look at the dynamics from far away. In a sense, all randomness is then just contained in the initial values u_0^i.

All Tchebyscheff maps with $N \geq 2$ are ergodic and mixing, with the natural invariant probability density

$$\rho(\Phi) = \frac{1}{\pi\sqrt{1-\Phi^2}} \qquad \Phi \in [-1,1], \tag{1.48}$$

describing the probability distribution of iterates under long-term iteration (see, e.g., [Beck et al. (1993)] for an introduction). This density is independent of N. Any expectation $\langle A(\Phi)\rangle$ of some observable A with respect to the chaotic dynamics can be calculated as

$$\langle A(\Phi)\rangle = \int_{-1}^{1} d\Phi \; \rho(\Phi) A(\Phi). \tag{1.49}$$

For example, from eq. (1.49) and (1.48) one obtains the even moments of Φ as

$$\langle \Phi^{2k}\rangle = 4^{-k} \begin{pmatrix} 2k \\ k \end{pmatrix}. \tag{1.50}$$

The odd moments, in particular the mean $\langle\Phi\rangle$, vanish. Most of the higher-order correlation functions $\langle \Phi_{n_1}\Phi_{n_2}\cdots\Phi_{n_r}\rangle$ vanish as well, with the exception of those that can be represented by certain types of graphs. This will be worked out in the next section.

1.7 * Graph theoretical method

The higher-order correlation functions of iterates Φ_n of Tchebyscheff maps are given by

$$\langle \Phi_{n_1}\Phi_{n_2}...\Phi_{n_r}\rangle = \int_{-1}^{1} d\Phi_0 \;\; \rho(\Phi_0)\Phi_{n_1}\Phi_{n_2}...\Phi_{n_r}, \tag{1.51}$$

where ρ is the natural invariant density (1.48). Substituting $\Phi_0 = \cos \pi u$ we obtain

$$\langle \Phi_{n_1}\Phi_{n_2}...\Phi_{n_r}\rangle = \int_{0}^{1} du \;\; cos\pi N^{n_1}u \ldots cos\pi N^{n_r}u, \tag{1.52}$$

and using $\cos u = \frac{1}{2}(e^{iu} + e^{-iu})$ this can be written as

$$\begin{aligned}\langle \Phi_{n_1}\Phi_{n_2}...\Phi_{n_r}\rangle &= \sum_\sigma \int_0^1 du \prod_{l=1}^{r} \frac{1}{2} exp\left(i\pi\sigma_l N^{n_l} u\right) \\ &= 2^{-r}\sum_\sigma \int_0^1 du\; exp\left(i\pi u \sum_{l=1}^{r} \sigma_l N^{n_l}\right). \qquad (1.53)\end{aligned}$$

Here $\sum_\sigma$ denotes the summation over all possible 'spin' configurations $(\sigma_1, ..., \sigma_r)$ with $\sigma_l = \pm 1$. Writing the Kronecker symbol $\delta_{i,j}$ as $\delta(i,j)$, we obtain

$$\langle \Phi_{n_1}\Phi_{n_2}...\Phi_{n_r}\rangle = 2^{-r}\sum_\sigma \delta(\sigma_1 N^{n_1} + \ldots + \sigma_r N^{n_r}, 0). \qquad (1.54)$$

Apparently, the non-vanishing correlations of the Tchebyscheff maps correspond to tuples $(n_1, \ldots, n_r)$ that are the solutions of the diophantic equations

$$\sum_{i=1}^{r} \sigma_i N^{n_i} = 0, \qquad \sigma_i = \pm 1. \qquad (1.55)$$

The tuples $(n_1, n_2, \ldots, n_r)$ that are a solution of at least one of the above 2^r equations can be represented by certain types of graphs, namely forests of double N-ary trees with r leaves in total. These graphs, denoted by $L_r^{(N)}$, are displayed in Fig. 1.9-1.12 for $N = 2, 3, 4, 5$.

Double N-ary trees are trees that are obtained by connecting two single N-ary trees at their roots. A double N-ary forest is a set containing one or several of these trees. The number of leaves of the forest is defined to be the sum of the number of leaves of the trees.

One has the following theorem which describes all higher-order correlations of Tchebyscheff maps:

Theorem.

$$\langle \Phi_{n_1}\Phi_{n_2}\cdots\Phi_{n_r}\rangle \neq 0 \Longleftrightarrow (n_1, n_2, \cdots, n_r) \in L_r^{(N)} \qquad (1.56)$$

For a proof, see [Hilgers et al. (2001)]. The theorem connects dynamics (on the left-hand side) with topological structure (on the right-hand side).

