Chapter 1

Introduction

In this chapter, we list both strong and weak points of the Gaussian MBS-theory, explain why it needs modification, describe informally the technique used in the book and present briefly the main results of the book.

1.1 The Gaussian Merton-Black-Scholes theory

1.1.1 *Strong points*

Ideally, an applied mathematical theory in Finance and Economics should possess the following merits:

(i) the theory has a wide range of applications;
(ii) in many situations of interest, the theory provides analytical answers in the form suitable for the analysis of the dependence of the endogenous variables on the exogenous parameters (comparative statics);
(iii) Mathematics used in the theory is accessible for the wide audience, in simplest situations at least;
(iv) moreover, it is strongly advisable that models of the theory can be explained and applied at the level of "the rule of the thumb";
(v) when the analytical solution is not known, effective numerical methods are available;
(vi) the basic principles of the theory and the answers it provides are in good agreement with the reality.

(If a model satisfies (ii), or better, (ii)–(iv), economists call it "tractable").

The Merton-Black-Scholes theory is so popular among practitioners and researches in Finance and Economics just because it satisfies criteria (i)–(v) almost to perfection. It has been successfully applied to most of the problems in Mathematical Finance, Financial Economics, theory of Real Options and in many other fields of Economics, essentially to any problem where the uncertainty matters, and in model situations the answers have been obtained in the form of fairly simple analytical expressions convenient for further analysis. Mathematics used in the MBS-theory is the theory of Partial Differential Equations (PDE), and in many cases of interest, only the knowledge of Ordinary Differential Equations (ODE) suffices. Being at the undergraduate level, ODE–technique is widely known, and its usage in the MBS-theory can be explained informally, as Dixit and Pindyck (1994) demonstrated. Finally, for PDE and ODE, a host of effective numerical methods is available.

1.1.2 *Drawbacks*

In the case of the MBS-theory, the motto: "Our vices are continuation of our virtues" [1] should be rephrased in the reverse order: "Our virtues are continuation of our vices". Both the simplicity and success of the MBS-theory stem from the possibility of reduction to boundary value problems for PDE or ODE, and this reduction becomes possible due to the choice of the simplest class of stochastic processes – Gaussian processes – to model the evolution of prices in financial and other markets. (The rival modern martingale approach, which tries to remain in the realm of Probability theory and avoid the usage of PDE and PDO whenever it can help it, also capitalizes on the extraordinary nice properties of Gaussian processes). In the basic Black-Scholes model, the price of a stock (or index, like the Standard & Poor 500 Index) follows the Geometric Brownian motion: $S_t = \exp X_t$, where X_t is the Brownian motion. Then the *probability density* $p_{\Delta t}(x)$ of the increments $X_{t+\Delta t} - X_t$ is given by

$$p_{\Delta t}(x) = \frac{1}{\sqrt{2\pi\sigma^2\Delta t}} \exp\left(-\frac{(x-\mu\Delta t)^2}{2\sigma^2\Delta t}\right), \tag{1.1}$$

where μ and σ^2 are called the *drift* and *volatility*. As Eq. (1.1) shows, the tails of the probability density decay faster than an exponenial function as

[1] Translated from Russian

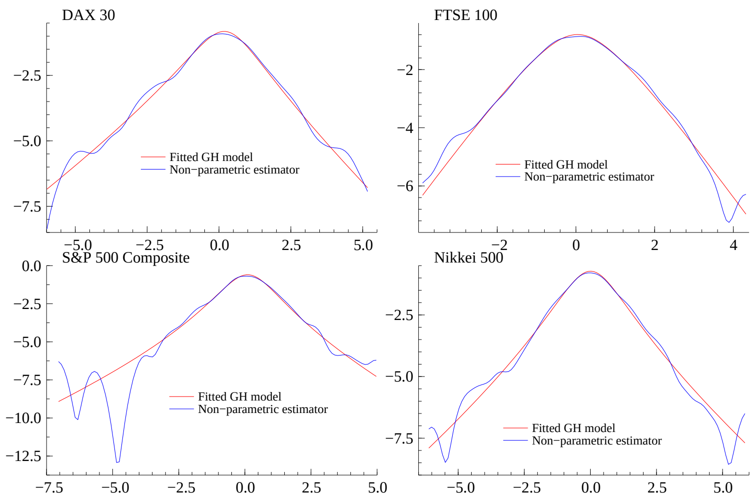

Fig. 1.1 Log-densities for major indices

$x \to \pm\infty$.

