## 1.4 The Generalized Black-Scholes equation

### 1.4.1 *The informal derivation*

Let $\mathbf{Q}$ be an EMM chosen by the market. Consider a contingent claim whose life span is a deterministic time interval $[0, T]$, with terminal payoff $g$. Let $\Delta t$ be small, and $t < T$, $t + \Delta t < T$. We can apply Eq. (1.22) with $t + \Delta t$ instead of $T$ and $f(X(t + \Delta t), t + \Delta t)$ instead of $g(X_T)$:

$$f(x, t) = \int_{-\infty}^{+\infty} \exp[ix\xi - \Delta t(r + \psi^{\mathbf{Q}}(\xi))]\hat{f}(\xi, t + \Delta t)d\xi, \tag{1.26}$$

where $\hat{f}$ is the Fourier transform of $f$ w.r.t. the first argument. By expanding in the power series in $\Delta t$, dividing by $\Delta t$ and passing to the limit as $\Delta t \to +0$, we obtain

$$\int_{-\infty}^{+\infty} e^{ix\xi}(-(r + \psi^{\mathbf{Q}}(\xi)) + \partial_t)\hat{f}(\xi, t)d\xi = 0. \tag{1.27}$$

By using the definition of PDO, we can write Eq. (1.27) as

$$\partial_t f(x, t) - (r + \psi^{\mathbf{Q}}(D_x))f(x, t) = 0, \quad t < T. \tag{1.28}$$

The same result can be obtained in a more transparent form, by writing Eq. (1.26) as

$$f(x, t) = \exp[-\Delta t(r + \psi^{\mathbf{Q}}(D))]f(x, t + \Delta t), \tag{1.29}$$

and expanding into the power series in Eq. (1.29) rather than in Eq. (1.26).

If we add to Eq. (1.28) the terminal condition

$$f(x, T) = g(x), \quad x \in \mathbf{R}, \tag{1.30}$$

and specify the type of the behaviour at the infinity (say, the price $f(x, t)$ of the call option is bounded by $e^x$, due to the evident no-arbitrage argument), we obtain the well-posed problem Eq. (1.28), Eq. (1.30). By solving it, we can recover Eq. (1.22) we started with. Certainly, there is not much sense in this procedure, but for more complicated contingent claims, no explicit pricing formula is known in advance, and so the reduction to an appropriate boundary problem with subsequent solution by analytical methods is reasonable.

Of course, to justify the formal derivation of Eq. (1.28), some regularity conditions must be imposed. We will not discuss them here, because the analytical derivation, which we have outlined, is inferior to the derivation based on Dynkin's formula or potential theory for Lévy processes: the latter can be applied to complex situations, where the life-time of a contingent claim is stochastic, whereas the naive derivation is applicable only in the case of the deterministic life-time.

Eq. (1.28) is the natural generalization of the Black-Scholes equation. To see this, recall that $D_x = -i\partial_x$, and substitute the characteristic exponent Eq. (1.19) into Eq. (1.28):

$$\partial_t f(x,t) - \left(r - (r - \frac{\sigma^2}{2})\partial_x - \frac{\sigma^2}{2}\partial_x^2\right) f(x,t) = 0,$$

or

$$\partial_t f + \frac{\sigma^2}{2}\partial_x^2 f + \left(r - \frac{\sigma^2}{2}\right)\partial_x f - rf = 0. \tag{1.31}$$

This is the Black-Scholes equation. In terms of the variable $S = e^x$, Eq. (1.31) assumes the standard form

$$\partial_t f + \frac{\sigma^2}{2} S^2 \partial_S^2 f + rS\partial_S f - rf = 0. \tag{1.32}$$

### 1.4.2 An outline of the reduction of the pricing of contingent claims to boundary value problems for the generalized Black-Scholes equation: barrier options

For the first illustration, consider the *down-and-out call* option; this is an example of a *barrier option*. Unlike a European option, the life-time of a barrier option is random: should the price of the stock fall below the specified value $H$ (the *barrier*) before the expiry date $T$, the option expires worthless or the option owner is entitled to some *rebate* $g^r(X_t, t)$. In the Gaussian case, the trajectories of the process are continuous, and hence the rebate must be specified at the barrier only but in the case of a Lévy process with discontinuous trajectories, the rebate must be specified for all values of $X_t = \ln S_t$ below the barrier $h := \ln H$. We conclude that the price of the down-and-out call option is the solution to the following boundary value problem:

$$\partial_t f(x,t) - (r + \psi^{\mathbf{Q}}(D_x))f(x,t) = 0, \quad t < T, \tag{1.33}$$

$$f(x,T) = (e^x - K)_+, \quad x > h; \tag{1.34}$$

$$f(x,t) = g^r(x,t), \quad x \le h, t \le T; \quad (1.35)$$
$$f(x,t) = e^x + O(1), \quad x \to +\infty. \quad (1.36)$$

The last condition is justified as follows: if there is no barrier, we have the call option, for which Eq. (1.36) holds. The influence of the barrier decreases as the distance from it grows, hence the introduction of the barrier can change only by the $O(1)$ term in Eq. (1.36).

