

*Proof.* Let  $g$  be the holomorphic local inverse to  $f$  at  $a$ . Then  $g(f(z)) = z$  for  $z$  in some neighborhood of  $a$ . As we saw the derivative of  $z$  is 1, differentiating, we get  $g'(f(z))f'(z) = 1$ . In particular, this holds at  $z = a$ . Therefore,  $f'(a) \neq 0$ .  $\square$

In Section 1.4 we shall see why the  $g$  above is holomorphic and that the converse of this corollary is also true.

## 1.3 Some important examples

Here, as in the previous section, we shall rely on various useful facts of real analysis.

We now define what is, taken in its most general form, the most important function in mathematics, namely the exponential function written  $e^z$  or  $\exp(z)$ .  $e^z$  is defined by the power series

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}, z \in \mathbb{C}.$$

This series converges absolutely and uniformly on compacta of  $\mathbb{C}$ . To check absolute convergence note that  $|z^n| = |z|^n$ , for every  $n$ . Hence  $\sum_{n=0}^{\infty} \frac{|z|^n}{n!} = e^{|z|}$ , the real exponential function, which converges absolutely. Also, if  $|z| \leq c$ , then the tail  $|\sum_{n=p}^q \frac{z^n}{n!}|$  can be estimated by  $\sum_{n=p}^q \frac{|z|^n}{n!} \leq \sum_{n=p}^q \frac{c^n}{n!}$  which is the tail of  $e^c$ ,  $c \in \mathbb{R}$ . Since this converges as  $p$  and  $q$  tend to infinity by completeness the series for  $e^z$  also converges uniformly on compacta. As we shall see in Section 1.7 these results are actually quite general.

**Corollary 1.3.1**  $e^{z+w} = e^z e^w$ .

*Proof.* Now  $e^{z+w} = \sum_{n=0}^{\infty} \frac{(z+w)^n}{n!}$ . By the binomial theorem,  $(z+w)^n = \sum_{k=0}^n \frac{n!}{k!(n-k)!} z^k w^{n-k}$ . Hence,  $e^{z+w} = \sum_{n=0}^{\infty} \sum_{k=0}^n \frac{1}{k!(n-k)!} z^k w^{n-k}$ . By absolute convergence we can rearrange the order of summation in this series. Letting  $m = n - k$  we get  $\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{z^n}{n!} \frac{w^m}{m!}$ . On the other hand, also by rearrangement this is  $\sum_{n=0}^{\infty} \frac{z^n}{n!} \sum_{m=0}^{\infty} \frac{w^m}{m!}$ , which is just  $e^z e^w$ .  $\square$

**Corollary 1.3.2** For all  $t \in \mathbb{R}$  and  $z \in \mathbb{C}$ ,  $|e^{tz}| = |e^z|^t$ .

*Proof.* For all  $z$  and  $w$ , by Corollary 1.3.1,  $e^z e^w = e^{z+w}$ . Therefore,  $e^{nz} = (e^z)^n$ , for  $z \in \mathbb{C}$  and  $n$  a positive integer. But also  $e^z e^{-z} = e^0 = 1$  so  $(e^z)^{-1} = e^{-z}$ . Hence for  $n$  a positive integer

$$e^{-nz} = e^{n(-z)} = e^{-zn} = ((e^z)^{-1})^n = (e^z)^{-n}.$$

Thus the conclusion holds for all  $z \in \mathbb{C}$  and  $n \in \mathbb{Z}$ .

Next let  $r = \frac{p}{q}$  be rational. Then  $(e^{\frac{p}{q}z})^q = e^{q\frac{p}{q}z} = e^{pz} = (e^z)^p$ . Hence  $|e^{\frac{p}{q}z}|^q = |e^z|^p$  and so  $|e^{\frac{p}{q}z}| = |e^z|^{\frac{p}{q}}$ . This shows  $|e^{rz}| = |e^z|^r$  for all  $z \in \mathbb{C}$  and  $r \in \mathbb{Q}$ .

Since for fixed  $z \in \mathbb{C}$  both sides of the equation in the statement of the Lemma are continuous functions of  $t$  and  $\mathbb{Q}$  is dense in  $\mathbb{R}$ , it holds for all  $t \in \mathbb{R}$ .  $\square$

**Exercise 1.2** Show  $\lim_{n \rightarrow \infty} (1 + \frac{z}{n})^n = e^z$ .

Now similar reasoning to that used to define  $\exp$  shows that the power series defining  $\sin$  and  $\cos$  in the real domain work just as well in the complex domain. Thus we get the functions

$$\sin z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}$$

and

$$\cos z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!},$$

all for  $z \in \mathbb{C}$ .