1.1.2.1 *Fat tails and anomalous skewness and kurtosis*

Unfortunately for the MBS-theory, empirical studies strongly reject the Gaussian model. Figure 1.1, kindly provided by Ole E. Barndorff-Nielsen and Neil Shephard, shows non-parametric estimators and fitted non-Gaussian Generalized Hyperbolic model for several major world indices. It is clearly seen that the tails of the log-densities decay approximately linearly; if the Gaussian model fitted well, the empirical log-densities would have been parabolas. Thus, BM fits poorly. In particular, the tails decay much slower than the BM model suggests: "the tails are fat". One may also notice that the central part is fitted well by the Lévy stable distribution (we will comment on the last observation in the next subsections).

The deviation from the Gaussianity can also be easily inferred from anomalous values of skewness and kurtosis of empirical $p_{\Delta t}$. Recall that the *mean*, m; *variance*, σ^2; *skewness*, λ_3; and *kurtosis*, $\kappa = \lambda_4$, of a probability distribution $p(x)dx$ are defined as follows:

$$m := \langle x \rangle := \int_{-\infty}^{+\infty} xp(x)dx; \quad \sigma^2 := \langle (x-m)^2 \rangle;$$

$$\lambda_3 := \frac{\langle (x-m)^3 \rangle}{\sigma^3}; \quad \kappa := \frac{\langle (x-m)^4 \rangle}{\sigma^4}.$$

For a Gaussian p, $\lambda_3 = 0$ and $\kappa = 3$, whereas for empirical $p(x)dx = p_{\Delta t}(x)dx$, the skewness $\lambda_{\Delta t,3}$ is non-zero, and the kurtosis $\kappa_{\Delta t}$ can be much larger than 3, especially for small Δt.

There exists another important feature of the financial markets which is not captured by the classical model of financial markets: the *quasi long range dependencies*. Whereas the estimated autocorrelation functions based on log price differences on stocks or currencies are generally closely consistent with an assumption of zero autocorrelation, the empirical autocorrelation functions of the absolute values or the squares of the returns may stay positive for many lags.

Thus, the model for the price process in the MBS-theory is unrealistic but it might not have been a very serious problem if the answers provided by the theory were in good agreement with reality. Much thinner tails of probability density of BM mean that under the Gaussian modelling, the extreme events such as large drops of a price are assigned negligible probabilities. This leads to under-pricing of financial risks with severe consequences like the Long Term Capital Management disaster. Even if one does not require that the theory takes the extreme events into account properly, one should expect that in relatively calm periods of the market activity the theory performs well, but the MBS-theory noticeably fails in this respect, too.

The most celebrated result of the theory, the Black-Scholes option pricing formula, seemed to produce fairly good approximations to observed option prices during the first several years after its invention, but later persistent deviations have been noticed. After October 1987 crash, the systematic errors increased, and in fact, for a long time practitioners have been using the Black-Scholes formula simply as a coding machine to re-express and interpret the observed option prices. To explain how the machine works, we describe briefly the Black-Scholes market and formula.

1.1.2.2 *The Black-Scholes market and formula*

Consider the market of a riskless bond yielding the constant rate of return $r > 0$, and an asset, whose price follows the Geometric Brownian motion: $S_t = \exp X_t$, where X_t is the Brownian motion with the *drift* and *diffusion coefficients* μ and σ^2; in Finance, σ^2 is called the *volatility*.

