1.5.4 *The case of the non-constant barrier and multi-asset contracts*

In this book, we consider only some special cases, when the reduction to one-dimensional problems on the line is possible. In the theory of PDO, there exists a well-developed machinery for handling multi-dimensional problems of this sort, approximate and numerical methods including.

1.5.5 *Pseudodifferential operators*

In all parts of the book but one, only PDO with constant symbols arise, and the corresponding part of the theory of PDO is, essentially, a part of Complex Analysis. However, in the study of NIG-like Feller processes, whose infinitesimal generators are PDO with non-constant symbols, all the main ingredients of the theory of PDO are needed.

1.6 An overview of the results covered in the book

1.6.1 *Elements of the theory of Lévy processes*

In Chapter 2, we list necessary definitions and results from the general theory of Lévy processes, and discuss in more detail the reduction of pricing problems to boundary value problems for the generalized Black-Scholes equation. In Chapter 3, RLPE are introduced, their main properties are derived, and model classes of RLPE are compared. The properties of the infinitesimal generators are discussed, and it is explained how one naturally comes to the definition of the class RLPE by using "naive" PDO-considerations. Explicit formulas for the factors in the Wiener-Hopf factorization formula are obtained.

1.6.2 *Option pricing*

In Chapter 4, we consider contingent claims of European type. We discuss the properties of the generalized Black-Scholes and the dependence of the properties on the choice of EMM. We calculate prices of several types of options, and produce numerical examples to show how the prices and volatility smiles depend on the choices of parameters of the model for the process and EMM. We derive an explicit formula for the locally risk-minimizing hedging ratio, which can be viewed as an analytical realization of M. Schweizer's idea of locally risk minimizing hedging. We show that for RLPE of order less than 2, the hedging ratio is Hölder continuous until expiry, even at the strike, whereas the Gaussian delta-hedge is discontinuous at the strike, at the expiry. This makes non-Gaussian RLPE-hedging much more stable than the Gaussian one. We produce numerical examples and compare the hedging ratio for different processes.

In Chapter 5, we consider perpetual American options. We formulate the optimal stopping problem, and the corresponding free boundary problem. We explicitly solve the problem for a fairly general payoff g, and show that in the case of the put options and similar more general options, the optimal exercise price $H = e^h$ is determined from the equation

$$(\phi_q^-(D)^{-1}g)(h) = 0,$$

where ϕ_q^- is determined from Eq. (1.49) with $q = r + \lambda$, where $\lambda \geq 0$ is the dividend rate. In particular, for the put, the equation reduces to

$$H = K\phi_q^-(-i) = KqE\left[\int_0^{+\infty} e^{-qt+N_t}dt\right],$$

where N_t is the infimum process.

In the case of calls, the exercise is optimal only if $\lambda > 0$ (which is the well-known fact), and then the optimal exercise price is determined by

$$H = K\phi_q^+(-i) = KqE\left[\int_0^{+\infty} e^{-qt+M_t}dt\right],$$

where M_t is the supremum process. For more general call-like options, the optimal exercise price is determined from the equation

$$(\phi_q^+(D)^{-1}g)(h) = 0.$$

By using these explicit formulas, we show that in some cases, the smooth pasting principle fails, and discuss its generalizations.

In Chapter 6, we consider the American options with the finite time horizon. The explicit formulas being non-available even in the Gaussian model, we consider RLPE-analogues of several approximate methods used in the Gaussian case. It is shown that the behaviour of the RLPE- price of the American put near expiry drastically differs from the one in the Gaussian case.

In Chapters 7 and 8, we derive explicit pricing formulas for touch-and-out options and barrier options, respectively. We consider the asymptotics of the price near the barrier, and explain why this is the place, where the Gaussian model differs most from non-Gaussian ones.

In Chapter 9, we consider simplest of the multi-asset contracts, and multi-asset hedging. We show that in the case of highly correlated assets in the portfolio, the difference between the Gaussian (seemingly) riskless portfolio and the RLPE-locally risk-minimizing one can be quite substantial. Notice that the main cause for the Long Term Capital Management disaster was the mispricing of risks, and the hedging model we suggest is not much more difficult to implement than the Gaussian hedging.

