Of course, many other, much more complicated examples of actions and field equations are possible [Ramond (1981); Itzykson et al. (1980)]. Stochastic quantization can also be done for fermionic fields, gauge fields, supersymmetric fields, gravitational fields, basically everything that is of interest (see [Damgaard et al. (1988)] for an overview). In realistic models the fields interact. For example, the classical field equations of quantum electrodynamics describe fermionic fields (electrons/positrons) and bosonic fields (photons) in various spin states, which interact and lead to 8-dimensional classical field equations (see any textbook on QED). To each component of the classical field equation one has to add noise. The action of the entire standard model of elementary particle physics, describing electroweak and strong interactions, contains a large number of interacting fermionic and bosonic fields (see, e.g., [Kane (1987)]). These are essentially the three quark and lepton families and the gauge and Higgs bosons in various interaction states. For each component of the classical field equation a corresponding noise field $L(x,t)$ is necessary for quantization.

1.2 Dynamical generation of the noise

We may now ask the following. If noise fields $L(x,t)$ are so important for the quantization of classical fields, where do these noise fields ultimately come from? Could they have a dynamical orign? Could these noise fields just be the result of some other field theory, describing a rapidly evolving field on a much smaller space-time scale, which looks like Gaussian white noise on a larger scale? Indeed, this is the general idea that will be worked out in this book.

For simplicity, let us first start with 0-dimensional field theories, i.e. theories where the field ϕ has no dependence on x. Then also the noise field L has no dependence on x — it only depends on t. Formally, Gaussian white noise $L(t)$ is the derivative of the Wiener process $W(t)$. The Wiener process can be regarded as a rescaled random walk. We may write

$$W(t) = \lim_{\tau \to 0} \tau^{1/2} \sigma^{-1} \sum_{n=0}^{\lfloor t/\tau \rfloor} \chi_n. \tag{1.7}$$

Here the χ_n are *independent* random variables with identical probability distribution and variance σ^2, τ is a small time scale parameter, and $\lfloor \ \rfloor$

denotes the integer part. Formally

$$L(t) = \frac{\partial W}{\partial t} = \lim_{\tau \to 0} \tau^{1/2} \sigma^{-1} \sum_{n=0}^{\infty} \chi_n \delta(t - n\tau). \tag{1.8}$$

If independent random variables χ_n are added and rescaled, the convergence to a Gaussian process is a consequence of the well-known Central Limit Theorem. The probability distribution of the χ_n is irrelevant (as long as the second moment exists), in eq. (1.7) we always get the same (universal) Gaussian process, the Wiener process $W(t)$, also called Brownian motion [van Kampen (1981)].

Now the assumption of independence of the χ_n is not necessary. In fact, there are more general versions of the Central Limit Theorem that apply to weakly dependent random variables [Billingsley (1968); Beck (1990b); Beck (1996); Chew et al. (2002); Shimizu (1990); Chernov (1995); Zygmund (1959); Salem et al. (1947); Spohn (1980)]. In particular, these 'random' variables can be the iterates of a chaotic mapping. The iterates of a mixing chaotic map are not independent, but just asymptotically independent for large time differences (this is just the meaning of the mixing property). So let us replace the independent χ_n by the iterates η_n of, for example, a deterministic 1-dimensional chaotic mapping T on some interval. We have, in a deterministic way,

$$\eta_{n+1} = T(\eta_n) \tag{1.9}$$

for some initial value η_0. Regarding the initial value η_0 as a random variable with some arbitrary (smooth) probability distribution, it can be rigorously proved that there is convergence to the Wiener process, respectively Gaussian white noise, provided T possesses the so-called φ-mixing property[Billingsley (1968); Beck et al. (1987)]. We have

$$W(t) = \lim_{\tau \to 0} W_\tau(t) \tag{1.10}$$

$$L(t) = \lim_{\tau \to 0} L_\tau(t), \tag{1.11}$$

where

$$W_\tau(t) = \lim_{\tau \to 0} \tau^{1/2} \sigma^{-1} \sum_{n=0}^{\lfloor t/\tau \rfloor} \eta_n \tag{1.12}$$

$$L_\tau(t) \quad = \quad \lim_{\tau \to 0} \tau^{1/2} \sigma^{-1} \sum_{n=0}^{\infty} \eta_n \delta(t - n\tau), \tag{1.13}$$