**Corollary 1.3.3**  $e^{iz} = \cos z + i \sin z$ .

To verify this, write the power series for

$$e^{iz} = \sum_{n=0}^{\infty} \frac{(iz)^n}{n!} = 1 + iz - \frac{z^2}{2!} - i\frac{z^3}{3!} + \frac{z^4}{4!} + \dots$$

As we have seen, this series converges absolutely and so can be rearranged. Combining the even and the odd terms we get the relation of the corollary.

In particular, taking  $z$  real in that relation we get Euler's relation. (Sometimes Corollary 1.3.3 itself is called Euler's relation).

$$e^{i\theta} = \cos \theta + i \sin \theta.$$

Thus,  $e^{i\theta}$  lies on the unit circle. In fact, it parameterizes the unit circle (modulo  $2\pi$ ).

An instance of this formula is  $e^{2\pi i} = 1$ . Combining this fact with Corollary 1.3.1 tells us for every integer  $n$ ,  $e^{z+2\pi in} = e^z$ . Thus  $e^z$  is periodic of period  $2\pi i$ . If  $e^z$  assumes a value, it must assume that value infinitely often. In particular, unlike the real situation,  $e^z$  is not invertible.

Another conclusion to be drawn from all this is if  $z = x + iy \in \mathbb{C}$ , then  $|e^z| = |e^{x+iy}| = |e^x e^{iy}| = |e^x| |e^{iy}|$ . Since  $e^x > 0$  and  $|e^{iy}| = 1$  we see

$$|e^z| = e^{\Re z}.$$

In particular,  $e^z$  is never zero. In fact, from Corollary 1.3.1 it follows that  $(e^z)^{-1} = e^{-z}$ . Also notice that, conversely, if  $e^z = 1$  for some  $z$ , then  $e^x e^{iy} = 1 \cdot 1$ . By the uniqueness of the polar decomposition  $e^x = 1$  and  $e^{iy} = 1$ , so  $x = 0$  and  $y$  is congruent to zero modulo  $2\pi$  and therefore  $z$  is congruent to zero modulo  $2\pi i\mathbb{Z}$ .

We now come to DeMoivre's theorem. Namely for  $z, w \in \mathbb{C}^\times$ ,

$$|zw| = |z||w|,$$

which we already know and

$$\arg zw = \arg z + \arg w \pmod{2\pi}.$$

This is because  $z = |z| \frac{z}{|z|} = |z| e^{i\theta}$  and similarly  $w = |w| e^{i\phi}$ . Therefore,

$$zw = |z||w| e^{i\theta} e^{i\phi} = |z||w| e^{i(\theta+\phi)}.$$

By uniqueness of the polar decomposition, both relations follow.

**Exercise 1.3** Let  $z$  be a non-zero complex number and  $n \geq 2$  an integer. Show that  $z$  has  $n$  distinct  $n^{\text{th}}$  roots. Taking  $n \geq 3$ , draw a diagram of these  $n$  distinct roots and show that they lie on a regular polygon with  $n$  sides.

From the calculation above we see that the range of  $e^z$  is all of  $\mathbb{C}^\times$ . For if  $w \neq 0$ , then  $w = |w|e^{i\phi}$  where  $|w| > 0$ . Therefore,  $|w| = e^t$  for some real  $t$ , and so  $w = e^t e^{i\phi} = e^{t+i\phi}$ .

A natural question is, when does  $e^z = e^{z_1}$ ? Since  $(e^{z_1})^{-1} = e^{-z_1}$ , this occurs if and only if  $e^z e^{-z_1} = 1$ . That is, iff  $e^{z-z_1} = 1$ . But then, as we saw,  $z$  is congruent to  $z_1$  modulo  $2\pi i\mathbb{Z}$ . In particular, this shows  $e^z$  is locally invertible because if  $z \in \mathbb{C}$  and we take an open disk  $D$  about  $z$  of radius  $2\pi$ , or even an open horizontal strip of width  $2\pi$ , then on this domain  $e^z$  is 1:1. Later we will construct holomorphic functions which locally invert  $e^z$ .

**Proposition 1.3.4**  $e^z$  is an entire function. Its derivative is itself.

*Proof.* First we note that, just as in calculus, by the change of variable  $h = z - a$  to see  $f$  is holomorphic at  $a \in \mathbb{C}$ , it is equivalent to check that

$$\lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}$$

exists. Here  $e^{a+h} = e^a e^h$  so that

$$\lim_{h \rightarrow 0} \frac{e^{a+h} - e^a}{h} = e^a \lim_{h \rightarrow 0} \frac{e^h - 1}{h}.$$

Thus  $e^z$  is entire if it is holomorphic at 0. We will show

$$\lim_{h \rightarrow 0} \frac{e^h - 1}{h} = 1.$$

Hence  $e^z$  is an entire function and its derivative is again  $e^z$ . Since  $e^h = \sum_{n=0}^{\infty} \frac{h^n}{n!}$ , we get  $\frac{e^h - 1}{h} = 1 + \frac{h}{2!} + \frac{h^2}{3!} \dots$ . Since this series also converges uniformly on compacta and the uniform limit on compacta of continuous functions is continuous (the partial sums being polynomials and therefore continuous), we see that this series gives a continuous function. Therefore its limiting value at  $h = 0$  is gotten by evaluating at 0. Here we get 1.  $\square$

