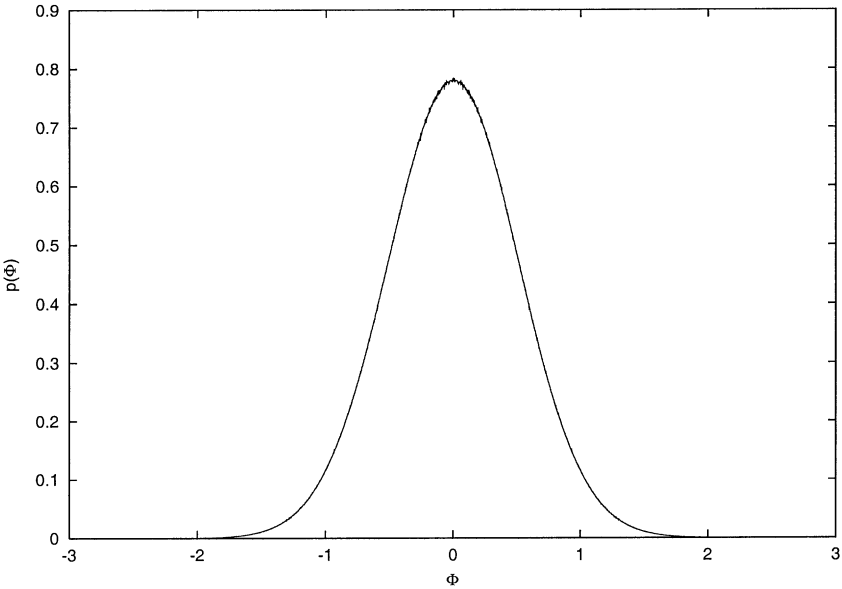

Fig. 1.5 Invariant probability density of a chaotically quantized real scalar field ϕ with 'noise' evolving as $\eta_{n+1} = 4\eta_n^3 - 3\eta_n$. The mass parameter is $m^2\tau = 0.05$.

where $\tilde{L}^T$ and $\tilde{L}^L$ are projections of the chaotic noise onto the transverse, respectively longitudinal direction. This shows that the transverse component of the gauge field behaves like a massless Klein-Gordon field with chaotic noise, and the same consideration as in the previous section applies, this time with $m = 0$. From a stochastic process point of view, the transverse component $\tilde{A}_\mu^T(k,t)$ is a generalized Ornstein-Uhlenbeck process, the Gaussian white noise being replaced by chaotic noise. The longitudinal component $\tilde{A}_\mu^L(k,t)$ is just the time-integrated chaotic noise field, and in this sense it is the chaotic generalization of a Wiener process.

1.6 Distinguished properties of Tchebyscheff maps

Let us now think about a suitable dynamics for the deterministic chaotic noise fields η_n^i. The most distinguished candidate is of course a dynamics that (in some appropriate sense) is closest to Gaussian white noise, though being completely deterministic. As we shall see in this and the following

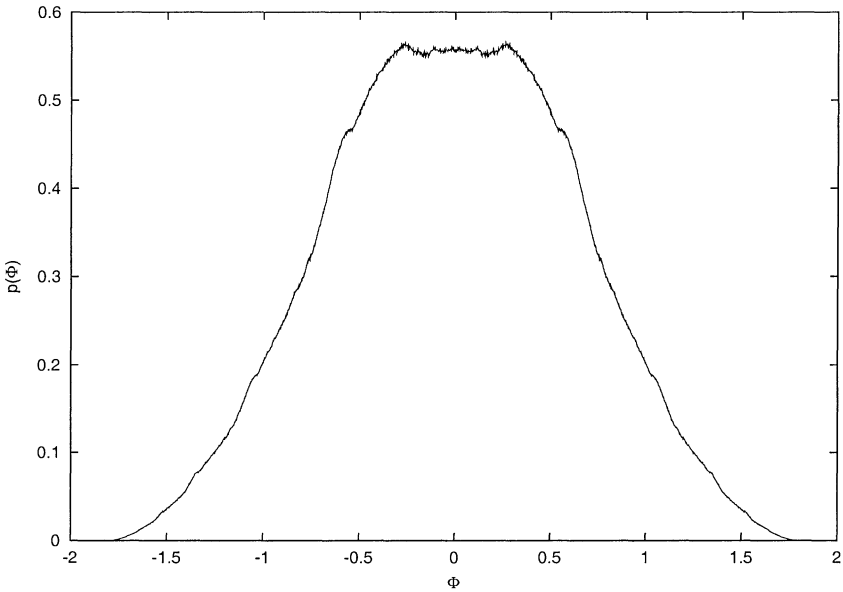

Fig. 1.6 Same as Fig. 1.5, but $m^2\tau = 0.5$.

