Exercise 1.7 *Show that, as above, an angle preserving linear map* T *of the plane (or indeed of* $\mathbb{R}^n$*) must be of the form a stretch followed by a rotation. (Notice that a stretch commutes with any other linear transformation).*

(Suggestion: First show that $\det T > 0$. Then consider $S = \frac{1}{(\det T)^{\frac{1}{n}}}T$. Show that S also preserves angles. Prove $\det S = 1$ and therefore S also preserves area (volume). Because S preserves both angles and area it must preserve congruences and since $\det S = 1$, it must actually be a rotation. Hence T is a positive multiple of a rotation. These are the infinitesimal conformal maps in $\mathbb{R}^n$.)

1.7 Power series

In this section we very briefly give the main elementary results on power series, usually referred to as Abel's lemma. To do so we first deal with the geometric series.

Proposition 1.7.1 *For* $|z| < 1$ *and* $b \in \mathbb{C}$ *the series*

$$\sum_{n=0}^{\infty} bz^n = \frac{b}{1-z}\,.$$

Proof. We may clearly assume $b = 1$. Then $z\sum_{n=0}^{N} z^n = \sum_{n=1}^{N+1} z^n$. Therefore the difference $\sum_{n=0}^{N} z^n - z\sum_{n=0}^{N} z^n = 1 - z^{N+1}$. Since $|z| < 1$, it follows easily that z^n tends to zero and so $1 - z^{N+1}$ tends to 1. This means that $(1-z)\sum_{n=0}^{N} z^n$ tends to 1 and since $z \neq 1$ that $\sum_{n=0}^{N} z^n$ tends to $\frac{1}{1-z}$. $\square$

Definition 1.7.2 In all that follows, if f is a bounded complex valued function on a space X we shall denote its sup norm over X by $\| f \|_X$.

Proposition 1.7.3 *For* $n \geq 0$ *let* M_n *be a sequence of positive numbers such that* $\sum_{n=0}^{\infty} M_n$ *is convergent. Suppose* $\sum_{n=0}^{\infty} f_n(x)$ *is a series of complex valued functions defined on a metric space* X *and for all* n, $\| f_n \| \leq M_n$. *Then* $\sum_{n=0}^{\infty} f_n(x)$ *is uniformly convergent on* X.

Proof. Since $\sum_{n=0}^{\infty} f_n(x) \leq \sum_{n=0}^{\infty} M_n$ and the latter is convergent evidently $\sum_{n=0}^{\infty} f_n(x)$ is convergent to $f(x)$ at every point $x \in X$. To see that the convergence is uniform we apply the Cauchy criterion. For all x,

$$\left|f(x) - \sum_{n=0}^{N} f_n(x)\right| = \left|\sum_{n=N+1}^{\infty} f_n(x)\right| \leq \sum_{n=N+1}^{\infty} |f_n(x)| \leq \sum_{n=N+1}^{\infty} M_n \,.$$

But since $\sum_{n=0}^{\infty} M_n$ is convergent its tail $\sum_{n=N+1}^{\infty} M_n$ tends to zero as $N \to \infty$. Hence if $\epsilon > 0$ is given, then for all $x \in X$. $|f(x) - \sum_{n=0}^{N} f_n(x)| < \epsilon$. Hence for N large enough $\| f - \sum_{n=0}^{N} f_n \|_X \leq \epsilon$. ☐

Lemma 1.7.4 *If the power series $\sum_{n=0}^{\infty} a_n(z_0 - a)^n$ is convergent for some z_0, then it converges absolutely for all $|z - a| < |z_0 - a|$.*

Proof. Because $\sum_{n=0}^{\infty} a_n(z_0 - a)^n$ is convergent its nth term tends to zero. So, given $B > 0$, eventually $|a_n||z_0 - a|^n < B$. By taking B larger we can absorb the finite number of earlier terms and get $|a_n||z_0 - a|^n < B$ for all n. Now

$$\sum_{n=0}^{\infty} |a_n|(|z - a|)^n = \sum_{n=0}^{\infty} |a_n| \frac{(|z - a|)^n}{|z_0 - a|^n} |z_0 - a|^n \leq B \sum_{n=0}^{\infty} \left|\frac{z - a}{z_0 - a}\right|^n .$$

But since $|z - a| < |z_0 - a|$, $|\frac{z-a}{z_0-a}| < 1$ and so we have a convergent geometric series. This means what is dominated by it must also converge. ☐

Theorem 1.7.5 *Let $f(z) = \sum_{n=0}^{\infty} a_n(z - a)^n$ be a convergent power series for $|z - a| < r_0$, where $0 < r_0 \leq \infty$. Then this series is absolutely and uniformly convergent for $|z - a| \leq r$, where $r < r_0$.*

Proof. Since $r < r_0$, the absolute convergence for $|z - a| \leq r$ follows from Abel's lemma. Therefore, $\sum_{n=0}^{\infty} |a_n| r^n < \infty$. Taking $M_n = |a_n| r^n$ for all $n \geq 0$ we see that $|a_n(z - a)^n| \leq M_n$ for all n and all $|z - a| \leq r$. Since $\sum_{n=0}^{\infty} M_n$ converges, it follows from the Weierstrass M-test that $\sum_{n=0}^{\infty} a_n(z - a)^n$ converges uniformly for $|z - a| \leq r$. ☐