

with smooth data is smooth up to the boundary; elliptic differential boundary value problems satisfy the transmission property, the stationary Black-Scholes equation being an example. If RLPE is not a Brownian motion, the transmission property fails, and typically, a solution behaves near the boundary point  $x_0$  like  $\sim \text{const}|x - x_0|^\kappa$ , where  $\kappa \in (0, 1)$  is determined by properties of the symbol. This observation indicates that near the boundary (e.g., near the barrier, for a barrier option), the difference between the answer the Gaussian model provides and the one, which an RLPE-model gives, can be very large indeed. It also allows us to obtain asymptotic formulas for a solution near the boundary. We use this possibility in Chapters 7-8 and 11, where we study the behaviour of prices of first-touch digitals and barrier options near the barrier, and junk bonds.

### 1.3 Pricing of contingent claims

*The aim of this section is to introduce the basic notions of Mathematical Finance, namely, no-arbitrage assumption, equivalent martingale measure, complete and incomplete markets, redundant securities and replication, and hedging. First, we explain these notions in simple discrete-time models with a discrete space of states, next describe briefly the situation in the continuous time Gaussian Black-Scholes model, and after that discuss the pricing and hedging in the Lévy market.*

#### 1.3.1 *Discrete time models with a discrete space of states: No-arbitrage and equivalent martingale measures*

Consider a two-period model of a financial market with  $n$  securities. Let  $M$  be the number of possible states of the market tomorrow labelled by  $\omega_1, \dots, \omega_m$ , and the probability of the state  $\omega_j$  is anticipated to be  $p_j$ . Thus,  $\sum_{j=1}^m p_j = 1, 0 < p_j < 1$ . The  $n$  securities are given by  $n \times m$  matrix  $D$ , with  $D_{jk}$  denoting the number of units of account paid by security  $j$  in state  $k$ . Thus,  $j$ -row  $D_j$  is the vector of *payoffs* of security  $j$ . Let  $S_j$  be the price of security  $j$ . Consider a *portfolio*  $\theta = (\theta_1, \dots, \theta_n) \in \mathbf{R}^n$ , where  $\theta_j$  denoted the number of shares of security  $j$ . If  $\theta_j > 0$  (resp.,  $\theta_j < 0$ ), the investor is said to have a *long position* (resp., *short position*) in security  $j$ ; both types of positions are allowed. Notice that  $\theta_j < 0$  means that the investor must deliver  $\theta_j$  shares of security  $j$  tomorrow. The portfolio  $\theta$

has the market value  $S \cdot \theta = \sum_{j=1}^n S_j \theta_j$  and payoff  $D^T \theta$ . We say that a portfolio  $\theta$  is an *arbitrage portfolio* if  $S \cdot \theta \leq 0$  and  $D^T \theta > 0$ , or  $S \cdot \theta < 0$  and  $D^T \theta \geq 0$ . If an arbitrage portfolio exists, there exists an opportunity of free lunches. In real financial markets, arbitrage opportunities may appear but they are promptly eliminated due to the activity of *arbitrageurs*, who make money by looking for those opportunities. Thus, the assumption of no free lunches is sufficiently realistic, and as we will see, it can be used as a cornerstone of the pricing theory.

Introduce an augmented payoff matrix

$$\mathcal{R} = \begin{bmatrix} -S \\ D^T \end{bmatrix}.$$

The *no-arbitrage* assumption implies that there does not exist a portfolio  $\theta$  such that  $\mathcal{R}\theta \geq 0$  and one of the components of  $\mathcal{R}\theta$  is positive. By the separating hyperplane theorem, there exists a row vector  $\tilde{\lambda} \in \mathbf{R}_{++}^{m+1}$  such that

$$\tilde{\lambda} \mathcal{R} = 0. \tag{1.12}$$

Since  $\tilde{\lambda}$  is defined up to a positive scalar multiple, we may normalize to one its first component. Define by  $\lambda \in \mathbf{R}_{++}^m$  the vector of the last  $m$  components of  $\tilde{\lambda}$ . Then the definition of  $\mathcal{R}$  and Eq. (1.12) imply together that

$$S = \lambda D^T. \tag{1.13}$$

The  $\lambda$  is called the vector of state prices. Notice that Eq. (1.13) determines prices but makes no use of the probabilities  $p_j$  of the states of the market tomorrow. In other words, for the no-arbitrage pricing, only the information about possible future events but not their probabilities matter. Another interpretation is: if investors agree on the set of future events (possible values of securities prices tomorrow) and there are no arbitrage opportunities, they may disagree on probabilities of those events.