Since for all  $z$ ,  $e^{iz} = \cos z + i \sin z$ , we also get  $e^{-iz} = \cos(-z) + i \sin(-z)$ . On the other hand inspection of the series defining  $\sin$  and  $\cos$  make clear that  $\sin$  is an odd function and  $\cos$  an even function. Thus  $\sin(-z) = -\sin z$  and  $\cos(-z) = \cos z$ . Substituting into the above gives  $e^{-iz} = \cos z - i \sin z$ . Adding to and subtracting this from  $e^{iz} = \cos z + i \sin z$  tells us that

$$\cos z = \frac{1}{2}(e^{iz} + e^{-iz})$$

and

$$\sin z = \frac{1}{2i}(e^{iz} - e^{-iz}).$$

Hence, we can also calculate the derivatives of these functions, getting

$$\frac{d}{dz} \sin z = \frac{1}{2i}(ie^{iz} + ie^{-iz}) = \frac{1}{2}(e^{iz} + e^{-iz}) = \cos z$$

and similarly,  $\frac{d}{dz} \cos z = -\sin z$ . Hence, these are entire functions.

We remark that since  $\frac{d}{dz} \sin z = \cos z$  everywhere, taking  $z = 0$  tells us since  $\sin 0 = 0$  and  $\cos 0 = 1$  that  $\lim_{z \rightarrow 0} \frac{\sin z}{z} = 1$ .

This statement is slightly stronger than the corresponding one for real functions because  $z$  can approach zero in more essentially different ways. Turning to the hyperbolic functions,  $\sinh z$  and  $\cosh z$ , these can also be defined by extending the usual real power series into the complex domain.

$$\sinh z = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!}$$

and

$$\cosh z = \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!}.$$

Similarly to  $\sin$  and  $\cos$ , it is easily checked that  $\sinh z = \frac{e^z - e^{-z}}{2}$  and  $\cosh z = \frac{e^z + e^{-z}}{2}$ . Exactly as above, from this we see  $\frac{d}{dz} \sinh z = \cosh z$  and  $\frac{d}{dz} \cosh z = \sinh z$ . Hence, these are also entire functions.

**Definition 1.3.5** We say a complex valued function defined on a domain is *analytic* if it is represented by a convergent power series. Similarly, if a real valued function of a real variable is represented by a convergent power series we call it a *real analytic* function.

We conclude this section with two remarks. First, all these differentiation results could also have been gotten by termwise differentiation of the appropriate power series. However, proof of this fact would have required further knowledge of power series than we presently have. In Chapter 3 we will see why this is so. Secondly, it is obvious that our process of extending real analytic functions defined on an interval, a half line, or the whole real axis by convergent power series to the corresponding complex domain is quite general and has nothing to do with the particulars of the functions we have been considering in this section (Figure 3.6). So, for example, if  $f(x) = \sum_{n=0}^{\infty} a_n(x-a)^n$  converges for  $|x-a| < r$ , where the  $x$ ,  $a_n$  and  $a$  are real and  $0 < r \leq \infty$ , then we get a complex power series  $f(z) = \sum_{n=0}^{\infty} a_n(z-a)^n$  which converges for all  $|z-a| < r$ , where  $0 < r \leq \infty$ .

**Exercise 1.4** Use the functional equation for  $e^z$  to prove the addition formulas for  $\sin$ ,  $\cos$ ,  $\sinh$  and  $\cosh$ .

## 1.4 The Cauchy-Riemann equations

Let  $f : \Omega \rightarrow \mathbb{C}$  be a map which is holomorphic at a point  $a$ . If  $z = x + iy \in \Omega$ , write  $f(z) = u(x, y) + iv(x, y)$ . We know  $\lim_{z \rightarrow a} \frac{f(z) - f(a)}{z - a} = f'(a)$  exists no matter how  $z \rightarrow a$ . We shall compute this limit in two different ways. First, let  $z \rightarrow a$  by keeping  $x$  constant and  $y$  varying and then by keeping  $y$  constant and  $x$  varying. Let  $a = b + ic$  and  $z = x + ic$ , where  $x \rightarrow b$ . Then,  $z - a = x - b$  so  $z \rightarrow a$  and

$$\frac{f(z) - f(a)}{z - a} = \frac{u(x, c) - u(b, c)}{x - b} + i \frac{v(x, c) - v(b, c)}{x - b}.$$

Taking limits as  $z \rightarrow a$  we get

$$f'(a) = \frac{\partial u}{\partial x}(b, c) + i \frac{\partial v}{\partial x}(b, c).$$