dividend-paying stock, there exists the early exercise boundary $S = H^*(t)$ such that the exercise is optimal when the spot price S_t reaches the level $H^*(t)$ or crosses it.

1.5 Analytical methods used in the book

1.5.1 *The Fourier transform, and Complex Analysis*

The Fourier transform is needed to write an analytic expression for the price of contingent claims of European type, in the form of the oscillating integrals; to calculate these integrals explicitly, simplify them or obtain approximate formulas, standard tools from Complex Analysis—the Cauchy theorem and the Residue theorem—are needed. All these tools are used in more complex situations as well.

1.5.2 *The Wiener-Hopf factorization and the Wiener-Hopf equation*

If the price of a claim is independent of time, the examples being perpetual American options and infinitely lived bonds, the generalized Black-Scholes equation becomes the stationary Black-Scholes equation, and we have to solve the boundary-value problems of the type:

$$(r + \psi^{\mathbf{Q}}(D))f(x) = g^o(x), \quad x > h; \tag{1.42}$$

$$f(x) = g^r(x), \quad x \le h. \tag{1.43}$$

By introducing a new unknown $u = f - g^r$, we reduce to the problem

$$(r + \psi^{\mathbf{Q}}(D))u(x) = G(x), \quad x > h; \tag{1.44}$$

$$u(x) = 0, \quad x \le h. \tag{1.45}$$

The problem Eq. (1.44)–Eq. (1.45) is called the *Wiener-Hopf equation*, and it can be solved by the *Wiener-Hopf factorization method*. The following scheme can be realized under fairly weak regularity assumptions on the data and the symbol $a(\xi) := r + \psi^{\mathbf{Q}}(\xi)$; in the case of an RLPE, the latter is sufficiently regular.

Step 1. Factorize $a(\xi)$, that is, represent it in the form

$$a(\xi) = a_+(\xi)a_-(\xi), \quad \xi \in \mathbf{R}, \tag{1.46}$$

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where a_+ (resp., a_-) admits the analytic continuation into the half-plane $\Im\xi > 0$ (resp., $\Im\xi < 0$), and does not vanish there. If $a_\pm$ and $1/a_\pm$ are polynomially bounded in the corresponding half-plane, the factors are uniquely defined up to scalar multiples, due to the Liouville theorem, so in some cases, they can be guessed.

The factorization problem can be solved for fairly wide classes of symbols, by using analytical methods, and these constructions admit generalizations for the multidimensional case, when one considers the Wiener-Hopf equation in the half-space $x_n > 0$, and the analytic continuation is made with respect to the dual variable ξ_n.

When $\psi^{\mathbf{Q}}$ is the characteristic exponent of a one-dimensional Lévy process, the Wiener-Hopf factorization has an important probabilistic interpretation in terms of the *supremum* and *infimum* processes $M_t = \sup_{0\le s\le t} X_s$ and $N_t = \inf_{0\le s\le t} X_s$:

$$r(r + \psi^{\mathbf{Q}}(\xi))^{-1} = \phi_r^+(\xi)\phi_r^-(\xi), \tag{1.47}$$

where

$$\phi_r^+(\xi) = rE\left[\int_0^{+\infty} e^{-rt}e^{i\xi M_t}dt\right], \tag{1.48}$$

$$\phi_r^-(\xi) = rE\left[\int_0^{+\infty} e^{-rt}e^{i\xi N_t}dt\right]. \tag{1.49}$$

Certainly, Eq. (1.48)–Eq. (1.49) are not explicit, and some additional efforts are to be made to produce analytical formulas.

Step 2. The solution to the problem Eq. (1.44)–Eq. (1.45) in a natural function class exists (for instance, this is the case if the data are smooth and of compact support, and the solution is sought in the class of continuous bounded functions); the solution is unique and given by

$$u = \phi_r^-(D)\mathbf{1}_{(h,+\infty)}\phi_r^+(D)r^{-1}lG, \tag{1.50}$$

where lG is any sufficiently regular continuation of G from $(h, +\infty)$ on $\mathbf{R}$.