So far, the derivation of the problem Eq. (1.33)–Eq. (1.36) was formal. The rigorous derivation is based on one of the fundamental results of the general theory of Markov processes: Dynkin's formula. First, we introduce the early expiration region $B := \{(x,t) \mid x \le h\}$, next, we denote by $\tau(B)$ the hitting time of $B$ by the two-dimensional process $\hat{X}_t = (X_t, t)$ (for the definitions, see Chapter 2), and then we express the price of the option as the sum of two expectations, the first one corresponding to the payoff in the case of the early expiration, and the second one to the case when the price of the underlying stock never crosses the barrier till the expiry date:

$$\begin{aligned} f(x,0) &= E\left[e^{-r\tau(B)}g^r(X(\tau(B)) \mid X(0) = x,\ \tau(B) \le T\right] \quad (1.37)\\ &+ E\left[e^{-rT}(e^{X(T)} - K)_+ \mid X(0) = x,\ \tau(B) > T\right]. \end{aligned}$$

Let $\hat{L} = \partial_t + L = \partial_t - \psi^{\mathbf{Q}}(D_x)$ be the infinitesimal generator of the two-dimensional process $\hat{X}$. Dynkin's formula reads, for any stopping time $\tau$:

$$\begin{aligned} f(x,0) &= E\left[\int_0^\tau e^{-rs}(r - \hat{L})f(X(s),s)ds \mid X(0) = x\right] \quad (1.38)\\ &+ E\left[e^{-r\tau}f(X(\tau),\tau) \mid X(0) = x\right]. \end{aligned}$$

Apply Eq. (1.38) with $\tau = \tau(B) \wedge T$, and compare with Eq. (1.37). In view of Eq. (1.34) and Eq. (1.35), we conclude that Eq. (1.33) must hold (this requires justification: see Chapter 2). Thus, the naive way of writing the boundary value problem for the price of an option is correct.

In many cases, for RLPE in particular, the justification is simplified significantly by using the potential theory of Lévy processes.

### 1.4.3 *The case of interest bearing securities*

Now suppose that during the life-time prior to expiry date $\tau$ (random or non-random), the owner of the security is entitled to a stream of revenues with the density $g^o(X(t), t)$. The typical examples are bonds of corporations: a typical bond pays some interest during its life-time; when the bond matures, it pays the principal, and in the (random) event of the default, the bond expires worthless or some amount is paid. In this case, the price of the security is the sum of the stochastic integral which expressed the present value of the stream of revenues, and the expectation of the payoff $g^e(X(\tau), \tau)$ on the expiration date:

$$\begin{aligned} f(x,0) &= E\left[\int_0^\tau e^{-rs} g^o(X(s),s)ds \mid X(0)=x\right] \\ &+ E\left[e^{-r\tau} g^e(X(\tau),\tau) \mid X(0)=x\right]. \end{aligned} \tag{1.39}$$

If $\tau$ is the hitting time of a closed region $B \subset \mathbf{R} \times [0, +\infty)$, then by comparing Eq. (1.39) with Eq. (1.38), we obtain the boundary problem

$$\partial_t f(x,t) - (r + \psi^{\mathbf{Q}}(D_x))f(x,t) = g^o(x,t),\ (x,t) \in B^c; \tag{1.40}$$

$$f(x,t) = g^e(x,t),\ (x,t) \in B. \tag{1.41}$$

### 1.4.4 *The generalized Merton-Black-Scholes theory*

The problem Eq. (1.40)-Eq. (1.41) is the general form of most of the problems which will be considered in this book, be they option pricing problems, investment problems in Real Options theory (the problem of the capital accumulation is similar but more complicated) or pricing of defaultable bonds. In each case, we know what the stream of revenues during the life-time is, and what is the payoff at the expiry. The revenue density appears in the RHS of the generalized Black-Scholes equation, and the payoff at the expiry – in the RHS of the boundary condition. This procedure can be applied formally though the rigorous treatment requires an additional justification. As in the case of the barrier option below, usually the second expectation in Eq. (1.39) admits a natural representation as a sum of two (or more) expectations, and then the boundary condition Eq. (1.41) can be naturally written as a system of boundary conditions. Usually, the first one comes from the terminal condition (the payoff at the deterministic moment $T$), and the second one represents the early exercise (or expiration) payoff.