Suppose that our two-state model describes an investor who buys a portfolio of securities at time 0, and liquidates (that is, sells) it at time 1, with no dividends paid in between. Then  $D_{jk}$  admits a natural interpretation as the price of the security  $S_j$  in state  $k$ , at time 1. Denote  $S_j(0) = S_j$ , and define a random variable  $S_j(1, \cdot)$  on the probability space  $\Omega = \{\omega_k \mid k = 1, \dots, m\}$  by  $S_j(1, \omega_k) = D_{kj}$ . Assume further that one of the securities, say,  $S_n$ , is the riskless bond (usually denoted by  $B$ ) yielding

a riskless return  $r$ ; thus,  $B(1, \omega) = (1+r)B(0)$ . In this case, the last of the equations in the system Eq. (1.13) is

$$B(0) = \sum_{k=1}^m \lambda_k (1+r)B(0)$$

(notice that  $S$  in Eq. (1.13) is a row vector, so the equations are written in a row), and therefore, vector  $q = (1+r)\lambda$  satisfies the following two conditions:

$$0 < q_k < 1, \text{ all } k; \quad \sum_{k=1}^m q_k = 1. \quad (1.14)$$

Thus,  $\{q_k\}_{k=1}^m$  can be viewed as new probabilities of the states  $\{\omega_k\}$ ; let  $\mathbf{Q}$  denote the new probability measure on  $\Omega$ , and  $E^{\mathbf{Q}}$  the expectation operator under  $\mathbf{Q}$ . Introduce the discounted prices  $S_j^*(1, \omega) = (1+r)^{-1}S_j(1, \omega)$ , and rewrite Eq. (1.13) as the expectation under the new measure:

$$S(0) = \sum_{k=1}^m q_k S^*(1, \omega_k) := E^{\mathbf{Q}}[S^*(1)]. \quad (1.15)$$

The measure  $\mathbf{Q}$  is called an *equivalent martingale measure (EMM)*. In examples below, the  $n$ -th security is the riskless bond, and hence,  $D_n = [1 + r \cdots 1 + r]$ .

**Example 1.3** Let  $n = m = 2$ . If the first security is risky, then  $d_{11} \neq d_{12}$ , hence  $D_1$  and  $D_2$  are linearly independent, and the system Eq. (1.13) has the unique solution: the EMM exists and it is unique.

**Example 1.4** Let  $n = 2, m = 3$ . If the first security is risky, then  $D_1$  and  $D_2$  are linearly independent, therefore solutions to Eq. (1.13) exist; since  $m > n$ , there are infinitely many of them. In this example, there are infinitely many EMM.

**Example 1.5** Let  $n = 3, m = 3$ . Suppose that  $D_1 = \alpha D_2$  ( $\alpha \in \mathbf{R} \setminus \{0\}$ ). Then generically, Eq. (1.13) has no solutions. In this example, there is no EMM generically, and hence, it is possible to construct an arbitrage portfolio. We leave the construction as an exercise for the reader.

In the case of a similar  $(T+1)$ -period model of  $n$  stocks paying no dividends, the probability space  $\Omega$  consists of all trajectories of the  $n$ -dimensional process  $S = \{S(t)\}_{0 \leq t \leq T}$ . For a fixed  $t$ , and a collection of subsets  $\{R(s)\}_{0 \leq s \leq t}$ ,

