with integer entries that labels the lattice sites. The η_n^i are chaotic variables living at lattice sites i, generated by deterministic maps.

We know that the complicated process $L_{l,\tau}$ reduces to spatio-temporal Gaussian white noise in the scaling limit $l \to 0, \tau \to 0$, t finite, if the η_n^i evolve in an uncoupled way under a φ-mixing dynamics T, since then the η_n^i are spatially independent and temporally φ-mixing. It is straightforward to conjecture that if we spatially couple the maps T, we still have convergence to spatio-temporal Gaussian white noise provided the coupling is small enough. For this one essentially has to show that the mixing property survives a small spatial coupling (see [Baladi et al. (1998); Baladi et al. (2001)] for some related work).

From a rigorous mathematical point of view, spatio-temporal Gaussian white noise in dimensions $d \geq 2$ is a somewhat singular object. Nevertheless, it is a very useful tool for the quantization of field theories. In any case, it is clear that if we look onto a weakly coupled spatio-temporal chaotic system from 'far away', meaning that τ and l are sufficiently small, it will look similar to spatio-temporal Gaussian white noise, since correlations fall off rapidly.

1.3 The free Klein-Gordon field with chaotic noise

As a simple example, let us consider a free Klein-Gordon field with chaotic noise

$$\frac{\partial}{\partial t}\phi(x,t) = (\partial^2 - m^2)\phi(x,t) + L_{l,\tau}(x,t) \tag{1.16}$$

($m^2 > 0$). Here $L_{l,\tau}(x,t)$ is some complicated spatio-temporal process where the η_n^i evolve according to some (so far arbitrary) chaotic dynamics. Choosing (for example) $d = 4$, the solution of eq. (1.16) can be written as

$$\phi(x,t) = \int_0^t ds \int d^4y\, G(x-y, t-s) L_{l,\tau}(y,s), \tag{1.17}$$

where G denotes the retarded Greens function, i.e. the solution of

$$\frac{\partial}{\partial t}G(x,t) = (\partial^2 - m^2)G(x,t) + \delta(x)\delta(t). \tag{1.18}$$

G has the Fourier representation

$$G(x,t) = \frac{1}{(2\pi)^4} \int d^4k \, e^{-(k^2+m^2)t + ik\cdot x} \theta(t). \tag{1.19}$$

The integrations over the variables $k_1, \ldots, k_4$ are elementary, and we obtain

$$G(x,t) = \frac{1}{16\pi^2 t^2} e^{-m^2 t} \exp\left\{ -\frac{x^2}{4t} \right\} \qquad (t > 0). \tag{1.20}$$

Remember that the probability density $P(x,t)$ of the 4-dimensional Wiener process with variance σ^2 is

$$P(x,t) = (2\pi\sigma^2 t)^{-2} \exp\left\{ -\frac{x^2}{2\sigma^2 t} \right\}. \tag{1.21}$$

Choosing $\sigma^2 = 2$ we can therefore write

$$G(x,t) = e^{-m^2 t} P(x,t) \qquad (t > 0), \tag{1.22}$$

i.e., we get a very transparent interpretation of the Klein-Gordon propagator of the Parisi-Wu approach: An initial signal at fictitious time $t = 0$ and space-time $x = 0$ evolves according to 4-dimensional Brownian motion, moreover, it is exponentially damped in fictitious time with relaxation time m^{-2}.

Let us now further evaluate eq. (1.17) for a lattice process of type (1.15) with $d = 4$. Putting eq. (1.15) into eq. (1.17), we obtain the result that the field can be written as a sum of iterates η_n^i multiplied by appropriate coefficients $a_n^i(x,t)$:

$$\phi(x,t) = \sum_i \sum_{n=0}^{\lfloor t/\tau \rfloor} a_n^i(x,t) \eta_n^i \tag{1.23}$$

$$a_n^i(x,t) = \tilde{\sigma}^{-1} \tau^{1/2} l^2 e^{-m^2(t-n\tau)} P(x - il, t - n\tau) \tag{1.24}$$

The a_n^i essentially describe how much the chaotic variable η_n^i at point il contributes to the field ϕ at point x. This depends on the spatial distance $x - il$ and the fictitious time distance $t - n\tau$.

Note that in natural units ($\hbar = c = 1$) the fictitious time t has dimension *energy*$^{-2}$, whereas a space-time coordinate x^μ has dimension *energy*$^{-1}$. The field $\phi(x,t)$ has dimension *energy* and the noise process $L_{l,\tau}(x,t)$ has dimension *energy*^{3}. The chaotic variables η_n^i are dimensionless.

Now suppose that τ and l are sufficiently small lattice constants. They should be so small that there are no notable differences to the ordinary formulation of the standard model with Gaussian white noise. For example, we may choose $\tau = O(m_{Pl}^{-2})$ and $l = O(m_{Pl}^{-1})$, where m_{Pl} denotes the Planck mass. There are two important limit cases in the above equations. Suppose that the mass parameter m^2 of the field ϕ satisfies $m^2 << \tau^{-1}$. In this case we just obtain an ordinary quantized free field ϕ with Gaussian properties, since the process $L_{l,\tau}(x,t)$ looks almost identical to Gaussian white noise on large scales and evolves on much faster time scales than m^{-2}. However, as soon as m^2 approaches a value of order τ^{-1}, deviations from ordinary quantum field theoretical behaviour will become evident. The field ϕ will then become a complicated chaotic process.