Additional complications arise when the owner of the security can choose the exercise time, that is, the region $B$ (usually subject to some restrictions), the typical example being American options, but in such cases, we also write down a problem of the form Eq. (1.40)-Eq. (1.41) and solve it by analytical means. This is what we call the generalized Merton-Black-Scholes theory.

### 1.4.5 *Optimal stopping problems and the smooth pasting condition*

Consider the *American put option*, with the strike price $K$ and the expiry date $T$. If exercised, it gives the option owner the payoff $g(x) = K - e^x$. Options of the European type can be exercised only at the expiry date, but the owner of an option of the American type has the right to exercise the option at any time $t \in [0, T]$. In other words, she chooses the optimal *early exercise boundary* $x = h(t)$ so that the option is exercised if and only if at time $t$, $X_t = \ln S_t$ reaches $h(t)$ or falls below $h(t)$, and in this case, she receives $K - e^x$. Introduce the exercise region $B := \{(x, t) \mid x \le h(t), t \le T\}$. Thus, we have a free boundary value problem, when both the exercise region $B$ and the solution to the problem Eq. (1.40)–Eq. (1.41) must be found so that the solution $f(x, t) = f(B; x, t)$ be maximal. In the Gaussian theory, one usually finds the candidate for the boundary of $B$ by the so-called *smooth pasting condition* (another name: the *smooth fit principle*). In the case of processes with jumps, this principle may fail, and so we solve the problem Eq. (1.40)–Eq. (1.41) for any candidate for the optimal exercise boundary, and choose the right one by using the explicit formula for $f(x, t)$. Since the explicit formula is available only in relatively simple cases, we manage to realize this program for *perpetual American options*, when the time horizon is infinite, and for some approximate discretization procedures for American options with the finite time horizon. Notice, however, that even in the Gaussian case the explicit solution for American options with the finite time horizon is not available.

### 1.4.6 *The case of a dividend-paying stock*

Suppose that the stock pays dividends at the constant rate $\lambda > 0$. Then the no-arbitrage argument shows that the process $\{e^{-(r+\lambda)t} S_t\}$ must be a martingale under the measure chosen by the market. In this case, we prefer to use the name a *risk-free measure* instead of EMM. The treatment of all the problems remains essentially the same, although in some cases, the results change. In particular, in the no-dividend case, it is non-optimal to exercise the American call option prior to expiry whereas in the case of a dividend-paying stock, there exists the early exercise boundary $S = H^*(t)$ such that the exercise is optimal when the spot price $S_t$ reaches the level $H^*(t)$ or crosses it.

## 1.5 Analytical methods used in the book

### 1.5.1 *The Fourier transform, and Complex Analysis*

The Fourier transform is needed to write an analytic expression for the price of contingent claims of European type, in the form of the oscillating integrals; to calculate these integrals explicitly, simplify them or obtain approximate formulas, standard tools from Complex Analysis—the Cauchy theorem and the Residue theorem—are needed. All these tools are used in more complex situations as well.

### 1.5.2 *The Wiener-Hopf factorization and the Wiener-Hopf equation*

If the price of a claim is independent of time, the examples being perpetual American options and infinitely lived bonds, the generalized Black-Scholes equation becomes the stationary Black-Scholes equation, and we have to solve the boundary-value problems of the type:

$$(r + \psi^{\mathbf{Q}}(D))f(x) = g^o(x), \quad x > h; \tag{1.42}$$

$$f(x) = g^r(x), \quad x \le h. \tag{1.43}$$

By introducing a new unknown $u = f - g^r$, we reduce to the problem

$$(r + \psi^{\mathbf{Q}}(D))u(x) = G(x), \quad x > h; \tag{1.44}$$

$$u(x) = 0, \quad x \le h. \tag{1.45}$$

The problem Eq. (1.44)–Eq. (1.45) is called the *Wiener-Hopf equation*, and it can be solved by the *Wiener-Hopf factorization method*. The following scheme can be realized under fairly weak regularity assumptions on the data and the symbol $a(\xi) := r + \psi^{\mathbf{Q}}(\xi)$; in the case of an RLPE, the latter is sufficiently regular.

*Step 1.* Factorize $a(\xi)$, that is, represent it in the form

$$a(\xi) = a_+(\xi)a_-(\xi), \quad \xi \in \mathbf{R}, \tag{1.46}$$