must be constant. In particular, if f takes only real, or purely imaginary values, or if $|f|$ is constant, then f must itself be constant.

Proof. Since the image of the domain is a curve and so of lower dimension, $J_f(z)$ must be singular at every point. As we know, $\det J_f(z) = |f'(z)|^2$. Hence $f'(z) = 0$ for all $z \in \Omega$. By Proposition 1.5.2, f must be constant. □

Later we shall see that this result also follows from the area theorem of Section 3.4.

Corollary 1.5.6 *Let $f : \Omega \to \mathbb{C}$ be a holomorphic map. Then the following are equivalent.*

(i) *f is constant.*
(ii) *$\Re(f)$ and $\Im(f)$ are constant.*
(iii) *$f' = 0$.*
(iv) *$|f|$ is constant.*

We already see that (i), (ii) and (iii) are all equivalent and they imply (iv). By Corollary 1.5.5, (iv) also implies (i).

1.6 Conformality

The definition of conformality actually has little to do with dimension 2 or with complex analysis for that matter. Let Ω be a domain in $\mathbb{R}^n$ and f a real C^∞ map $\Omega \to \mathbb{R}^n$ which is locally invertible.

Definition 1.6.1 We say f is *conformal at a point* $a \in \Omega$ if for each pair of smooth curves γ and δ in Ω passing through a, say at $t = t_0$, the angle between their tangent vectors $\gamma'(t_0)$ and $\delta'(t_0)$ equals the angle between the tangent vectors $(f \circ \gamma)'(t_0)$ and $(f \circ \delta)'(t_0)$, their images under f. If f is conformal at every point in Ω, we just say f is *conformal.*

Our purpose here is to produce a large number of conformal mappings in $\mathbb{R}^2 = \mathbb{C}$ and actually to characterize conformal maps in $\mathbb{R}^2$. In this connection a useful application of the chain rule is the following:

Corollary 1.6.2 *Let $f : \Omega \to \Omega'$ be a holomorphic function between domains and γ be a smooth curve in Ω. Then $f \circ \gamma$ is a smooth curve in Ω' and $f \circ \gamma'(t_0)$, its tangent vector at t_0, is $f'(\gamma(t_0))\gamma'(t_0)$.*

The main result here is the following:

Theorem 1.6.3 *Let $f : \Omega \to \mathbb{C}$ be a regular holomorphic map. Then f is conformal. Conversely, if $f : \Omega \to \mathbb{C}$ is conformal, then f is holomorphic and regular on Ω.*

Proof. Suppose f is holomorphic at a, and γ and δ are smooth curves in Ω passing through a at t_0. By the chain rule,

$$(f \circ \gamma)'(t_0) = f'(\gamma(t_0))\gamma'(t_0)$$

and

$$(f \circ \delta)'(t_0) = f'(\delta(t_0))\delta'(t_0)\,.$$

Hence,

$$\arg(f \circ \gamma)'(t_0) = \arg(f'(\gamma(t_0))) + \arg(\gamma'(t_0))$$

and similarly

$$\arg(f \circ \delta)'(t_0) = \arg(f'(\delta(t_0))) + \arg(\delta'(t_0))\,.$$

Since these curves both pass through a at t_0 it follows that

$$\arg(f'(\gamma(t_0))) = \arg(f'(\delta(t_0)))\,.$$

So we get

$$\arg(f \circ \gamma)'(t_0) - \arg(f \circ \delta)'(t_0) = \arg(\gamma'(t_0)) - \arg(\delta'(t_0))\,.$$

This means the angle between $\gamma'(t_0)$ and $\delta'(t_0)$ equals the angle between $(f \circ \gamma)'(t_0)$ and $(f \circ \delta)'(t_0)$. Since the smooth curves are arbitrary, as is $a \in \Omega$, f is conformal.

Conversely, suppose f is conformal. Linear algebra tells us that the tangent mapping $J_f(a)$ at a is given by $J_f(a) = \lambda R_\theta$, where $\lambda > 0$ and R_θ is rotation by angle θ. Therefore, the Cauchy-Riemann equations are satisfied at each point. Since f is a C^∞ map, by Corollary 1.4.6 it is holomorphic at a. Because f is locally invertible, by Corollary 1.2.2, f is also regular at a. □

Exercise 1.7 *Show that, as above, an angle preserving linear map T of the plane (or indeed of $\mathbb{R}^n$) must be of the form a stretch followed by a rotation. (Notice that a stretch commutes with any other linear transformation).*

(Suggestion: First show that $\det T > 0$. Then consider $S = \frac{1}{(\det T)^{\frac{1}{n}}} T$. Show that S also preserves angles. Prove $\det S = 1$ and therefore S also preserves area (volume). Because S preserves both angles and area it must preserve congruences and since $\det S = 1$, it must actually be a rotation. Hence T is a positive multiple of a rotation. These are the infinitesimal conformal maps in $\mathbb{R}^n$.)

1.7 Power series

In this section we very briefly give the main elementary results on power series, usually referred to as Abel's lemma. To do so we first deal with the geometric series.

Proposition 1.7.1 *For $|z| < 1$ and $b \in \mathbb{C}$ the series*

$$\sum_{n=0}^{\infty} bz^n = \frac{b}{1-z}.$$

Proof. We may clearly assume $b = 1$. Then $z \sum_{n=0}^{N} z^n = \sum_{n=1}^{N+1} z^n$. Therefore the difference $\sum_{n=0}^{N} z^n - z \sum_{n=0}^{N} z^n = 1 - z^{N+1}$. Since $|z| < 1$, it follows easily that z^n tends to zero and so $1 - z^{N+1}$ tends to 1. This means that $(1-z) \sum_{n=0}^{N} z^n$ tends to 1 and since $z \neq 1$ that $\sum_{n=0}^{N} z^n$ tends to $\frac{1}{1-z}$. □

Definition 1.7.2 In all that follows, if f is a bounded complex valued function on a space X we shall denote its sup norm over X by $\| f \|_X$.

Proposition 1.7.3 *For $n \geq 0$ let M_n be a sequence of positive numbers such that $\sum_{n=0}^{\infty} M_n$ is convergent. Suppose $\sum_{n=0}^{\infty} f_n(x)$ is a series of complex valued functions defined on a metric space X and for all n, $\| f_n \| \leq M_n$. Then $\sum_{n=0}^{\infty} f_n(x)$ is uniformly convergent on X.*