$R(s) \in \mathbf{R}_{++}^n$ , denote by  $A = A(R(0), \dots, R(t))$  the set of trajectories, which pass via  $R(s)$  at time  $s = 0, \dots, t$ . Let  $\mathcal{F}_t$  denote the subalgebra of the algebra of subsets of  $\Omega$  generated by these sets of trajectories.  $\mathcal{F}_t$  can be interpreted as the algebra of events corresponding to the information available at time  $t$ . We have  $\mathcal{F}_s \subset \mathcal{F}_t$ ,  $s \leq t$ ; the collection  $\mathcal{F} := \{\mathcal{F}_t\}$  is called the *filtration*, and  $\Omega$  is called a *filtered space*. Suppose that a probability measure  $\mathbf{P}$  is introduced on the measure space  $(\Omega, \mathcal{F}_T)$ . We assume that any set in  $\mathcal{F}_0$  has the probability 0 or 1, that is, the prices today are known for sure.

An investor chooses an initial portfolio  $\theta(0)$ , and adjusts the portfolio  $\theta(t)$  at each moment  $t = 1, \dots, T - 1$  so that  $\theta(t - 1) \cdot S(t) = \theta(t) \cdot S(t)$  (we assume that she has no additional source of income). In making her decisions, the investor takes into account the information available at time  $t$ ; hence,  $\theta(t)$  is an  $(\Omega, \mathcal{F}_t)$ -random variable; one says that a *trading strategy*  $\theta = \{\theta(t)\}_{0 \leq t \leq T-1}$  is *adapted* to the filtration  $\mathcal{F}$ .

An adapted trading strategy  $\theta$  provides an *arbitrage opportunity* if and only if one of the following two conditions is satisfied:

- 1)  $S(0) \cdot \theta(0) \leq 0$ , and  $S(T) \cdot \theta(T - 1) \geq 0$  always, and  $E^{\mathbf{P}}[S(T) \cdot \theta(T - 1)] > 0$ ;
- 2)  $S(0) \cdot \theta(0) < 0$  and  $E^{\mathbf{P}}[S(T) \cdot \theta(T - 1)] \geq 0$ .

Assume that one of the securities is a riskless bond with the dynamics  $B(t) = (1 + r)^t B(0)$ . Introduce the discounted price process

$$S^*(t) = (1 + r)^{-t} S(t).$$

The same sort of argument as in the two-period model above shows that if there is no arbitrage opportunity, then there exists a new probability measure  $\mathbf{Q}$  on the same measure space  $(\Omega, \mathcal{F}_T)$  such that for all  $0 \leq s < t \leq T$ ,

$$S^*(s) = E^{\mathbf{Q}}[S^*(t) \mid \mathcal{F}_s], \tag{1.16}$$

where  $E^{\mathbf{Q}}[S \mid \mathcal{F}_s]$  denotes the conditional expectation of a random variable  $S$  given  $\mathcal{F}_s$ . Since the set of events has not changed, the measures  $\mathbf{P}$  and  $\mathbf{Q}$  are equivalent, and Eq. (1.16) means that the discounted price processes are martingales under  $\mathbf{Q}$ . We see that if there is no arbitrage, there exists an EMM. It can be shown that the converse is also true: if there exists an EMM, there is no arbitrage.

**Example 1.6** In the multi-period setting, analogues of Examples 1.3-

1.4 are binomial model and trinomial model, respectively. In the binomial model, at each time step,  $S(t+1)/S(t)$  assumes the value  $u$  with probability  $p \in (0, 1)$ , and the value  $d$  with probability  $1 - p$ ; in the trinomial model,  $S(t+1)/S(t)$  can assume three values, with probabilities  $p_j \in (0, 1)$ ,  $j = 1, 2, 3$ .

In the binomial model, the EMM is unique, and in the trinomial, there are infinitely many EMM. Clearly, the latter is more flexible in the sense that it is much easier to adjust its parameters to data.

