tails[3] have been used to model stock returns and price options: *Variance Gamma Processes* (*VGP*), used by D. Madan with co-authors; *Normal Inverse Gaussian Processes* (*NIG*), used by O.E. Barndorff-Nielsen's group; *Hyperbolic Processes* and *Generalized Hyperbolic Processes* (HP and GHP) used by E. Eberlein's group; *Truncated Lévy Processes* (*TLP*, constructed by Koponen [4], used by J.-P. Bouchaud and his group, and extended by Boyarchenko and Levendorskiǐ; and *Normal Tempered Stable Lévy Processes* (*NTS Lévy* processes); we delegate the detailed discussion and references to Chapter 3. As A.N. Shiryaev remarked, the name TLP was misleading, and so we replace it with the name "*KoBoL processes*".

Processes of all the families listed above have been shown to fit better to the dynamics of historic prices, and pricing formulas for European options, based on these processes, also perform better than earlier models. Until recently, almost no effective analytical formulas in more difficult situations than the pricing of the European options and even simpler forwards and futures have been known, though by now, there are many papers and a couple of books devoted to various aspects of the general theory of modelling, pricing and hedging under Lévy processes. The main goal of the book is to partially fill in this gap, while remaining as close as possible to the initial Merton-Black-Scholes framework.

1.2 Regular Lévy Processes of Exponential type

When working with an empirical set of data, one specifies the type of the process, that is, chooses a parameterized family of processes, and then fits the parameters of the process to data. For a general theory like a non-Gaussian analogue of the MBS-theory, too specific information is not needed; in fact, unnecessary specification of the process makes the theory harder to understand. There are two ways to describe the properties of the process, which are sufficient but by no means necessary for our approach to work, and the classes of Lévy processes with exponentially decaying tails used in empirical studies of Financial markets and described in Subsection 1.1.3 enjoy these properties. From the point of view of Probability

[3]By semi-heavy or exponentially decaying tails we mean that the probability density behaves, for $x \to \pm\infty$, as $const \cdot |x|^{\rho_\pm} \exp(-\sigma_\pm |x|)$, for some $\rho_\pm \in \mathbf{R}$ and $\sigma_\pm > 0$.

[4]Non-infinitely divisible *truncated Lévy distributions* had been constructed earlier by R.N. Mantegna

Theory, the most natural description comes in terms of the Lévy density, which can be visualized as the density of jumps of the process, but from the point of view of PDO-theory and Analysis in general, it is more convenient to work with the characteristic exponent of the Lévy process.

1.2.1 *Characteristic function and exponent, Lévy measure and Lévy-Khintchine formula*

Let $X = \{X_t\}_{t\ge 0}$ be a one-dimensional Lévy process on a probability triple $(\Omega, \mathcal{F}, \mathbf{P})$, and let $E = E^{\mathbf{P}}$ be the expectation operator under $\mathbf{P}$[5]. The reader can view Ω as a set of all trajectories of the process, and the σ-algebra $\mathcal{F}$ of subsets of Ω as a collection of all possible events. $(\Omega, \mathcal{F}, \mathbf{P})$ is a *filtered space*, in the sense that $\mathcal{F} = \cup_{t\ge 0}\mathcal{F}_t$, $\mathcal{F}_s \subset \mathcal{F}_t$ for $s \le t$, and all $\mathcal{F}_t$ are σ-subalgebras of $\mathcal{F}$. Each $\mathcal{F}_t$ is generated by X_s, $0 \le s \le t$, that is, $\mathcal{F}_t$ is the smallest σ-subalgebra such that for any $0 \le s \le t$ and $-\infty \le a < b \le +\infty$, the preimage $X_s^{-1}((a, b))$ is in $\mathcal{F}_t$. Thus, $\mathcal{F}_t$ can be viewed as the information revealed at time t (we know the values, which the process X assumed up to the moment t).

Let X be a Lévy process on $\mathbf{R}$. Then the characteristic function of the distribution of the random variable X_t can be represented in the form $E[e^{i\xi X_t}] = \exp(-t\psi(\xi))$. The function ψ is called the *characteristic exponent* of X. By the *Lévy-Khintchine formula*, ψ admits the representation

$$\psi(\xi) = \frac{\sigma^2}{2}\xi^2 - i\gamma\xi + \int_{-\infty}^{+\infty}(1 - e^{i\xi x} + i\xi x\mathbf{1}_{[-1,1]}(x))\Pi(dx), \tag{1.3}$$

where $\sigma \ge 0$ and $\gamma \in \mathbf{R}$ are constants, and Π is a measure on $\mathbf{R} \setminus \{0\}$ with $\int_{-\infty}^{+\infty}\min\{1, x^2\}\Pi(dx) < +\infty$. The parameters σ^2 and Π, appearing in Eq. (1.3), are called the *Gaussian coefficient*, and the *Lévy measure*, and the triple (σ^2, γ, Π) is called the *generating triplet*. The density of Π is called the *Lévy density*. If $\Pi = 0$, the process is Gaussian, and if $\sigma^2 = 0$, the Lévy process is a pure non-Gaussian process, without the diffusion component.