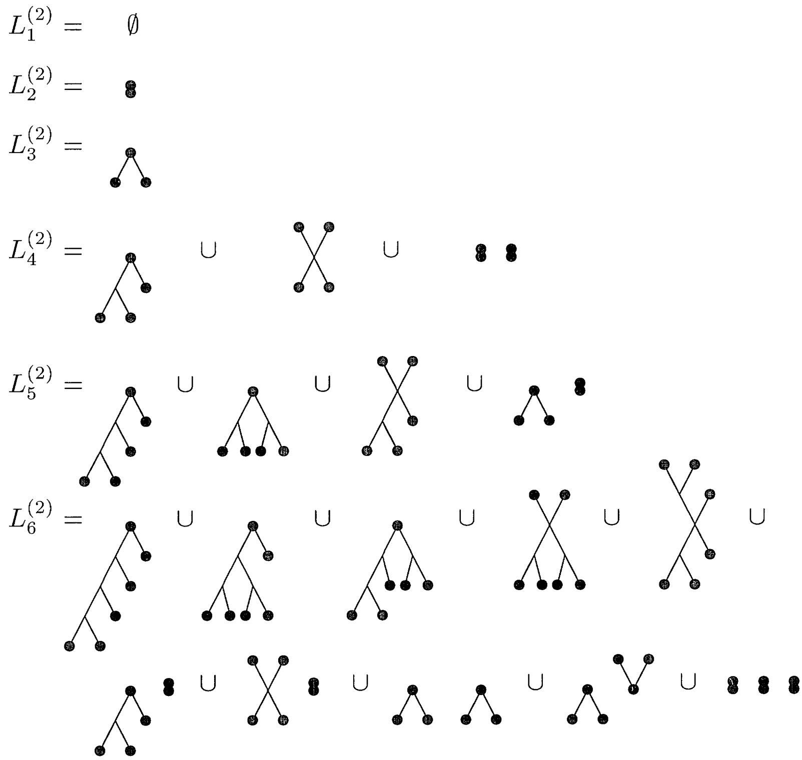

Fig. 1.9 Forests describing the higher-order correlations of $T_2(\Phi)$.

Still we have to explain how the forests are identified with certain tuples $(n_1, n_2, \ldots, n_r)$ of integers. Note that each forest consists of one or several double N-ary trees. Consider one of the trees. The point where all branching processes start from is called the 'root' of the tree. The black dots at the endpoints are called 'leaves'. Given a tree with distances $d_1, d_2, \ldots, d_r$ of the leaves from the root, it is identified with tuples of the form $(n_1, n_2, \ldots, n_r) = (m - d_1, m - d_2, \ldots, m - d_j)$, including all possible permutations. m is an arbitrary integer. For example, $L_3^{(2)}$ in Fig. 1.9 stands for tuples of the form $(n_1, n_2, n_3) = (m-1, m-1, m)$ plus all possible permutations. These solve the diophantic equation $2^{n_1} + 2^{n_2} = 2^{n_3}$. The

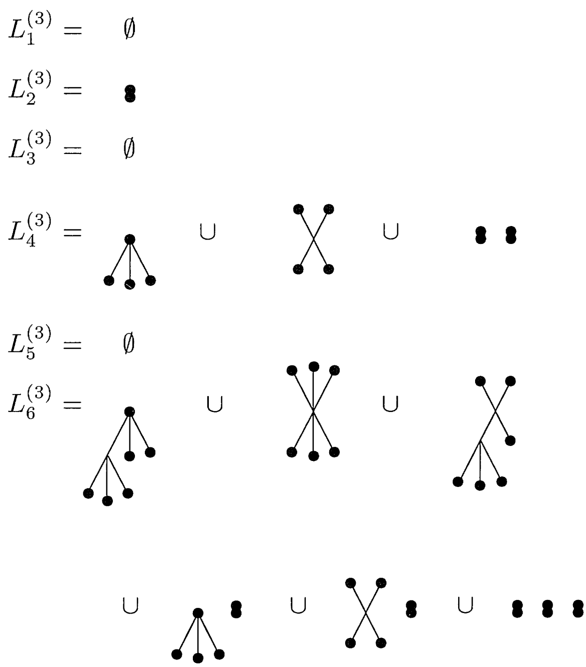

Fig. 1.10 Forests describing the higher-order correlations of $T_3(\Phi)$.