### 1.3.2 *Discrete time models with a discrete space of states: Completeness of the market, and pricing of derivative securities*

For simplicity, consider the two-period model. A *contingent claim* is a contract with the specified payoff  $F(\omega)$  for each state of the market tomorrow. The market is called *complete* if any contingent claim can be *replicated*, that is, if  $\text{Im}D^T = \mathbf{R}^m$ . In other words, for any contingent claim, one can construct a portfolio (*replicating portfolio*)  $\theta$  such that  $F^T = D^T\theta$ . An equivalent condition is: the payoff vectors of basic securities (that is, the rows of the payoff matrix  $D$ ) span  $\mathbf{R}^m$ , which implies  $n \geq m$ . In other words, the number of spanning securities is not less than the number of future states of the world. In a complete market, Eq. (1.13) has a unique solution, and hence, there exists a unique EMM. It is clear that EMM is unique if and only if  $n \geq m$  and  $\text{rank}D = m$  is maximal. If the market is complete, and  $n > m$ , we can choose a basis of  $\mathbf{R}^m$  from the rows of the payoff matrix  $D$ . Suppose that the first  $m$  rows constitute a basis. Then the first  $m$  securities can be used to span the others; thus, the latter are *redundant*.

**Example 1.7** In the two securities-two states model, suppose that the payoff on the risky security in the first state of the world tomorrow is greater than the one in the second state, that is,  $d_{11} > d_{12}$ . Introduce a call option on the risky security, with the strike price  $K$ ,  $d_{12} < K < d_{11}$ . An option owner will exercise the option and buy 1-security tomorrow if the first of the possible states of the world materializes, and will not otherwise. Hence, the payoff row  $F(1)$  is  $[d_{11} - K \ 0]$ , and the price  $F(0)$  of the call option today is

$$F(0) = \lambda \cdot F(1)^T = \lambda_1(d_{11} - K) + \lambda_2 0 = \lambda_1(d_{11} - K), \quad (1.17)$$

where  $\lambda$  are determined from Eq. (1.13).

In a complete financial market, it is possible to perfectly *hedge*. To hedge means to reduce risk against market fluctuations by making appropriate transactions. In a complete market, the risk can be completely eliminated: let  $F$  be a contingent claim, and  $\theta$  a replicating portfolio. Then the portfolio  $(F, -\theta)$  is riskless: in each possible state of the world tomorrow, the payoff of this portfolio is 0, and  $-\theta$  is the perfect hedge for  $F$ .

If the market is incomplete, then  $\text{rank}D < n$ , and one can introduce additional securities in order to obtain a complete market. The creation of *derivative securities*, e.g., options of different kind, in real financial markets may be seen as attempts to make the market more complete and increase the possibility to hedge.

**Example 1.8** Consider the two securities-three states model, which is incomplete, and hence it is impossible to hedge against some financial risks in the market: if  $F^T \notin \text{Im}D^T$ , then there is no hedge for  $F$ . Suppose that  $d_{11} \geq d_{12} > d_{13}$ , and introduce the call option on the risky security, with the strike price  $K$ ,  $d_{13} < K < d_{12}$ . Now we have the market with three securities and the payoff matrix

$$D = \begin{bmatrix} d_{11} & d_{12} & d_{13} \\ d_{11} - K & d_{12} - K & 0 \\ 1 + r & 1 + r & 1 + r \end{bmatrix}$$

The reader can easily verify that  $\text{rank}D = 3$  if and only if  $d_{11} \neq d_{12}$ . Thus, in the case  $d_{11} = d_{12}$  the option is redundant, and in the case  $d_{11} > d_{12}$  its introduction makes the market complete.

Had the real financial markets been complete, there would have been no need in the creation of derivative securities. This observation implies that models of incomplete financial markets are more realistic than models of complete markets.

In multi-period models, the situation is similar, only the role of a portfolio is played by a trading strategy (another name: dynamic portfolio). For details, see references in the review of literature at the end of the chapter.

### 1.3.3 *Absence of arbitrage, EMM and completeness in the Gaussian Black-Scholes model market*

Under certain regularity assumptions, all of the properties listed above—the no-arbitrage, completeness, in particular, possibility of replication of options and perfect hedge of options, and the existence of the unique EMM—hold in the Black-Scholes market. The no-arbitrage is equivalent to the existence of EMM. Moreover, it is possible to derive the formula for EMM by looking for the perfect hedging strategy. So, the Black-Scholes model has all the nice features one can imagine but it implies that there is no need to introduce options at all: all of them are redundant.

The reader can learn the Gaussian theory of financial markets from many excellent books - see the review of literature.