Example 1.1 Let $\nu \in (0, 2)$, and $\psi(\xi) = c|\xi|^\nu$. This is the characteristic exponent of the *stable Lévy process* of order ν, and the Lévy measure is

$$\Pi(dx) = c\Gamma(-\nu)^{-1}|x|^{-\nu-1}dx.$$

[5] For the list of basic definitions of Probability Theory, see Chapter 2

For $\nu = 2$, $\psi(\xi) = |\xi|^\nu$ is the characteristic exponent of the standard Brownian motion, and $\Pi = 0$.

1.2.2 *Definition of RLPE in 1D*

Loosely speaking, a Lévy process X is called a *Regular Lévy Process of Exponential type* (RLPE) if its Lévy density has a polynomial singularity at the origin and decays exponentially at the infinity. An almost equivalent loose definition is: the characteristic exponent is holomorphic in a strip $\Im\xi \in (\lambda_-, \lambda_+)$, continuous up to the boundary of the strip, and admits the representation

$$\psi(\xi) = -i\mu\xi + \phi(\xi),$$

where $\phi(\xi)$ stabilizes to a positively homogeneous function at the infinity:

$$\phi(\xi) \sim c|\xi|^\nu, \quad \text{as } \xi \to \infty, \quad \text{in the strip } \Im\xi \in (\lambda_-, \lambda_+). \tag{1.4}$$

"Almost" means that though processes of BM, NIG, HP, GHP, KoBoL and NTS Lévy families satisfy conditions of both definitions, VGP satisfies the conditions of the first definition but not the second one, since the characteristic exponent behaves like $const \cdot \ln|\xi|$, as $\xi \to \infty$. For pricing of contingent claims of European type, the additional property Eq. (1.4) is not essential, but it is needed to obtain effective explicit formulas for the factors in the Wiener-Hopf factorization formula, which we need in the study of perpetual American options and barrier options. This is the reason why we will mainly use the second definition. The adjective "exponential" needs no explanation, and "regular" indicates that from the analytical point of view, RLPE is the most tractable subclass of Lévy processes, if the Brownian motion is not available (notice that BM is an RLPE). We will call ν the *order of the process*, λ_- and λ_+ the *steepness parameters*, and c the *intensity parameter* of the process. The λ_- (resp., λ_+) characterizes the rate of the exponential decay of the right (resp., left) tail of the probability densities, and c plays the part similar to the variance of the Brownian motion.

Example 1.2 For $\nu \in (0, 2]$, $\delta > 0, \alpha > |\beta|$, and $\mu \in \mathbf{R}$,

$$\psi(\xi) = -i\mu\xi + \delta[(\alpha^2 - (\beta + i\xi)^2)^{\nu/2} - (\alpha^2 - \beta^2)^{\nu/2}], \tag{1.5}$$

is the characteristic exponent of an RLPE of order ν, with the steepness parameters $\lambda_- = -\alpha + \beta$ and $\lambda_+ = \alpha + \beta$. With $\nu = 2$, we obtain

the characteristic exponent of the Brownian motion, and with $\nu = 1$, the characteristic exponent of the model NIG. When $\nu \neq 1, 2$, Eq. (1.5) gives the characteristic exponent of an NTS Lévy process.
Notice that if $\mu = 0$, then in the limit $\alpha \to 0$, Eq. (1.5) defines the characteristic exponent of the stable Lévy process.

1.2.3 *Infinitesimal generators of RLPE as PDO*

Let f belong to the space $C_0^2(\mathbf{R})$ of twice continuously differentiable functions vanishing at the infinity. Then for each $x \in \mathbf{R}$, there exists a limit

$$(Lf)(x) := \lim_{t\downarrow 0} \frac{E[f(x+X_t)] - f(x)}{t}, \tag{1.6}$$

and Lf is in $C_0(\mathbf{R})$, the space of continuous functions vanishing at the infinity. The map $f \mapsto Lf$ is called the *infinitesimal generator* of the process X. The infinitesimal generator admits an explicit representation in terms of the generating triplet:

$$Lf(x) = \frac{\sigma^2}{2} f''(x) + \gamma f'(x) + \int_{-\infty}^{+\infty} (f(x+y) - f(x) - \mathbf{1}_{\{|y|\leq 1\}}(y) f'(x)) \Pi(dy). \tag{1.7}$$

