. In particular, if p has odd degree it must have a real root.*