### 1.3.4 *Sufficient condition for no-arbitrage in a Lévy market and incompleteness of a Lévy market. The pricing formula for contingent claims of European type and the problem of a choice of EMM*

Consider a continuous-time model market of a riskless bond, the riskless rate of return being  $r > 0$ , and a risky stock. Suppose that the price of the stock evolves as  $S(t) = \exp X(t)$ , where  $X$  is a Lévy process under the *historic measure*<sup>8</sup>  $\mathbf{P}$ . From general results due to Delbaen and Schachermayer (1994), it follows that the existence of EMM  $\mathbf{Q}$ , which is absolutely continuous with respect to the historic measure, is equivalent to the no-arbitrage condition. So, as in the situations above, we can calculate prices by using an EMM, but if  $X$  is neither the Brownian motion nor the Poisson process, an EMM is not unique, and the market is incomplete. Moreover, typically there are infinitely many different EMM; we discuss the restrictions on the choice of EMM from a given class in Chapter 4. The first restriction is quite universal; we will refer to it as the *EMM-condition*.

In the continuous-time models, the discounted price process is given by  $S^*(t) = e^{-rt}S(t)$ , therefore, by applying Eq. (1.16) with  $s = 0$  to the riskless bond and to the stock with the price dynamics  $B(t) = B(0)e^{rt}$  and  $S(t) = S(0)\exp X(t)$ , respectively, and using the definition of the characteristic

<sup>8</sup>That is, the measure inferred from the observations of returns

exponent, we obtain, for each  $t \geq 0$ :

$$B(0) = B(0)e^{-t\psi^{\mathbf{Q}}(0)}, \quad \text{and} \quad S(0) = S(0)e^{-t(r+\psi^{\mathbf{Q}}(-i))},$$

hence  $\psi^{\mathbf{Q}}(0) = 0$ , which is satisfied for any process without killing, and

$$r + \psi^{\mathbf{Q}}(-i) = 0. \tag{1.18}$$

We call Eq. (1.18) the EMM-condition. There are more subtle restrictions; in particular, parameters  $c$  and  $\nu$  in Eq. (1.4) must be the same for the historic measure and an EMM. Still, for any model class of RLPE, free parameters remain, and one can introduce additional degrees of freedom by considering mixtures of models processes. In Section 4, we produce numerical results to show how one can change the price of an option and the shape of the smile by playing with parameters of KoBoL.

Notice that if  $X$  is assumed to be Gaussian both under the historic measure and an EMM, then the condition:  $c$  in Eq. (1.4) is fixed means that the variance of the process does not change under the change of the measure, and the EMM-requirement Eq. (1.18) fixes the drift  $\mu$  of an EMM:

$$r - \mu - \frac{\sigma^2}{2} = 0,$$

that is,

$$\psi^{\mathbf{Q}}(\xi) = \frac{\sigma^2}{2}\xi^2 - i(r - \frac{\sigma^2}{2})\xi. \tag{1.19}$$

The oldest variant of EMM is the *Esscher transform*, which have been used in Actuarial Science for several decades, and in Financial Mathematics, from the beginning of the last decade: in terms of the characteristic exponent, one looks for  $\psi^{\mathbf{Q}}$  in the form

$$\psi^{\mathbf{Q}}(\xi) = \psi^{\mathbf{P}}(\xi - i\theta) - \psi^{\mathbf{P}}(-i\theta), \tag{1.20}$$

where  $\theta$  is real, and the EMM-requirement Eq. (1.18) leads to the equation for  $\theta$

$$r + \psi^{\mathbf{P}}(-i(\theta + 1)) - \psi^{\mathbf{P}}(-i\theta) = 0. \tag{1.21}$$

For a chosen EMM  $\mathbf{Q}$  and an European option with the expiry date  $T$  and the *terminal payoff*  $g(X_T)$ , the pricing formula Eq. (1.16) can be written as

$$F(S_t, t) = (2\pi)^{-1} \int \exp[ix\xi - \tau(r + \psi^{\mathbf{Q}}(\xi))] \hat{g}(\xi) d\xi, \tag{1.22}$$

where  $x = \ln S_t$ ,  $\tau = T - t$ , with the integration over an appropriate line  $\Im\xi = \sigma$ ;  $\sigma \in (\lambda_-, \lambda_+)$  is determined by the type of the growth of the payoff  $g$  at the infinity. To deduce Eq. (1.22), it suffices to decompose  $g$  into the Fourier integral

$$\hat{g}(\xi) = \int_{-\infty}^{+\infty} e^{-ix\xi} g(x) dx, \quad (1.23)$$

substitute into Eq. (1.16) and use the definition of the characteristic exponent.