section, under the assumption of smoothness this criterium leads to a local dynamics generated by Tchebyscheff maps. We denote these distinguished chaotic noise fields η_n^i by Φ_n^i. The (spatially uncoupled) dynamics is given by

$$\Phi_{n+1}^i = T_N(\Phi_n^i) \tag{1.42}$$

where T_N is a Tchebyscheff polynomial of degree $N \geq 2$ and $\Phi_i^0 \in [-1, 1]$. The Tchebyscheff polynomials $T_N(\Phi)$ are defined as

$$T_N(\Phi) = \cos(N \arccos \Phi). \tag{1.43}$$

The first few polynomials are given by

$$\begin{aligned}
T_1(\Phi) &= \Phi, \\
T_2(\Phi) &= 2\Phi^2 - 1, \\
T_3(\Phi) &= 4\Phi^3 - 3\Phi, \\
T_4(\Phi) &= 8\Phi^4 - 8\Phi^2 + 1,
\end{aligned}$$

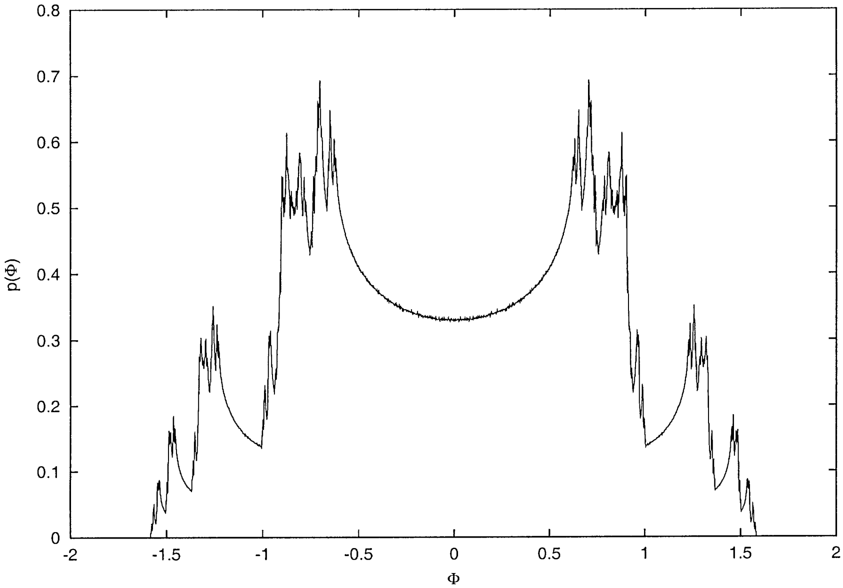

Fig. 1.7 Same as Fig. 1.5, but $m^2\tau = 1$.

$$\begin{aligned} T_5(\Phi) &= 16\Phi^5 - 20\Phi^3 + 5\Phi, \\ T_6(\Phi) &= 32\Phi^6 - 48\Phi^4 + 18\Phi^2 - 1 \\ &\cdots \\ &\cdots \end{aligned} \tag{1.44}$$

The most remarkable and distinguished feature of Tchebyscheff maps T_N with $N \geq 2$ is that these maps are semi-conjugated to a Bernoulli shift with N symbols [Ulam et al. (1947)]. Defining $\Phi_0 = \cos \pi u_0$ the iterates $\Phi_n = T_N(\Phi_{n-1})$ can be written as