Step 3. Write down the RHS in Eq. (1.50) explicitly, by using the definition of PDO, and calculate it by using tools of Complex Analysis.

We would like to make several general comments.

(1) Though the Wiener-Hopf equation is not widely known in Economics and Finance, Steps 1–2 are easy to memorize and apply.

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(2) The third step requires the knowledge of some technique of the reader, and in the end the final answer has to be computed numerically by using some integration procedure.
(3) Fortunately, for wide classes of processes and parameters values observed in real financial markets, effective approximate formulas are available, which we demonstrate in the main part of the book.
(4) The very form of Eq. (1.50) allows us to derive an algebraic equation for the optimal exercise price of the American perpetual options for fairly general class of payoffs, and for the investment threshold in Real Option theory; this equation is new even in the Gaussian case.
(5) The same form allows us to formulate the correct form of the Marshallian Law and naturally separate the two factors which influence the capital accumulation. These results are important for Economics, and are new in the Gaussian case as well.
(6) Finally, if the reader is willing to work with non-Gaussian Lévy processes, there is no hope to use differential equations as in the Black-Scholes theory anyway: Steps 1-3 are the simplest general scheme available.

In the case of American options with the finite time horizon, the explicit analytical methods are not available, but after the discretization of time, the Wiener-Hopf method can be applied as well.

1.5.3 *The case of the non-stationary Black-Scholes equation and the constant barrier*

Here the model example is the problem Eq. (1.33)–Eq. (1.36), with the possible inclusion of a non-zero term in the RHS of Eq. (1.33). If the data are independent of t, one can make the Fourier transform w.r.t. t (or the Laplace transform, depending on the taste of the reader) and reduce the problem to the family of problems on the line, which can be solved by the Wiener-Hopf method. By making the inverse transform, we obtain the result. In the case of more general payoffs, we introduce a new unknown satisfying the homogeneous boundary condition, and consider the resulting problem as the Cauchy problem on $(-\infty, T)$ with the data at $t = T$, for the ordinary differential operator of the first order, with the operator-valued coefficient. Once again, the problem can be solved by inverting a family of the Wiener-Hopf problems and two integration procedures.

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1.5.4 *The case of the non-constant barrier and multi-asset contracts*

In this book, we consider only some special cases, when the reduction to one-dimensional problems on the line is possible. In the theory of PDO, there exists a well-developed machinery for handling multi-dimensional problems of this sort, approximate and numerical methods including.

1.5.5 *Pseudodifferential operators*

In all parts of the book but one, only PDO with constant symbols arise, and the corresponding part of the theory of PDO is, essentially, a part of Complex Analysis. However, in the study of NIG-like Feller processes, whose infinitesimal generators are PDO with non-constant symbols, all the main ingredients of the theory of PDO are needed.

1.6 An overview of the results covered in the book

1.6.1 *Elements of the theory of Lévy processes*

In Chapter 2, we list necessary definitions and results from the general theory of Lévy processes, and discuss in more detail the reduction of pricing problems to boundary value problems for the generalized Black-Scholes equation. In Chapter 3, RLPE are introduced, their main properties are derived, and model classes of RLPE are compared. The properties of the infinitesimal generators are discussed, and it is explained how one naturally comes to the definition of the class RLPE by using "naive" PDO-considerations. Explicit formulas for the factors in the Wiener-Hopf factorization formula are obtained.

1.6.2 *Option pricing*

In Chapter 4, we consider contingent claims of European type. We discuss the properties of the generalized Black-Scholes and the dependence of the properties on the choice of EMM. We calculate prices of several types of options, and produce numerical examples to show how the prices and volatility smiles depend on the choices of parameters of the model for the process and EMM. We derive an explicit formula for the locally risk-minimizing hedging ratio, which can be viewed as an analytical realization of M. Schweizer's

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