theorem says that for the iterates of the second order Tchebyscheff map T_2 one has $\langle \Phi_{n-1}^2 \Phi_n \rangle \neq 0$. This property of the chaotic noise is clearly different from that of independent random variables χ_n with zero mean, where one would have $\langle \chi_{n-1}^2 \chi_n \rangle = \langle \chi_{n-1}^2 \rangle \langle \chi_n \rangle = 0$. Similarly, the first tree in $L_5^{(2)}$ indicates that $\langle \Phi_{n-3}^2 \Phi_{n-2} \Phi_{n-1} \Phi_n \rangle \neq 0$, and so on. For the 3rd-order Tchebyscheff map the graphs and hence the correlations are different, for example the first tree in $L_4^{(3)}$ indicates that one has $\langle \Phi_{n-1}^3 \Phi_n \rangle \neq 0$.

Generally, if there are s trees forming one forest, then the various tuples corresponding to the single trees are combined into one tuple with independent values $m_1, \ldots, m_s$ corresponding to the roots of the s trees. For example, the last forest in $L_6^{(2)}$ consists of 3 (trivial) trees. This forest describes tuples of the form $(n_1, n_2, n_3, n_4, n_5, n_6) = (m_1, m_1, m_2, m_2, m_3, m_3)$ plus

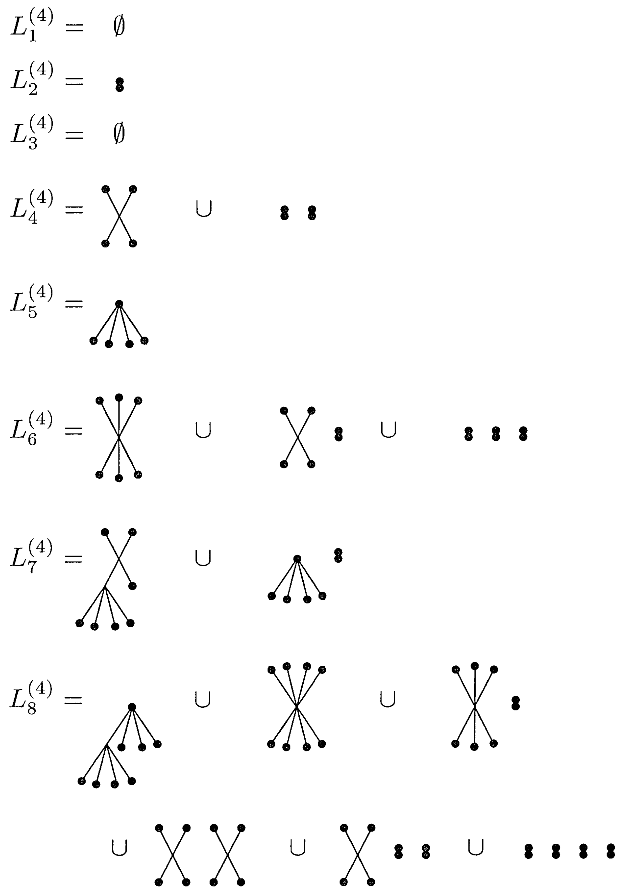

Fig. 1.11 Forests describing the higher-order correlations of $T_4(\Phi)$.

all possible permutations. The graph describes trivial higher-order correlations of Tchebyscheff maps that independent random variables χ_n would have as well.

Let us compare the correlations of the Tchebyscheff polynomials T_N with those of other maps $\tilde{T}$ conjugated to a Bernoulli shift. We assume