Consider a contract which gives to its holder the right, but not the obligation, to buy the underlying asset for the specified price (the *strike price*), K, at the specified *expiry date*, T. This contract is called the *European call option*. Denote its price at time $t < T$, conditioned on the current price (*spot price*), S_t, of the underlying asset by $F_{\text{call}}(S_t, t)$. The *Black-Scholes formula* reads

$$F_{\text{call}}(S_t, t) = S_t N(d_1) - Ke^{-r\tau} N(d_2), \tag{1.2}$$

where $\tau = T - t$ is the *time to expiry*,

$$d_1 = \frac{\ln(S_t/K) + (r + \sigma^2/2)\tau}{\sigma\sqrt{\tau}}, \quad d_2 = d_1 - \sigma\sqrt{\tau},$$

and

$$N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2} dt$$

is the distribution function of the standard normal random variable. It is seen that for fixed r, T, t and S_t, Eq. (1.2) establishes a one-to–one correspondence between the option price and the volatility σ^2. Given the observed option price, we can use Eq. (1.2) to calculate the volatility which leads to this price. The volatility inferred in this way is called the *implied volatility*, as opposed to the *historic volatility* inferred from the observations of the dynamics of the price of the underlying asset. If the Black-Scholes formula were correct, the implied volatility, σ_{imp}, would have been independent of both the time to the expiry, τ, and the strike price, K, and equal to the historic volatility, σ. In reality, this is not the case.

1.1.2.3 *Volatility smile and volatility surface*

Figure 1.2 is a stylised graph of the implied volatility as a function of the moneyness S/K, τ being fixed. The reader can easily understand why it is called the *volatility smile*. If the Black-Scholes formula were correct, the graph would have been a straight line shown on the picture. Smiles of similar shapes were typically observed before October 1987 crash.

Later, the shape of the smile changed; sometimes, the smile is more pronounced, in other cases, it is downward sloping or upward sloping. It can even be a *frown*. In all cases, the shape of the smile depends on the time to the expiry, and becomes more pronounced for small τ. The graph of the implied volatility as a function of (K, τ) is called the *volatility surface*. The reader can find several examples of volatility surfaces in Chapter 4.

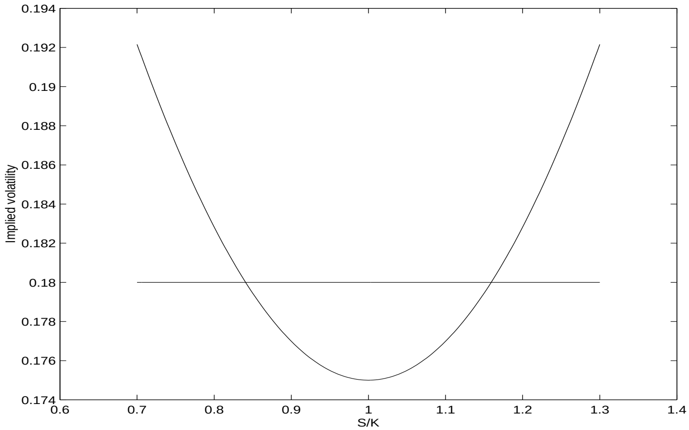

Fig. 1.2 Volatility smile

1.1.3 *Remedies for the MBS-theory*

1.1.3.1 *General remarks*

The volatility smile clearly demonstrates that the Black-Scholes model is not adequate. Some researches try to deduce the smile from various frictions and distortions in the market while keeping the Gaussian assumption intact, whereas others invent heuristic models to describe dynamics of the volatility surface. We believe that it is more reasonable to start with more realistic models for prices of the underlying securities rather than trying to mend the MBS-theory while remaining in the Gaussian world.

1.1.3.2 *Jump-diffusion models and stochastic volatility models*

The MBS-theory becomes more flexible if additional stochastic factors are introduced, as in the *jump-diffusion models*, where X_t is modelled as a mixture of independent BM and Poisson processes, or the diffusion coefficient itself is modelled as a stochastic process (stochastic volatility models). There is a good rationale in introducing jumps since jumps in prices are easy to notice, but jump-diffusion models either do not fit well to data, if the number of jumps is small, or they are not very tractable, if many jumps are allowed. *Stochastic volatility models* (*SV-models*) have the advantage of using an additional observable, the volatility, which increases the accuracy of the pricing, and they provide a modelling of the important quasi long range dependence typically observed in the financial markets, which is impossible with Lévy processes, but there is a price to pay: the usage of SV-models leads to PDE in the space of higher dimensions, and these PDE have variable coefficients. As a result, analytical formulas are usually not available, though there are some exceptional tractable cases for European options - see the commentary in the end of the chapter; for contracts with early exercise features, analytical formulas are unknown, and one has to resort to numerical methods from the very beginning.