$$\Phi_n = \cos \pi N^n u_0 \qquad (u_0 \in [0,1]). \tag{1.45}$$

If we write the initial value u_0 as

$$u_0 = \sum_{j=1}^{\infty} s_j N^{-j}, \tag{1.46}$$

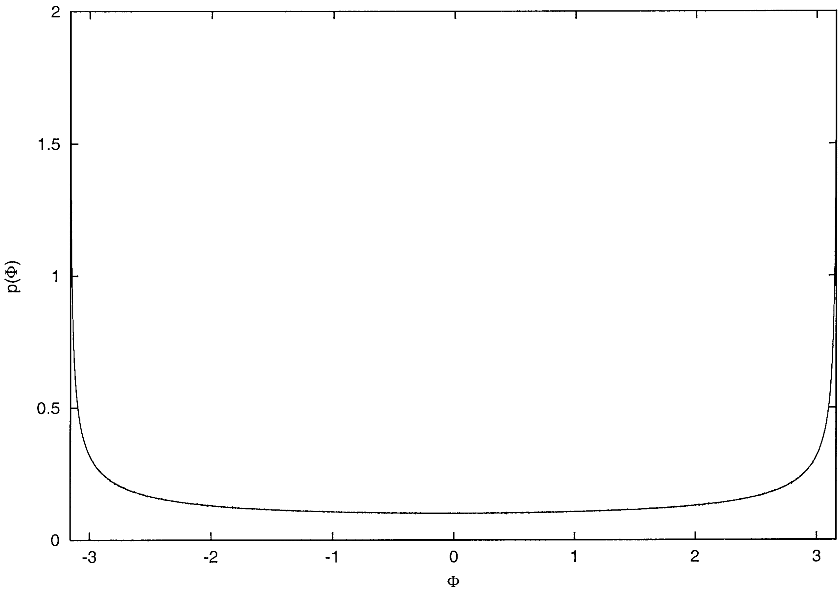

Fig. 1.8 Same as Fig. 1.5, but $m^2\tau = 10$.

where the integers

$$s_j \in \{0, 1, 2, \ldots, N-1\} \tag{1.47}$$

correspond to the N-ary representation of u_0, then each iteration step of T_N shifts all the symbols s_j by one step to the left, i.e. $s_j \to s_{j-1}$, and the information on the first symbol s_1 is lost. We have $\Phi_n = \cos \pi u_n$ with $u_n = \sum_{j=1}^{\infty} s_{j+n} N^{-j}$. In this sense iteration of Tchebyscheff maps is just a shift of information, encoded with N different symbols.

For any map conjugated to a Bernoulli shift, rescaled sums of the iterates satisfy a functional central limit theorem guaranteeing the convergence to the Wiener process in the limit $l \to 0, \tau \to 0, t = n\tau$ finite (see section 1.2). Since the derivative of the Wiener process is Gaussian white noise, in this limit the deterministic chaotic Φ_n^i may perfectly serve as a source of Gaussian white noise for any field theory quantized according to the Parisi-Wu approach. This means that we can use the Φ_n^i of this section as the η_n^i of the previous sections, and have convergence to the Wiener process (and thus to ordinary quantized field theories) under rescaling, i.e. if we look at the dynamics from far away. In a sense, all randomness is then just contained in the initial values u_0^i.

All Tchebyscheff maps with $N \geq 2$ are ergodic and mixing, with the natural invariant probability density

$$\rho(\Phi) = \frac{1}{\pi\sqrt{1-\Phi^2}} \qquad \Phi \in [-1,1], \tag{1.48}$$

describing the probability distribution of iterates under long-term iteration (see, e.g., [Beck et al. (1993)] for an introduction). This density is independent of N. Any expectation $\langle A(\Phi)\rangle$ of some observable A with respect to the chaotic dynamics can be calculated as

$$\langle A(\Phi)\rangle = \int_{-1}^{1} d\Phi \; \rho(\Phi) A(\Phi). \tag{1.49}$$

For example, from eq. (1.49) and (1.48) one obtains the even moments of Φ as

$$\langle \Phi^{2k}\rangle = 4^{-k} \binom{2k}{k}. \tag{1.50}$$

The odd moments, in particular the mean $\langle\Phi\rangle$, vanish. Most of the higher-order correlation functions $\langle \Phi_{n_1}\Phi_{n_2}\cdots\Phi_{n_r}\rangle$ vanish as well, with the exception of those that can be represented by certain types of graphs. This will be worked out in the next section.

1.7 * Graph theoretical method

The higher-order correlation functions of iterates Φ_n of Tchebyscheff maps are given by

$$\langle \Phi_{n_1}\Phi_{n_2}...\Phi_{n_r}\rangle = \int_{-1}^{1} d\Phi_0 \;\; \rho(\Phi_0)\Phi_{n_1}\Phi_{n_2}...\Phi_{n_r}, \tag{1.51}$$

where ρ is the natural invariant density (1.48). Substituting $\Phi_0 = \cos \pi u$ we obtain

$$\langle \Phi_{n_1}\Phi_{n_2}...\Phi_{n_r}\rangle = \int_{0}^{1} du \;\; cos\pi N^{n_1}u \ldots cos\pi N^{n_r}u, \tag{1.52}$$