By using the Lévy-Khintchine formula, we derive from Eq. (1.7) the following formula for the action of L on oscillating exponents: for $\xi \in \mathbf{R}$,

$$Le^{ix\xi} = -\psi(\xi) e^{ix\xi}. \tag{1.8}$$

If f is sufficiently regular, for instance, f belongs to $\mathcal{S}(\mathbf{R})$, the space of functions of the class $C^\infty(\mathbf{R})$, rapidly decreasing at the infinity with all their derivatives, then we can decompose f into the Fourier integral

$$f(x) = (2\pi)^{-1} \int_{-\infty}^{+\infty} e^{ix\xi} \hat{f}(\xi) d\xi, \tag{1.9}$$

where $\hat{f}$ is the *Fourier transform* of f:

$$\hat{f}(\xi) = \int_{-\infty}^{+\infty} e^{-ix\xi} f(x) dx.$$

By applying Eq. (1.8) to Eq. (1.9), we obtain

$$Lf(x) = (2\pi)^{-1} \int_{-\infty}^{+\infty} e^{ix\xi} (-\psi(\xi)) \hat{f}(\xi) d\xi. \tag{1.10}$$

An operator of the form

$$Af(x) = (2\pi)^{-1} \int_{-\infty}^{+\infty} e^{ix\xi} a(x,\xi) \hat{f}(\xi) d\xi \tag{1.11}$$

is called a *pseudodifferential operator (PDO)* with the *symbol* a. Thus, L is a PDO with the symbol $-\psi$, and therefore one can use the well-developed machinery of the theory of PDO[6].

In the theory of PDO, the properties of symbols of operators are crucial: the more regular symbols are, the more explicit results can be obtained, and here is the list of important properties, which characteristic exponents of RLPE enjoy.

- From the point of view of the theory of PDO, Lévy processes are convenient since the symbols of the infinitesimal generators are independent of the state variable, x (in the PDO-language, these are *constant* symbols), and therefore, many results can be obtained by using the simplest tools of Complex Analysis.
- To study boundary value problems, explicit formulas for the factors in the Wiener-Hopf factorization are needed, and Eq. (1.4) allows one to derive relatively simple formulas. This explains why the regularity condition in the definition of RLPE is important [7].
- The derivatives of characteristic exponents of processes of model classes of RLPE grow at the infinity slower than the characteristic exponents themselves (cf. Eq. (1.5)); this property allows one to generalize the class RLPE and construct a general class of Lévy-like Feller processes with good properties (see Chapter 14).
- Moreover, in empirical studies of Financial markets, the characteristic exponents are characterized by a large parameter, like the parameter α in Eq. (1.5), and so it is possible to develop approximate effective formulas. We will pursue this possibility in Chapters 7-8 and 10-11, where we study first-touch digital options, barrier options, investment under uncertainty and endogenous default.

So, the characteristic exponents of RLPE's enjoy almost all desirable properties but one: the so-called *transmission property*, which ensures that a solution of a regular boundary value problem, e.g., the Dirichlet problem,

[6] Of course, all these definitions and constructions generalize to the n-dimensional case

[7] Eq. (1.4) can be relaxed: see Chapter 15

with smooth data is smooth up to the boundary; elliptic differential boundary value problems satisfy the transmission property, the stationary Black-Scholes equation being an example. If RLPE is not a Brownian motion, the transmission property fails, and typically, a solution behaves near the boundary point x_0 like $\sim const|x - x_0|^\kappa$, where $\kappa \in (0, 1)$ is determined by properties of the symbol. This observation indicates that near the boundary (e.g., near the barrier, for a barrier option), the difference between the answer the Gaussian model provides and the one, which an RLPE-model gives, can be very large indeed. It also allows us to obtain asymptotic formulas for a solution near the boundary. We use this possibility in Chapters 7-8 and 11, where we study the behaviour of prices of first-touch digitals and barrier options near the barrier, and junk bonds.

1.3 Pricing of contingent claims

The aim of this section is to introduce the basic notions of Mathematical Finance, namely, no-arbitrage assumption, equivalent martingale measure, complete and incomplete markets, redundant securities and replication, and hedging. First, we explain these notions in simple discrete-time models with a discrete space of states, next describe briefly the situation in the continuous time Gaussian Black-Scholes model, and after that discuss the pricing and hedging in the Lévy market.

1.3.1 *Discrete time models with a discrete space of states: No-arbitrage and equivalent martingale measures*

Consider a two-period model of a financial market with n securities. Let M be the number of possible states of the market tomorrow labelled by $\omega_1, \dots, \omega_m$, and the probability of the state ω_j is anticipated to be p_j. Thus, $\sum_{j=1}^{m} p_j = 1, 0 < p_j < 1$. The n securities are given by $n \times m$ matrix D, with D_{jk} denoting the number of units of account paid by security j in state k. Thus, j-row D_j is the vector of *payoffs* of security j. Let S_j be the price of security j. Consider a *portfolio* $\theta = (\theta_1, \dots, \theta_n) \in \mathbf{R}^n$, where θ_j denoted the number of shares of security j. If $\theta_j > 0$ (resp., $\theta_j < 0$), the investor is said to have a *long position* (resp., *short position*) in security j; both types of positions are allowed. Notice that $\theta_j < 0$ means that the investor must deliver θ_j shares of security j tomorrow. The portfolio θ