There is another objection to the usage of jump-diffusion models, of a more fundamental nature: the empirical studies of high-frequency data suggest that the processes typically observed in Financial markets do not have a diffusion component at all. Notice that there are important situations, when the pricing under processes with a Gaussian component differs from the one under purely discontinuous processes not only quantitatively but qualitatively as well, the typical example being the pricing of a barrier option or touch-and-out option near the barrier.

1.1.3.3 *Lévy processes*

As a process with the independent stationary increments, the Brownian motion is a member of a wide family of *Lévy processes*. As early as in 1963, Mandelbrot suggested to use the *Lévy stable distributions* to model the returns in the Financial markets. Mandelbrot's idea is corroborated to some extent by the shape of the observed probability densities in the central part, where they are fitted well by Lévy stable distributions, but the tails of the Lévy stable distributions are too fat: polynomially decaying, whereas many empirical studies suggest that the tails decay exponentially; even worse, non-Gaussian stable distributions have infinite second moments, which contradicts the observed convergence of empirical probability distributions to Gaussian ones over longer time scale[2]. Finally, if X is the stable Lévy process under the measure chosen by the market, then the expectation of the stock price $E[e^{X_t}] = \infty$, which makes the model unsuitable for consistent pricing.

From the beginning of the 90th, several families of Lévy processes with probability densities having semi-heavy, that is, exponentially decaying

[2]The last observation explains why the Gaussian MBS-theory performs much better for options far from the expiry

tails[3] have been used to model stock returns and price options: *Variance Gamma Processes* (*VGP*), used by D. Madan with co-authors; *Normal Inverse Gaussian Processes* (*NIG*), used by O.E. Barndorff-Nielsen's group; *Hyperbolic Processes* and *Generalized Hyperbolic Processes* (HP and GHP) used by E. Eberlein's group; *Truncated Lévy Processes* (*TLP*, constructed by Koponen [4], used by J.-P. Bouchaud and his group, and extended by Boyarchenko and Levendorskiǐ; and *Normal Tempered Stable Lévy Processes* (*NTS Lévy* processes); we delegate the detailed discussion and references to Chapter 3. As A.N. Shiryaev remarked, the name TLP was misleading, and so we replace it with the name "*KoBoL processes*".

Processes of all the families listed above have been shown to fit better to the dynamics of historic prices, and pricing formulas for European options, based on these processes, also perform better than earlier models. Until recently, almost no effective analytical formulas in more difficult situations than the pricing of the European options and even simpler forwards and futures have been known, though by now, there are many papers and a couple of books devoted to various aspects of the general theory of modelling, pricing and hedging under Lévy processes. The main goal of the book is to partially fill in this gap, while remaining as close as possible to the initial Merton-Black-Scholes framework.

1.2 Regular Lévy Processes of Exponential type

When working with an empirical set of data, one specifies the type of the process, that is, chooses a parameterized family of processes, and then fits the parameters of the process to data. For a general theory like a non-Gaussian analogue of the MBS-theory, too specific information is not needed; in fact, unnecessary specification of the process makes the theory harder to understand. There are two ways to describe the properties of the process, which are sufficient but by no means necessary for our approach to work, and the classes of Lévy processes with exponentially decaying tails used in empirical studies of Financial markets and described in Subsection 1.1.3 enjoy these properties. From the point of view of Probability

[3] By semi-heavy or exponentially decaying tails we mean that the probability density behaves, for $x \to \pm\infty$, as $const \cdot |x|^{\rho_\pm} \exp(-\sigma_\pm |x|)$, for some $\rho_\pm \in \mathbf{R}$ and $\sigma_\pm > 0$.

[4] Non-infinitely divisible *truncated Lévy distributions* had been constructed earlier by R.N. Mantegna