Since by the chain rule the tangent mapping of f^{-1} is $J_f(z)^{-1}$, it follows from the result immediately above that f^{-1} is holomorphic. □

Because even for real functions a local diffeomorphism is open, we get the following:

Corollary 1.4.8 *Let $f : \Omega \to \mathbb{C}$ be a regular holomorphic map. Then f is an open map. In particular, $f(\Omega)$ is open.*

Exercise 1.5 *Let f be a holomorphic function in a domain, Ω and Δ be the Laplacian. Show that*

$$\Delta(|f(z)|^2) = 4|f'(z)|^2.$$

Hence if $\{f_1, \ldots, f_k\}$ are holomorphic functions and $\sum_{j=1}^k |f_j(z)|^2$ is harmonic, each f_j is constant.

1.5 Some elementary differential equations

From this point on we shall require that domains Ω be connected open sets and we shall assume this is understood without it being explicitly mentioned in the sequel.

We observe that if Ω is merely open, then each connected component is a domain. Also notice that if γ is a continuous curve in an open set that starts out at say a, then it remains in the component of Ω containing a throughout its trajectory. Thus, there is no harm that can come from this restriction. The reader is invited to prove these statements.

Lemma 1.5.1 *Let $f : D \to \mathbb{C}$ be a holomorphic map, where $D(a, r)$ is the disk centered at a of radius $r > 0$. If $f' \equiv 0$, then f is constant.*

Proof. Let $z \in D$. By convexity, the line segment $\gamma(t) = ta + (1-t)z$ lies in D for $0 \le t \le 1$. This is because $|\gamma(t) - a| = |(t-1)a + (1-t)z| = |1-t||z-a| \le |z-a| < r$. Now as we saw, $\frac{d}{dt} f(\gamma(t)) = f'(\gamma(t))\gamma'(t)$. Since $f' \equiv 0$ we see $\frac{d}{dt} f(\gamma(t)) = 0$ for all t. Applying the mean value theorem to the components f_j of $f(\gamma(t))$, we conclude $f_j(a) - f_j(z) = f_j(\gamma(1)) - f_j(\gamma(0)) = 0$ for each j. Hence $f(z) = f(a)$, a constant. □

Proposition 1.5.2 *Let $f : \Omega \to \mathbb{C}$ be a holomorphic map. If $f' \equiv 0$, then f is constant.*

Proof. Let z_0 be fixed and z a variable point in Ω. Since Ω is connected and open, it is arcwise connected. That is, these two points can be joined by a continuous arc $\gamma : [a, b] \to \Omega$. By compactness of $[a, b]$ and continuity of γ we know the trajectory, $\gamma([a, b])$, is compact. At each point $\gamma(t)$ of the trajectory choose a small disk D_t centered at $\gamma(t)$ and contained in Ω. This covering of the trajectory has a finite subcover, say $D_{t_1}, \ldots, D_{t_n}$. By Lemma 1.5.1, f is constant on each of these disks. But each successive pair of these disks must overlap or together they could not cover the trajectory. Now the union of two connected sets with a point in common must itself be connected so $D_{t_j} \cup D_{t_{j+1}}$ is connected. Hence, so is its image under f. Therefore, f is constant on the union. It follows that f is constant on the trajectory of γ and in particular, $f(z_0) = f(\gamma(a)) = f(\gamma(b)) = f(z)$. □

Exercise 1.6

1. *Show if the domain were not connected this result would be false.*
2. *Prove that a connected space such as Ω is arcwise connected.*

Corollary 1.5.3 *Let f, $g : \Omega \to \mathbb{C}$ be holomorphic maps. If $f' = g'$, then $f - g$ is constant.*

This is clear since $(f - g)' = f' - g' = 0$. Therefore $f - g$ is constant.

Corollary 1.5.4 *f is a polynomial if and only if $f^{(n)} = 0$ for some n.*

Proof. Here we take $\Omega = \mathbb{C}$. If f is a polynomial of degree $n - 1$, then as was mentioned earlier, $f^{(n)} = 0$. Conversely, suppose $f^{(n)} = 0$ for some n. Since $f^{(n-1)'} = 0$, $f^{(n-1)}$ is constant, say c. Let $g = f^{(n-2)}$ and $h(z) = cz$. Since the derivatives of g and h are equal, $g - h = b$, a constant, so $f^{(n-2)} = cz + b$. Continuing in this way by induction, we see that f is a polynomial of degree at most $n - 1$. □

Corollary 1.5.5 *Let $f : \Omega \to \mathbb{C}$ be a holomorphic map and γ be a smooth curve in Ω. If $f(\Omega)$ is contained in the trajectory of γ, then f*

must be constant. In particular, if f takes only real, or purely imaginary values, or if $|f|$ is constant, then f must itself be constant.

Proof. Since the image of the domain is a curve and so of lower dimension, $J_f(z)$ must be singular at every point. As we know, $\det J_f(z) = |f'(z)|^2$. Hence $f'(z) = 0$ for all $z \in \Omega$. By Proposition 1.5.2, f must be constant. □

Later we shall see that this result also follows from the area theorem of Section 3.4.

Corollary 1.5.6 *Let $f : \Omega \to \mathbb{C}$ be a holomorphic map. Then the following are equivalent.*

(i) *f is constant.*
(ii) *$\Re(f)$ and $\Im(f)$ are constant.*
(iii) *$f' = 0$.*
(iv) *$|f|$ is constant.*

We already see that (i), (ii) and (iii) are all equivalent and they imply (iv). By Corollary 1.5.5, (iv) also implies (i).

1.6 Conformality

The definition of conformality actually has little to do with dimension 2 or with complex analysis for that matter. Let Ω be a domain in $\mathbb{R}^n$ and f a real C^∞ map $\Omega \to \mathbb{R}^n$ which is locally invertible.

Definition 1.6.1 We say f is *conformal at a point $a \in \Omega$* if for each pair of smooth curves γ and δ in Ω passing through a, say at $t = t_0$, the angle between their tangent vectors $\gamma'(t_0)$ and $\delta'(t_0)$ equals the angle between the tangent vectors $(f \circ \gamma)'(t_0)$ and $(f \circ \delta)'(t_0)$, their images under f. If f is conformal at every point in Ω, we just say f is *conformal.*

Our purpose here is to produce a large number of conformal mappings in $\mathbb{R}^2 = \mathbb{C}$ and actually to characterize conformal maps in $\mathbb{R}^2$. In this connection a useful application of the chain rule is the following: