Definition 1.3.5 We say a complex valued function defined on a domain is *analytic* if it is represented by a convergent power series. Similarly, if a real valued function of a real variable is represented by a convergent power series we call it a *real analytic* function.

We conclude this section with two remarks. First, all these differentiation results could also have been gotten by termwise differentiation of the appropriate power series. However, proof of this fact would have required further knowledge of power series than we presently have. In Chapter 3 we will see why this is so. Secondly, it is obvious that our process of extending real analytic functions defined on an interval, a half line, or the whole real axis by convergent power series to the corresponding complex domain is quite general and has nothing to do with the particulars of the functions we have been considering in this section (Figure 3.6). So, for example, if $f(x) = \sum_{n=0}^{\infty} a_n(x-a)^n$ converges for $|x-a| < r$, where the x, a_n and a are real and $0 < r \leq \infty$, then we get a complex power series $f(z) = \sum_{n=0}^{\infty} a_n(z-a)^n$ which converges for all $|z-a| < r$, where $0 < r \leq \infty$.

Exercise 1.4 *Use the functional equation for e^z to prove the addition formulas for* sin, cos, sinh *and* cosh.

1.4 The Cauchy-Riemann equations

Let $f : \Omega \to \mathbb{C}$ be a map which is holomorphic at a point a. If $z = x + iy \in \Omega$, write $f(z) = u(x,y) + iv(x,y)$. We know $\lim_{z\to a} \frac{f(z)-f(a)}{z-a} = f'(a)$ exists no matter how $z \to a$. We shall compute this limit in two different ways. First, let $z \to a$ by keeping x constant and y varying and then by keeping y constant and x varying. Let $a = b + ic$ and $z = x + ic$, where $x \to b$. Then, $z - a = x - b$ so $z \to a$ and

$$\frac{f(z)-f(a)}{z-a} = \frac{u(x,c)-u(b,c)}{x-b} + i\frac{v(x,c)-v(b,c)}{x-b}\,.$$

Taking limits as $z \to a$ we get

$$f'(a) = \frac{\partial u}{\partial x}(b,c) + i\frac{\partial v}{\partial x}(b,c)\,.$$

On the other hand if $z = b + iy$ where $y \to c$, then $z - a = i(y - c)$; so z also approaches a and

$$\frac{f(z) - f(a)}{z - a} = \frac{u(b, y) - u(b, c)}{i(y - c)} + i\frac{v(b, y) - v(b, c)}{i(y - c)}.$$

Since $i^{-1} = -i$, taking limits this time we get

$$f'(a) = -i\frac{\partial u}{\partial y}(b, c) + \frac{\partial v}{\partial y}(b, c).$$

Thus $\frac{\partial u}{\partial x}(a) = \frac{\partial v}{\partial y}(a)$ and $\frac{\partial v}{\partial x}(a) = -\frac{\partial u}{\partial y}(a)$. These are called the Cauchy-Riemann equations at a.

In general, when $f : \Omega \to \mathbb{R}^2$ is a smooth function, the Jacobian matrix $J_f(a)$ has the form

$$\begin{pmatrix} \dfrac{\partial u}{\partial x}(a) & \dfrac{\partial u}{\partial y}(a) \\ \dfrac{\partial v}{\partial x}(a) & \dfrac{\partial v}{\partial y}(a) \end{pmatrix}$$

However, because of the Cauchy-Riemann equations, in the holomorphic case we have:

Corollary 1.4.1 *If f is holomorphic at a, then $J_f(a)$ has the form*

$$\begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}.$$

Corollary 1.4.2 *If f is holomorphic at a, then $|f'(a)|^2 = \det J_f(a)$. In particular, f is regular at a if and only if f is locally invertible there.*

Proof. As we saw, $f'(a) = \frac{\partial u}{\partial x}(a) + i\frac{\partial v}{\partial x}(a)$. Hence $|f'(a)|^2 = \frac{\partial u}{\partial x}(a)^2 + \frac{\partial v}{\partial x}(a)^2$. But by the Cauchy-Riemann equations, this is exactly $\det J_f(a)$. In particular, f is regular iff $J_f(a)$ is invertible. By the real inverse function theorem this is equivalent to f being locally invertible as a real mapping. □

Corollary 1.4.3 *If f is holomorphic on Ω, then f is C^1 and $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$, $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$. Also, $|f'(z)|^2 = \det J_f(z)$. In particular, f is regular on Ω if and only if f is locally invertible everywhere on Ω.*

Corollary 1.4.3 gives us a way of testing whether or not a complex function is holomorphic. For example, conjugation is not holomorphic since $u_x = 1$, but $v_y = -1$.

A real valued function u defined on a domain Ω is called *harmonic* if $u_{xx} + u_{yy} = 0$ everywhere on the domain. The operator $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ is called the Laplacian.

Corollary 1.4.4 *If f is holomorphic on Ω and f is C^2, then its real and imaginary parts are harmonic.*

We remark that in Chapter 3 we shall see f must actually be C^∞.

Proof. By the Cauchy-Riemann equations $u_x = v_y$ and $v_x = -u_y$. Differentiating we get $u_{xx} = v_{xy}$ and $u_{yy} = -v_{yx}$. But since v (and u) are both C^2, we know $v_{yx} = v_{xy}$. Therefore $u_{xx} + u_{yy} = 0$. Similarly, $v_{xx} + v_{yy} = 0$. □

We now turn to the converse of the Cauchy-Riemann equations.

