

## 1.2 Holomorphic functions

Since we have a (metric) topology on  $\mathbb{C}$  we can talk about open sets. Just as in real analysis, these will be the domains of differentiable functions. Let  $\Omega$  be a non-empty open set in  $\mathbb{C}$ . For example,  $\Omega$  could be  $\mathbb{C}$  itself, or  $\mathbb{C}$  with say a finite number of points removed (such as  $\mathbb{C}^\times$ ), or the interior of a disk, or a half plane. Analogously to real analysis we consider functions  $f : \Omega \rightarrow \mathbb{C}$ . To distinguish the situation from that of real functions (and as we shall see, it is quite different) we use the term *holomorphic function*.

**Definition 1.2.1** Let  $f : \Omega \rightarrow \mathbb{C}$  and  $a \in \Omega$ . We say  $f$  is holomorphic at  $a$  if

$$\lim_{z \rightarrow a} \frac{f(z) - f(a)}{z - a}$$

exists. Since if limits exist they are unique, we can give this one the name  $f'(a)$ , called the derivative of  $f$  at  $a$ . If  $f$  is holomorphic at every point of  $\Omega$ , we say it is a holomorphic function on  $\Omega$ . In this case the derivative is denoted by  $f'$ . If we wish to take several derivatives we denote by  $f^{(k)}$  the  $k$ th derivative of  $f$ .

Before proceeding we make a few observations. Just as in the case of real functions of a real variable, items 1 and 2 below follow from the definitions and elementary properties of limits in the complex domain. Together they go by the name, the linear approximation theorem. Item 3 follows immediately from item 1.

1. If  $f$  is differentiable at  $a$ , then

$$f(z) = f(a) + f'(a)(z - a) + \epsilon(z)(z - a),$$

where  $\epsilon(z)$  tends to zero as  $z$  tends to  $a$ .

2. If  $f$  is defined in a neighborhood of  $a$  and

$$f(z) = f(a) + c(z - a) + \epsilon(z)(z - a),$$

where  $c$  is a constant and  $\epsilon(z)$  tends to zero as  $z$  tends to  $a$ , then  $f$  is differentiable at  $a$  and  $f'(a) = c$ .

3. If  $f$  is differentiable at  $a$ , then  $f$  is continuous at  $a$ .

Notice that, just as in the real case, being holomorphic is a local property. Moreover, if we have two functions  $f$  and  $g$  both differentiable at  $a$ , then

1.  $f \pm g$  is differentiable at  $a$  and  $(f \pm g)'(a) = f'(a) \pm g'(a)$ .
2. Similarly,  $fg$  is differentiable at  $a$  and  $(fg)'(a) = f'(a)g(a) + f(a)g'(a)$ .
3. If, in addition,  $g(a) \neq 0$ , then  $\frac{f}{g}$  is also differentiable at  $a$  and 
$$\left(\frac{f}{g}\right)'(a) = \frac{f'(a)g(a) - f(a)g'(a)}{g(a)^2}.$$

We ask the student to check all these facts by looking at the analogous material in a calculus book.

Since the derivative of the constant function is clearly zero, from the second of these formulas it follows that the derivative of a constant times  $f$  is  $c \cdot f'$ . Also from the second formula, we see by induction that the function  $f(z) = z^n$  is holomorphic on all of  $\mathbb{C}$  with  $f'(z) = nz^{n-1}$ . Hence by the first formula, together with the above, all polynomials are holomorphic on  $\mathbb{C}$ . Also, if  $f$  is a polynomial of degree  $n$ , then taking  $n + 1$  successive derivatives gives zero. Functions that are holomorphic on all of  $\mathbb{C}$  have a special name; they are called *entire*.

The third formula gives examples of holomorphic functions on domains other than  $\mathbb{C}$ . Let  $f(z) = \frac{p(z)}{q(z)}$  where  $p$  and  $q$  are polynomials. These are the so-called rational functions. By the third formula these are holomorphic everywhere on  $\mathbb{C}$ , except at the zeros of  $q$ .

Another fact whose statement and proof are completely analogous to that of the real case is the chain rule. If  $f : \Omega \rightarrow \Omega' \subseteq \mathbb{C}$  is holomorphic at  $a$  and  $g : \Omega' \rightarrow \mathbb{C}$  is holomorphic at  $f(a)$ , then  $g \circ f : \Omega \rightarrow \mathbb{C}$  is also holomorphic at  $a$  and  $(g \circ f)'(a) = g'(f(a))f'(a)$ . Just as in the real case this follows from items 1 and 2 above.

We shall say a holomorphic function is *regular* on  $\Omega$  if  $f'(a) \neq 0$  for every  $a \in \Omega$ .

**Corollary 1.2.2** *If  $f$  is locally invertible at  $a$ , then  $f$  is regular there.*

*Proof.* Let  $g$  be the holomorphic local inverse to  $f$  at  $a$ . Then  $g(f(z)) = z$  for  $z$  in some neighborhood of  $a$ . As we saw the derivative of  $z$  is 1, differentiating, we get  $g'(f(z))f'(z) = 1$ . In particular, this holds at  $z = a$ . Therefore,  $f'(a) \neq 0$ .  $\square$

In Section 1.4 we shall see why the  $g$  above is holomorphic and that the converse of this corollary is also true.

## 1.3 Some important examples

Here, as in the previous section, we shall rely on various useful facts of real analysis.

We now define what is, taken in its most general form, the most important function in mathematics, namely the exponential function written  $e^z$  or  $\exp(z)$ .  $e^z$  is defined by the power series

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}, z \in \mathbb{C}.$$

This series converges absolutely and uniformly on compacta of  $\mathbb{C}$ . To check absolute convergence note that  $|z^n| = |z|^n$ , for every  $n$ . Hence  $\sum_{n=0}^{\infty} \frac{|z|^n}{n!} = e^{|z|}$ , the real exponential function, which converges absolutely. Also, if  $|z| \leq c$ , then the tail  $|\sum_{n=p}^q \frac{z^n}{n!}|$  can be estimated by  $\sum_{n=p}^q \frac{|z|^n}{n!} \leq \sum_{n=p}^q \frac{c^n}{n!}$  which is the tail of  $e^c$ ,  $c \in \mathbb{R}$ . Since this converges as  $p$  and  $q$  tend to infinity by completeness the series for  $e^z$  also converges uniformly on compacta. As we shall see in Section 1.7 these results are actually quite general.

**Corollary 1.3.1**  $e^{z+w} = e^z e^w$ .

*Proof.* Now  $e^{z+w} = \sum_{n=0}^{\infty} \frac{(z+w)^n}{n!}$ . By the binomial theorem,  $(z+w)^n = \sum_{k=0}^n \frac{n!}{k!(n-k)!} z^k w^{n-k}$ . Hence,  $e^{z+w} = \sum_{n=0}^{\infty} \sum_{k=0}^n \frac{1}{k!(n-k)!} z^k w^{n-k}$ . By absolute convergence we can rearrange the order of summation in this series. Letting  $m = n - k$  we get  $\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{z^n w^m}{n! m!}$ . On the other hand, also by rearrangement this is  $\sum_{n=0}^{\infty} \frac{z^n}{n!} \sum_{m=0}^{\infty} \frac{w^m}{m!}$ , which is just  $e^z e^w$ .  $\square$