least one of the diophantic equations

$$\sum_{i=1}^{r} k_i N^{n_i} = 0, \qquad k_i \in \mathbf{Z}. \tag{1.58}$$

Now clearly eq. (1.58) has a larger number of solutions than eq. (1.55), since generally the k_i take on more values than just -1 and 1, as for the Tchebyscheff maps. Thus these other maps $\tilde{T}$ have more non-vanishing higher-order correlations than the Tchebyscheff maps. The number of non-vanishing correlations is minimal for Tchebyscheff maps because for those only the coefficients a_1 and a_{-1} are non-zero in the Fourier representation of the conjugating function f. We see that Tchebyscheff maps are distinguished by a minimum skeleton of higher-order correlations. In that sense they are closest to uncorrelated Gaussian white noise, as close as possible for a smooth deterministic system. This makes them an attractive model for a deterministic dynamics that generates 'noise' at the smallest quantum mechanical scales.

1.8 * Perturbative approach

The graph theoretical method is important for chaotic quantization, because we saw in sections 1.3–1.5 that within this quantization method ordinary standard model fields such as the free Klein-Gordon field $\phi(x,t)$ are linear combinations of chaotic variables η_n^i, now denoted as Φ_n^i:

$$\phi(x,t) = \sum_{n,i} a_n^i(x,t)\Phi_n^i \tag{1.59}$$

If the Φ_n^i are generated by Tchebyscheff maps then these types of sums are lacunary trigonometric series, and various rigorous results are known [Salem et al. (1947); Zygmund (1959)]. To evaluate moments or correlation functions of r-th order of the standard model field ϕ, one has to sum over all non-vanishing r-point functions of the chaotic variables:

$$\langle \phi^r \rangle = \sum_{n_1,i_1,\ldots,n_r,i_r} a_{n_1}^{i_1} \cdots a_{n_r}^{i_r} \langle \Phi_{n_1}^{i_1} \cdots \Phi_{n_r}^{i_r} \rangle \tag{1.60}$$

For spatially uncoupled variables Φ_n^i the higher-order correlations in space are trivial, and those in time n can be evaluated using the double N-ary forests. Each graph yields a certain contribution. A kind of perturbation theory can then be developed, where the small parameter is $m^2\tau$, where m is the mass of the standard model field, and τ is the lattice constant of the fictitious time. In other words, if we assume that τ is of the order m_{Pl}^{-2} then the small parameter is the dimensionless gravitational coupling $g \sim (m/m_{Pl})^2$ (see section 1.3). The larger the number of trees in a forest, the more important is the contribution of this forest. One can easily check that a forest with s trees yields a contribution of order $g^{\frac{r}{2}-s}$. In the limit $g \sim m^2\tau \to 0$, only the contribution of the trivial forests survives, and we then obtain ordinary stochastically quantized standard model fields on a large scale. More details on this perturbative approach can be found in [Beck (1991a); Hilgers et al. (2001)].

Here we just mention a perturbative result for the 0-dimensional field displayed in Fig. 1.1-1.8. In order to get something finite for $\tau \to 0$ we had rescaled it with $\sqrt{m^2\tau}$, i.e., the field was given by

$$
\begin{aligned}
\phi &= \sqrt{g}\sum_{j=0}^{n} \lambda^{n-j}\Phi_n && (1.61)\\
\Phi_{n+1} &= T_N(\Phi_n) && (1.62)\\
\lambda &= e^{-g} && (1.63)\\
g &= m^2\tau \sim \left(\frac{m}{m_{Pl}}\right)^2 . && (1.64)
\end{aligned}
$$

Summing over the relevant graphs the final result of a longer calculation are the following formulas for the invariant density $p^{(N)}(\phi)$ of ϕ for $n \to \infty$:

$$
\begin{aligned}
p^{(N\geq 4)}(\phi) &= \sqrt{\frac{2}{\pi}}\left(1 + g(-\phi^4 + \frac{7}{2}\phi^2 - \frac{11}{16})\right) e^{-2\phi^2} + O(g^{J/2}) && (1.65)\\
p^{(N=3)}(\phi) &= \sqrt{\frac{2}{\pi}}\left(1 + g(\frac{1}{3}\phi^4 + \frac{3}{2}\phi^2 - \frac{7}{16})\right) e^{-2\phi^2} + O(g^2) && (1.66)\\
p^{(N=2)}(\phi) &= \sqrt{\frac{2}{\pi}}\left(1 + g^{\frac{1}{2}}(-2\phi + \frac{8}{3}\phi^3)\right.\\
&+ \left. g(\frac{32}{9}\phi^6 - \frac{31}{3}\phi^4 + \frac{15}{2}\phi^2 - \frac{37}{48})\right) e^{-2\phi^2} + O(g^{3/2}) && (1.67)
\end{aligned}
$$

(see [Hilgers et al. (1999b); Hilgers et al. (2001)] for details). Here $J = 3$ for $N = 4$ and $J = 4$ for $N \geq 5$. Note that the first-order correction to the Gaussian behaviour is of order $\sqrt{g}$ for $N = 2$ and of order g for $N \geq 3$. For

$N \geq 4$, up to second order in $\sqrt{g}$, only trivial trees contribute, and as a result of this the leading order perturbative expression for $p^{(N\geq 4)}(\phi)$ is the same as generated by independent discrete random variables χ_n. In other words, only $N = 2$ and $N = 3$ yield nontrivial behaviour in leading order of chaotic quantization.