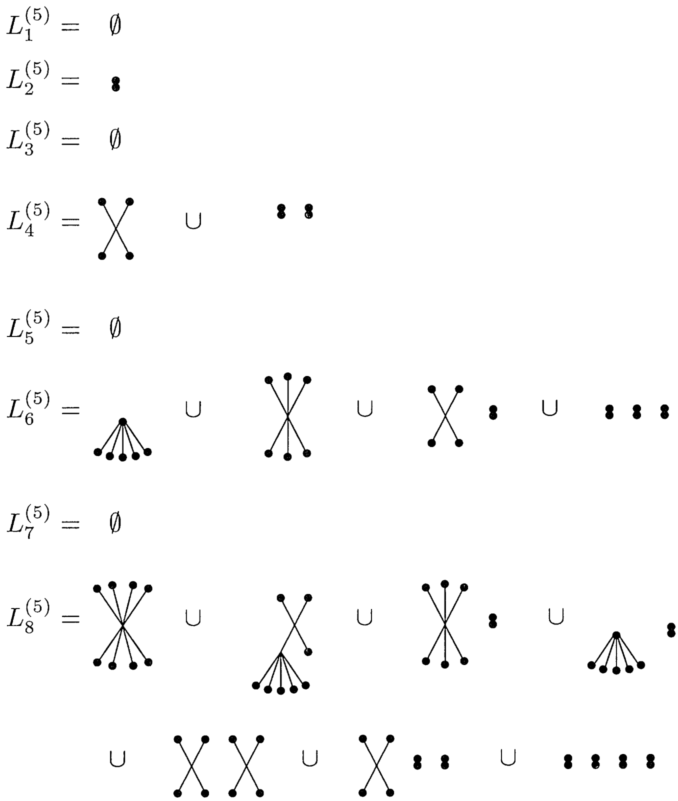

Fig. 1.12 Forests describing the higher-order correlations of $T_5(\Phi)$.

that the iterates $\tilde{\Phi}_n$ of $\tilde{T}$ can also be written as $\tilde{\Phi}_n = f(N^n u)$, but this time with some other periodic function f that is smooth (remember that for Tchebyscheff maps f is the cosine). If $f(u)$ has the Fourier representation $\sum_{k=-\infty}^{\infty} a_k exp(i\pi k u)$, we obtain for the r-point correlations

$$\langle \tilde{\Phi}_{n_1} \tilde{\Phi}_{n_2} ... \tilde{\Phi}_{n_r} \rangle = \sum_{k_1=-\infty}^{\infty} \cdots \sum_{k_r=-\infty}^{\infty} a_{k_1} \ldots a_{k_r} \delta(k_1 N^{n_1} + \ldots + k_r N^{n_r}, 0). \tag{1.57}$$

Non-vanishing correlations occur if the tuple $(n_1, ... n_r)$ is a solution of at least one of the diophantic equations

$$\sum_{i=1}^{r} k_i N^{n_i} = 0, \qquad k_i \in \mathbf{Z}. \tag{1.58}$$

Now clearly eq. (1.58) has a larger number of solutions than eq. (1.55), since generally the k_i take on more values than just -1 and 1, as for the Tchebyscheff maps. Thus these other maps $\tilde{T}$ have more non-vanishing higher-order correlations than the Tchebyscheff maps. The number of non-vanishing correlations is minimal for Tchebyscheff maps because for those only the coefficients a_1 and a_{-1} are non-zero in the Fourier representation of the conjugating function f. We see that Tchebyscheff maps are distinguished by a minimum skeleton of higher-order correlations. In that sense they are closest to uncorrelated Gaussian white noise, as close as possible for a smooth deterministic system. This makes them an attractive model for a deterministic dynamics that generates 'noise' at the smallest quantum mechanical scales.

1.8 * Perturbative approach

The graph theoretical method is important for chaotic quantization, because we saw in sections 1.3–1.5 that within this quantization method ordinary standard model fields such as the free Klein-Gordon field $\phi(x,t)$ are linear combinations of chaotic variables η_n^i, now denoted as Φ_n^i:

$$\phi(x,t) = \sum_{n,i} a_n^i(x,t)\Phi_n^i \tag{1.59}$$

If the Φ_n^i are generated by Tchebyscheff maps then these types of sums are lacunary trigonometric series, and various rigorous results are known [Salem et al. (1947); Zygmund (1959)]. To evaluate moments or correlation functions of r-th order of the standard model field ϕ, one has to sum over all non-vanishing r-point functions of the chaotic variables:

$$\langle \phi^r \rangle = \sum_{n_1,i_1,\ldots,n_r,i_r} a_{n_1}^{i_1} \cdots a_{n_r}^{i_r} \langle \Phi_{n_1}^{i_1} \cdots \Phi_{n_r}^{i_r} \rangle \tag{1.60}$$

For spatially uncoupled variables Φ_n^i the higher-order correlations in space are trivial, and those in time n can be evaluated using the double N-ary forests. Each graph yields a certain contribution. A kind of perturbation