**Example 1.9** Consider a European call option with the strike price  $K$  and the expiry date  $T$ . The terminal payoff is  $g(X(T)) = (e^{X(T)} - K)_+$ , and the integral in Eq. (1.23) is well-defined for  $\xi$  in the half-plane  $\Im\xi < -1$ . Take any  $\sigma < -1$ , and calculate, for  $\Im\xi = \sigma$ :

$$\begin{aligned} \hat{g}(\xi) &= \int_{\ln K}^{+\infty} (e^{(-i\xi+1)x} - Ke^{-i\xi x}) dx & (1.24) \\ &= \frac{Ke^{-i\xi \ln K}}{i\xi - 1} - \frac{Ke^{-i\xi \ln K}}{i\xi} \\ &= \frac{Ke^{-i\xi \ln K}}{(i\xi - 1)i\xi} \\ &= -\frac{Ke^{-i\xi \ln K}}{(\xi + i)\xi}. \end{aligned}$$

By substituting into Eq. (1.22), we obtain

$$F(S_t, t) = -\frac{K}{2\pi} \int_{-\infty+i\sigma}^{+\infty+i\sigma} \frac{\exp[ix\xi - \tau(r + \psi^{\mathbf{Q}}(\xi))]}{(\xi + i)\xi} d\xi. \quad (1.25)$$

The reader may be tempted to decompose the integral in the RHS of Eq. (1.22), as one does in the Gaussian case but she should not do that: it makes any numerical integration procedure to perform worse, since the integrand vanishes at infinity more slowly after the decomposition has been made.

The integral in Eq. (1.22), Eq. (1.25) being an example, can be calculated by using the Fast Fourier Transform (FFT) or its modifications; there are cases when FFT performs poorly, and other methods must be used. In some cases, the probability densities of probability distributions of a process can be calculated explicitly, and then the pricing formula can be written

essentially in the same spirit as in the Gaussian case: only instead of the tabulated normal distribution another distribution is used.

When fitting the model to real data, the observed prices of the option are compared with the ones calculated from Eq. (1.22). There are many papers devoted to the derivation of the formula for EMM from the formula for the historic measure by using different heuristic and economic arguments (see a discussion in Chapter 4). We believe that the parameters of EMM should be inferred from data on both the stock and options, and not derived by some formal reasoning.

The perfect hedging in the Lévy market is impossible, and an investor can only try to minimize the risk, after some measure of the risk is chosen.

### **1.3.5 *On pricing based on the utility maximization***

The reader acquainted with Mathematical Finance has noticed that we never mentioned the utility maximization of an investor. Apart from the desire to make the book as short as possible, there is a more serious reason for this omission. Though the concept of the utility of an economic agent is very important for Economics and can be used to obtain many important *qualitative* conclusions, it is hardly suitable for the realistic *quantitative* analysis. In Economic literature, there is a host of utility functions with very different properties, some criticism of the concept of utility optimization per se, and the disagreement about time horizon of economic agents, but even if we assume that the class of the utility functions for the investor is known and the investor optimizes her life-time utility as it is usually assumed (because it greatly simplifies the problem, not that it is very realistic), one may notice that the utility is not directly observable, and it is impossible to fit the parameters of the utility function to the data with any reasonable degree of accuracy. The no-arbitrage assumption and parameter fitting for EMM from a chosen class seems to be a much more reasonable procedure, and traders are known to dislike the utility optimization approach as well. On the other hand, one needs utility optimization in models of financial markets with frictions, for instance, for option pricing under transaction costs. The study of such situations goes beyond the scope of this book, anyway.