Theorem 1.4.5 *Let $f : \Omega \to \mathbb{C}$ be a map, $f(z) = u(x,y) + iv(x,y)$, where u and v are C^1 functions. If the Cauchy-Riemann equations are satisfied at a point a, then f is holomorphic at a.*

Proof. By the linear approximation theorem for real valued functions of several real variables we see that

$$u(z) - u(a) = u_x(a)(x-b) + u_y(a)(y-c) + \epsilon_1(z)(x-b) + \epsilon_2(z)(y-c)$$

and

$$v(z) - v(a) = v_x(a)(x-b) + v_y(a)(y-c) + \epsilon_3(z)(x-b) + \epsilon_4(z)(y-c),$$

where, as above, $a = (b,c)$ and each ϵ_j tends to zero as $z \to a$. Therefore,

$$\begin{aligned} f(z) - f(a) = u_x(a)(x-b) &+ u_y(a)(y-c) + \epsilon_1(z)(x-b) \\ &+ \epsilon_2(z)(y-c) + i(v_x(a)(x-b) + v_y(a)(y-c)) \\ &+ \epsilon_3(z)(x-b) + \epsilon_4(z)(y-c)) \,. \end{aligned}$$

That is,

$$\frac{f(z)-f(a)}{z-a} = (u_x(a)+iv_x(a))\frac{x-b}{z-a} - i(u_y(a)+iv_y(a))i\frac{y-c}{z-a}$$
$$+ (\epsilon_1(z)+i\epsilon_3(z))\frac{x-b}{z-a} + (\epsilon_2(z)+i\epsilon_4(z))\frac{y-c}{z-a}\,.$$

By the Cauchy-Riemann equations,

$$\frac{f(z)-f(a)}{z-a} = (u_x(a)+iv_x(a))\frac{z-a}{z-a} + (\epsilon_1(z)+i\epsilon_3(z))\frac{x-b}{z-a}$$
$$+ (\epsilon_2(z)+i\epsilon_4(z))\frac{y-c}{z-a}\,.$$

Taking into account that each ϵ_j tends to zero and $|\frac{x-b}{z-a}|$ and $|\frac{y-c}{z-a}| \leq 1$, we see that

$$\lim_{z\to a}\frac{f(z)-f(a)}{z-a} = u_x(a)+iv_x(a).$$

□

Corollary 1.4.6 *Let $f:\Omega\to\mathbb{C}$ be a map, where $\Re f$ and $\Im f$ are C^1 functions. If f satisfies the Cauchy-Riemann equations on Ω, then f is holomorphic.*

Corollary 1.4.7 *Let $f:\Omega\to\mathbb{C}$ be a regular holomorphic map. Then local inverses of f are holomorphic.*

Proof. Since f is regular, we know f is locally invertible by a smooth real function. The only question is whether the local inverse is holomorphic. But we know that for each $z\in\Omega$, $J_f(z)$ has the form

$$\begin{pmatrix}\alpha & \beta\\ -\beta & \alpha\end{pmatrix}.$$

As is well known from linear algebra, $J_f(z)^{-1}$ has the form

$$\begin{pmatrix}\dfrac{\alpha}{\alpha^2+\beta^2} & -\dfrac{\beta}{\alpha^2+\beta^2}\\[2ex] \dfrac{\beta}{\alpha^2+\beta^2} & \dfrac{\alpha}{\alpha^2+\beta^2}\end{pmatrix}.$$

Since by the chain rule the tangent mapping of f^{-1} is $J_f(z)^{-1}$, it follows from the result immediately above that f^{-1} is holomorphic. $\square$

Because even for real functions a local diffeomorphism is open, we get the following:

Corollary 1.4.8 *Let $f : \Omega \to \mathbb{C}$ be a regular holomorphic map. Then f is an open map. In particular, $f(\Omega)$ is open.*

Exercise 1.5 *Let f be a holomorphic function in a domain, Ω and Δ be the Laplacian. Show that*

$$\Delta(|f(z)|^2) = 4|f'(z)|^2.$$

Hence if $\{f_1, \ldots, f_k\}$ are holomorphic functions and $\sum_{j=1}^{k} |f_j(z)|^2$ is harmonic, each f_j is constant.

1.5 Some elementary differential equations

From this point on we shall require that domains Ω be connected open sets and we shall assume this is understood without it being explicitly mentioned in the sequel.

We observe that if Ω is merely open, then each connected component is a domain. Also notice that if γ is a continuous curve in an open set that starts out at say a, then it remains in the component of Ω containing a throughout its trajectory. Thus, there is no harm that can come from this restriction. The reader is invited to prove these statements.

Lemma 1.5.1 *Let $f : D \to \mathbb{C}$ be a holomorphic map, where $D(a, r)$ is the disk centered at a of radius $r > 0$. If $f' \equiv 0$, then f is constant.*

Proof. Let $z \in D$. By convexity, the line segment $\gamma(t) = ta + (1-t)z$ lies in D for $0 \leq t \leq 1$. This is because $|\gamma(t) - a| = |(t-1)a + (1-t)z| = |1-t||z-a| \leq |z-a| < r$. Now as we saw, $\frac{d}{dt} f(\gamma(t)) = f'(\gamma(t))\gamma'(t)$. Since $f' \equiv 0$ we see $\frac{d}{dt} f(\gamma(t)) = 0$ for all t. Applying the mean value theorem to the components f_j of $f(\gamma(t))$, we conclude $f_j(a) - f_j(z) = f_j(\gamma(1)) - f_j(\gamma(0)) = 0$ for each j. Hence $f(z) = f(a)$, a constant. $\square$