the η_n being given by eq. (1.9). The variance σ^2 is given by

$$\sigma^2 = \int d\eta_0 \rho(\eta_0) \eta_0^2 + 2 \sum_{n=1}^{\infty} \int d\eta_0 \rho(\eta_0) \eta_0 \eta_n, \tag{1.14}$$

where ρ denotes the natural invariant density of the map T, i.e. the probability density of iterates for generic initial values. Once again let us remark that the convergence to the Wiener process is a non-trivial property, and the proof is not simple. Namely, the iterates of a dynamical system T are *dependent* random variables. Even for simple standard examples of chaotic maps such as Tchebyscheff maps there are complicated higher-order correlations (see section 1.7). Nevertheless, for φ-mixing systems one can rigorously estimate that the dependence is weak enough to guarantee the convergence to the Wiener process. This chaotically generated Wiener process can then be used to simulate path integrals [Roepstorff (1994); Schulman (1981)] in various ways [Beck (1991b); Beck (1995a); Beck (1995c)].

The φ-mixing property is a slightly stronger property than the usual mixing property. Examples of φ-mixing maps are all maps (semi-)conjugated to a Bernoulli shift by a smooth function, such as the 2nd-order Tchebyscheff polynomial $T(\eta) = 2\eta^2 - 1$ ($\sigma^2 = 1/2$), the 3rd-order Tchebyscheff polynomial $T(\eta) = 4\eta^3 - 3\eta$ ($\sigma^2 = 1/2$), or generally the N-th order Tchebyscheff polynomial with $N \geq 2$. Other examples are the binary shift map $T(\eta) = 2\eta \; mod \; 1$ ($\sigma^2 = 1/4$) with substracted mean, or generally N-ary shift maps. Another φ-mixing map is the continued fraction map $T(\eta) = 1/\eta \bmod 1$ with substracted mean.

For our application in quantum field theory we still have to generalize to higher dimensions. To obtain a spatio-temporal noise field on a d-dimensional lattice, we may generalize eq. (1.13) as follows:

$$L_{l,\tau}(x,t) = l^{d/2} \tau^{1/2} \tilde{\sigma}^{-1} \sum_i \sum_{n=0}^{\infty} \eta_n^i \delta(x - il) \delta(t - n\tau) \tag{1.15}$$

Here x is a d-dimensional vector, l is a small spatial lattice constant, $\tilde{\sigma}$ is a suitable variance parameter, and $i = (i_1, \ldots, i_d)$ is a d-dimensional vector with integer entries that labels the lattice sites. The η_n^i are chaotic variables living at lattice sites i, generated by deterministic maps.

We know that the complicated process $L_{l,\tau}$ reduces to spatio-temporal Gaussian white noise in the scaling limit $l \to 0, \tau \to 0$, t finite, if the η_n^i evolve in an uncoupled way under a φ-mixing dynamics T, since then the η_n^i are spatially independent and temporally φ-mixing. It is straightforward to conjecture that if we spatially couple the maps T, we still have convergence to spatio-temporal Gaussian white noise provided the coupling is small enough. For this one essentially has to show that the mixing property survives a small spatial coupling (see [Baladi et al. (1998); Baladi et al. (2001)] for some related work).

From a rigorous mathematical point of view, spatio-temporal Gaussian white noise in dimensions $d \geq 2$ is a somewhat singular object. Nevertheless, it is a very useful tool for the quantization of field theories. In any case, it is clear that if we look onto a weakly coupled spatio-temporal chaotic system from 'far away', meaning that τ and l are sufficiently small, it will look similar to spatio-temporal Gaussian white noise, since correlations fall off rapidly.

1.3 The free Klein-Gordon field with chaotic noise

As a simple example, let us consider a free Klein-Gordon field with chaotic noise

$$\frac{\partial}{\partial t}\phi(x,t) = (\partial^2 - m^2)\phi(x,t) + L_{l,\tau}(x,t) \tag{1.16}$$

$(m^2 > 0)$. Here $L_{l,\tau}(x,t)$ is some complicated spatio-temporal process where the η_n^i evolve according to some (so far arbitrary) chaotic dynamics. Choosing (for example) $d = 4$, the solution of eq. (1.16) can be written as

$$\phi(x,t) = \int_0^t ds \int d^4y \; G(x-y, t-s) L_{l,\tau}(y,s), \tag{1.17}$$

where G denotes the retarded Greens function, i.e. the solution of

$$\frac{\partial}{\partial t}G(x,t) = (\partial^2 - m^2)G(x,t) + \delta(x)\delta(t). \tag{1.18}$$