Chapter 1

Chaotic quantization of field theories

We will first give a short introduction to the stochastic quantization method, and then generalize to noise fields with a dynamical origin. This extended quantization method is called 'chaotic quantization' since the noise field is generated by a deterministic chaotic dynamics on a very small scale.

1.1 Stochastic quantization

Stochastic quantization [Parisi et al. (1981); Damgaard et al. (1987)] is an elegant method to quantize a classical field ϕ. It is an alternative to other quantization methods with many advantages. By quantization, we actually mean second quantization. That means, our 'classical' field equation is already a quantum mechanical expression given, for example, by the Klein-Gordon or Dirac equation. Conceptually, stochastic quantization is a very simple and straightforward method. The basic idea is to add uncorrelated Gaussian white noise to the classical field equation, obtaining a stochastic differential equation. Quantum mechanical expectations can then be calculated as expectations with respect the various realizations of the noise.

A field theory is usually determined by some action functional $S[\phi]$. The field ϕ is a function of the space-time coordinates and may, in general, be vector-valued. The classical field equation can be written as

$$\frac{\delta S}{\delta \phi} = 0, \tag{1.1}$$

meaning that the action has an extremum.

In the Parisi-Wu appraoch of stochastic quantization one proceeds from the classical field equation to a quantized theory by means of the following Langevin equation:

$$\frac{\partial}{\partial t}\phi(x,t) = -\frac{\delta S}{\delta \phi}(x,t) + L(x,t) \tag{1.2}$$

Here $x = (x^1, x^2, x^3, x^4) = x^\mu$ is a point in Euclidean space-time, t denotes a fictitious time variable (different from the physical time x^4), and $L(x,t)$ denotes spatio-temporal Gaussian white noise. It satisfies

$$\begin{aligned} \langle L(x,t)\rangle &= 0 \\ \langle L(x,t)L(x',t')\rangle &= 2\delta(x-x')\delta(t-t'), \end{aligned} \tag{1.3}$$

where $\langle \cdots \rangle$ denotes the expectation. All even higher-order correlation functions of L can be expressed by appropriate products of two-point functions, whereas the odd correlation functions vanish.

The fictitious time t is just introduced as an artificial fifth coordinate. It is different from the physical time. What is of physical relevance is the stationary solution of the Langevin equation in the limit $t \to \infty$. It is the quantized field, a stochastic process. All quantum mechanical expectations of the field $\phi(x)$ can be calculated as expectations with respect to the realizations of the Langevin process in the limit $t \to \infty$.

As a standard example, we may consider a self-interacting scalar Klein-Gordon field in Euclidean space-time. For a ϕ^4-theory the action is

$$S[\phi] = \int d^4x \left(\frac{1}{2}\partial_\mu \phi \partial^\mu \phi - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4}\phi^4 \right). \tag{1.4}$$

m is a mass parameter and λ is a quartic coupling, describing the strength of self-interaction of the field. We have chosen natural units where $\hbar = c = 1$. The classical field equation is

$$(-\partial^2 + m^2)\phi(x) + \lambda\phi^3(x,t) = 0. \tag{1.5}$$

For $\lambda = 0$ this is the Klein-Gordon equation, describing a free scalar field of mass m in Euclidian metric. After quantization, we obtain the Langevin equation

$$\frac{\partial}{\partial t}\phi(x,t) = (\partial^2 - m^2)\phi(x,t) - \lambda\phi^3(x,t) + L(x,t). \tag{1.6}$$

Of course, many other, much more complicated examples of actions and field equations are possible [Ramond (1981); Itzykson et al. (1980)]. Stochastic quantization can also be done for fermionic fields, gauge fields, supersymmetric fields, gravitational fields, basically everything that is of interest (see [Damgaard et al. (1988)] for an overview). In realistic models the fields interact. For example, the classical field equations of quantum electrodynamics describe fermionic fields (electrons/positrons) and bosonic fields (photons) in various spin states, which interact and lead to 8-dimensional classical field equations (see any textbook on QED). To each component of the classical field equation one has to add noise. The action of the entire standard model of elementary particle physics, describing electroweak and strong interactions, contains a large number of interacting fermionic and bosonic fields (see, e.g., [Kane (1987)]). These are essentially the three quark and lepton families and the gauge and Higgs bosons in various interaction states. For each component of the classical field equation a corresponding noise field $L(x,t)$ is necessary for quantization.

1.2 Dynamical generation of the noise

We may now ask the following. If noise fields $L(x,t)$ are so important for the quantization of classical fields, where do these noise fields ultimately come from? Could they have a dynamical orign? Could these noise fields just be the result of some other field theory, describing a rapidly evolving field on a much smaller space-time scale, which looks like Gaussian white noise on a larger scale? Indeed, this is the general idea that will be worked out in this book.

For simplicity, let us first start with 0-dimensional field theories, i.e. theories where the field ϕ has no dependence on x. Then also the noise field L has no dependence on x — it only depends on t. Formally, Gaussian white noise $L(t)$ is the derivative of the Wiener process $W(t)$. The Wiener process can be regarded as a rescaled random walk. We may write

$$W(t) = \lim_{\tau \to 0} \tau^{1/2} \sigma^{-1} \sum_{n=0}^{\lfloor t/\tau \rfloor} \chi_n. \tag{1.7}$$

Here the χ_n are *independent* random variables with identical probability distribution and variance σ^2, τ is a small time scale parameter, and $\lfloor \ \rfloor$