PREFACE

This book is an introduction to the theory and application of modes. Modes are algebras that are idempotent (i.e. each element forms a subalgebra) and entropic (i.e. each operation is a homomorphism). Modes appear in many different branches of mathematics, for instance in affine geometry, convex analysis, linear programming, differential geometry, combinatorics, and program semantics. They find application in computer science, economics, physics, and biology. Examples of modes include semilattices and more general normal bands, many interesting classes of groupoids appearing in combinatorics and geometry, convex sets and their generalization to barycentric algebras, affine spaces under affine combinations, and various reducts of such algebras.

Modes were studied under several guises by many authors since the late nineteen-thirties, and have attracted much attention over recent years. For a long time, many results were scattered throughout the literature, and were often obtained independently of the work of others in the field. The first systematic approach to modes (and the closely related algebras known as modals) was presented in our 1985 monograph "Modal Theory, an Algebraic Approach to Order, Geometry and Convexity," published by Heldermann, Berlin. We will refer to this book as [MT] throughout the present volume. The subsequent development of modal theory has proceeded in many directions, essentially enlarging almost all the areas covered in [MT], and bringing many new applications. Surveys have appeared in [Romanowska 1992] and [Smith 1999]. This new book is intended as an introduction to the current state of the theory of modes. Bearing in mind that [MT] was formulated in the condensed style of a research monograph, we decided that the present volume should begin at a gentler pace, more readily accessible to non-specialists and beginning graduate students. There is a gradual progression to the more demanding and "sophisticated" latter chapters. Some of the material from [MT] is reprised here, albeit in different form, but the emphasis is on topics that were not covered in [MT].

Readers are assumed to have a solid grounding in undergraduate mathematics, especially in the classical algebraic structures: groups, rings, fields, and vector spaces. Following a brief summary of some preliminaries, we begin with a chapter discussing the basic concepts of general algebra, illustrated by descriptions of some of the non-classical algebraic structures to be studied in more detail later. The next chapter reviews the basic tools of category theory that will be required. Chapter 3 covers the fundamentals of the theory of varieties, prevarieties, and quasivarieties. Taken together, the first three chapters should provide the basis for a full understanding of the subsequent material, but we will occasionally refer the reader to other sources, especially to our monograph "Post-Modern Algebra" (Wiley, 1999), cited throughout as [PMA].

Chapter 4 provides a general theory for the construction of new algebras as "sums" of component algebras, usually better known algebras of the same type, and indexed by elements of another algebra. These sums provide an excellent method for representing modes in certain quasivarieties. The true introduction to modes starts in Chapter 5, where we discuss the basic properties and basic examples. Later chapters proceed to more advanced topics. After a brief introduction to general Mal'cev varieties, Chapter 6 studies Mal'cev modes. The main theorem of this chapter demonstrates the equivalence between Mal'cev modes and affine spaces. Subsequent sections are devoted to certain classes of groupoids and quasigroups equivalent to affine spaces. Chapter 7 discusses subreducts of affine spaces, and their sums. The general results are applied to describe the structure of barycentric algebras and commutative groupoid modes, and to classify the quasivarieties of barycentric algebras. The main theorems show that each cancellative mode embeds as a subreduct into an affine space, and more generally that certain sums of cancellative modes embed as subreducts into functorial sums of affine spaces. Chapter 8 studies the algebraic properties of binary or groupoid modes. It describes the free binary modes, characterizes the simple objects, and discusses the classification of varieties. Binary modes provide a prototype for more general modes. Until one has reached a full understanding of the intricacies of binary modes, it will be hard to proceed to an effective analysis of the general case.

The ninth chapter is devoted to one sample application of the theory of modes, namely the development of hierarchical statistical mechanics. The hierarchy here is understood in the truly qualitative sense, not just in the quantitative sense where the levels of the hierarchy are differentiated only by orders of magnitude of comparable quantities. Traditional statistical mechanics is ultimately founded on convex sets. The hierarchical statistical mechanics discussed in Chapter 9 is founded on barycentric algebras. Modal theory shows how barycentric algebras decompose as sums of convex sets indexed by a semilattice, the semilattice replica of the barycentric algebra. This decomposition is the key to the whole application. The semilattice replica of a barycentric algebra describes the structure of the underlying hierarchy of a complex system.

The book concludes with Chapter 10, discussing further topics, recent developments, and open problems that are the subject of current research. As mentioned earlier, much of the work on modes is scattered throughout the literature. It is not always easy to find, and not always readily accessible. At the end of each chapter from the fourth to the eighth, we have tried to collate at least the most essential references in a series of "historical" notes. Nevertheless, these notes are by no means intended to be definitive or comprehensive. The general bibliography appears at the end of the book, but again we make no claim to completeness. References to the bibliography are by name of author(s) and year of publication (with additional letters where necessary to distinguish multiple citations of the same author(s) in the same year). An internal reference such as "Theorem 6.2.3" denotes Theorem 2.3 of Chapter 6. Occasionally, references such as "Proposition 6.4.4.2" are necessary, denoting part of the discussion of Example 6.4.4 apppearing in the fourth section of Chapter 6.

A wide range of exercises is offered throughout most chapters of the book, usually placed at the end of each section and indexed alphabetically. Some exercises are simply designed to familiarize readers with the notation and concepts appearing. Others are more difficult, extending the content of the sections in which they appear, and intended to give a foretaste of further research. Some of the material in the book has been used in a one-semester graduate-level course at Warsaw University of Technology. We are grateful to our colleague, Barbara Roszkowska, and to our students, particularly Katarzyna Matczak, Krzystof Pszczoła, Michał Stronkowski, and Anna Zamojska, for correcting various mistakes in earlier versions of the text. The mathematics departments of Warsaw University of Technology and Iowa State University contributed substantially towards the timely completion of this book by granting the authors the leave and facilities necessary for its production. In particular, Anna Romanowska was partially supported by a series of Warsaw University of Technology Statutory Grants.[1] It is also a great pleasure to acknowledge the contributions of Ruth DeBoer, Krzysztof Pszczoła and Agnieszka Świątkiewicz in the preparation of the typescript and bibliography, and to thank E.H. Chionh of World Scientific for her fine editorial work that brought the project to fruition.

[1]Numbers 504/052/0178, 504G/1172/1120/0004/000, 504G/1120/0015/000, and 504G/1120/0027/000.