We may also consider the other extreme case $m^2 >> \tau^{-1}$. In this case the exponential in eq. (1.24) decays rapidly. Essentially, only the coefficient $a_n^i(x,t)$ with $t - n\tau = 0$ and $x = il$ will contribute. The surprising result is that in this limit the field $\phi(x,t)$ just degenerates to the chaotic variable η_n^i, up to a trivial scale factor.

To elucidate this transition scenario as a function of m^2, consider for simplicity the case of a homogeneous field $\phi(x,t) = \phi(t)$ with no spatial dependence. With a suitably chosen variance parameter $\tilde{\sigma}$ the field can be written as $\phi = \sqrt{m^2\tau}\sum_{n=0}^{n'} \lambda^{n'-n}\eta_n$ where $\lambda := e^{-m^2\tau}$. The invariant probability density ($n' \to \infty$) of this quantity is plotted in Fig. 1.1-1.8 for various values of $m^2\tau$ and for two different types of mappings T, the 2nd- and 3rd-order Tchebyscheff polynomial. The density can be easily obtained by iterating the 2-dimensional map

$$\eta_{n+1} = T(\eta_n) \tag{1.25}$$

$$\Phi_{n+1} = \lambda\Phi_n + \sqrt{m^2\tau}\eta_n \tag{1.26}$$

and making a histogram of the Φ-variable [Kaplan et al. (1979); Beck (1990a); Beck (1996)].

For $m^2\tau \to 0$ the density $p(\phi)$ becomes Gaussian, hence the ordinary behaviour of a free field in quantum field theory is recovered. For $m^2\tau \to \infty$ the (properly rescaled) density approaches the invariant density $\pi^{-1}(1-\phi^2)^{-1/2}$ of the Tchebyscheff maps. Inbetween, there is complicated fractal and selfsimilar behaviour.

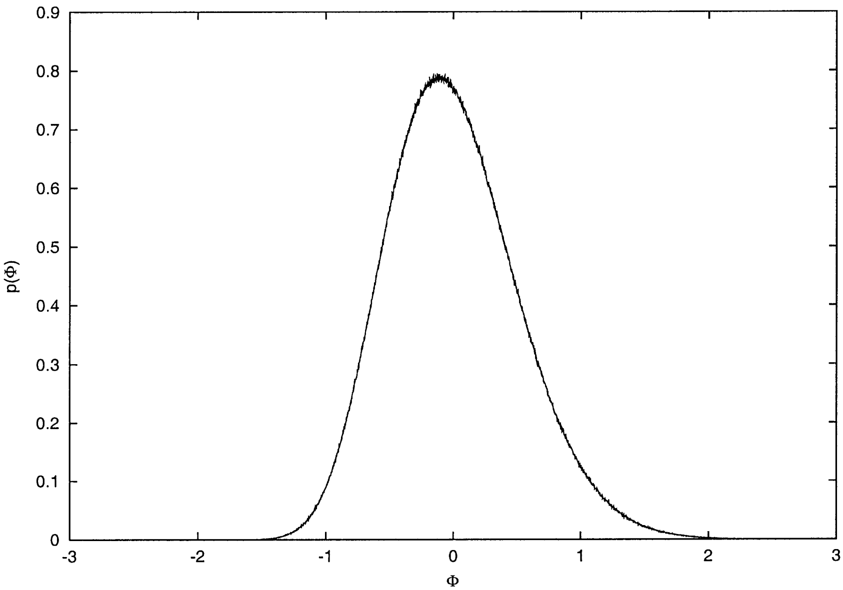

Fig. 1.1 Invariant probability density of a chaotically quantized real scalar field ϕ. The 'noise' evolves as $\eta_{n+1} = 2\eta_n^2 - 1$. The mass parameter is given by $m^2\tau = 0.05$.

1.4 * Chaotic quantization in momentum space

Alternatively, we may perform chaotic quantization in momentum space. Let us illustrate this again for the example of the free Klein-Gordon field. We may Fourier transform eq. (1.16) to obtain

$$\frac{\partial}{\partial t}\tilde{\phi}(k,t) = -(k^2 + m^2)\tilde{\phi}(k,t) + \tilde{L}_{l,\tau}(k,t). \tag{1.27}$$

Here $\tilde{\phi}(k,t)$ is the 4-dimensional Fourier transform of the field $\phi(x,t)$,

$$\tilde{\phi}(k,t) = \frac{1}{(2\pi)^2}\int d^4x e^{-ikx}\phi(x,t), \tag{1.28}$$

and $\tilde{L}_{l,\tau}(k,t)$ denotes the 4-dimensional Fourier transform of the chaotic noise field $L_{l,\tau}(x,t)$,

$$\tilde{L}_{l,\tau}(k,t) \quad = \quad \frac{1}{\tilde{\sigma}(2\pi)^2}\tau^{1/2}l^2\int d^4x e^{-ikx}L_{l,\tau}(x,t) \tag{1.29}$$