1.6.3 *Investment under uncertainty and capital accumulation*

In Chapter 10, we consider a risk-neutral, competitive, and value maximizing firm under demand uncertainty. The firm chooses optimal investment strategies; the investment is irreversible. This is a typical problem of Real Options theory (for the Gaussian variant of the theory, see Dixit and Pindyck (1996)). The problems of irreversible entry (resp., exit) from the market are quite similar to the option pricing problem for perpetual calls (resp., puts), if the possibility of the further expansion is ruled out. In the model of sequential capital accumulation, additional subtle points arise.

About a century ago, Marshall suggested a rule that a firm should *invest as long as the present value of expected marginal revenue is not less than the marginal cost of investment.* However, as Dixit and Pindyck (1996) pointed out, this rule did not take into consideration option-like characteristics of investment opportunities. The methods we introduce allow us to restate the *Marshallian law* as follows. Starting with the original price process, define a new process, called the infimum process for the price of the firm's output: $N_t = \inf_{0\le s\le t} P_t$. Then the correct investment rule is: *in the formula for the profit function, replace the price process with the infimum process started at the current level of the price and invest as long as the present value of expected marginal revenue is not less than the marginal cost of investment.* This rule is applicable when the price can move in both directions.

We also write down an analytic formula for the expected level of the capital stock in terms of the infimum and supremum processes.

1.6.4 *Endogenous default and pricing of the corporate debt*

In Chapter 11, we extend the structural approach to credit risk modelling for the case of non-Gaussian processes. The central feature of this approach is an attempt to model explicitly the evolution of the assets of the firm. We focus on the choice of optimal capital structure and bankruptcy level by a firm which keeps a constant profile of debt, but we model the evolution of the assets of the firm as a Lévy process. Notice that models of credit risk relying on diffusion process for dynamics of the firms' assets cannot capture the basic features of credit risk observed in practice. In particular, empirical studies show that the credit spreads on corporate bonds are too high to be matched by diffusion approach. Also, under a diffusion process, firms never default by surprise because a sudden drop in the firm's value is impossible. Therefore, if a firm is not currently in financial distress, the probability of its default on very short-term debt is zero. This fact implies that the credit spreads tend to zero as the maturity of debt tends to zero, which contradicts empirical evidence.

We suggest using RLPE for problems of credit risk modelling and endogenous default as more realistic than diffusion processes and almost as tractable analytically as the latter. For a benchmark model, where the debt has infinite maturity, we present analytical solutions for the endogenous variables. In more complicated situations, an asymptotic solution may be possible.

We study the firm near the bankruptcy and suggest two types of approximate formulas for all endogenous variables. The first set of approximate formulas can be used very close to the default level, and the second set of approximate formulas is valid for Lévy processes with large truncation parameters, i.e., for the processes with not very fat tails, and the firm not very close to the bankruptcy level. These approximations do not neglect the effects of non-Gaussianity on one hand, and on the other hand, they allow to perform comparative statics analysis. Using the approximate solutions, we derive the optimal leverage for the firm. We produce several series of numerical examples which show that if the firm's value is not very far from the default level, the difference between the Gaussian and non-Gaussian models can be sizable, and very close to the bankruptcy level – very large indeed.

1.6.5 *Numerical methods*

In the literature on the non-Gaussian option pricing, one sees the statements like: “The Fast Fourier Transform (FFT) can be used”. In Chapter 12, we discuss the limitations of FFT and suggest a very fast method for calculation of prices of European options under model classes of RLPE of order $\nu \in (0, 1]$, that is, NIG, and KoBoL and NTS Lévy processes of order $\nu \in (0, 1)$. We have not studied the new method for HP and GHP different from NIG. For calculation of individual option prices, the method is several hundred times (for options out of the money or deep in the money) faster than FFT. For parameter fitting purposes, it is necessary to calculate the prices of options for many values of the stock price, and here FFT with its ability to calculate the prices at many points at once may have some advantage. However, if one is satisfied with fitting of option prices for a hundred values of the stock prices or so, which is usually the case, then the new method is faster than FFT for processes of order less than one. For options near expiry, it is several times faster; the relative advantage of IAC decreases as the time to the expiry increases, and increases as the order of the process $\nu \downarrow 0$. For KoBoL of order 0.2, say, IAC can be 10-20 times faster than FFT, and by using additional devices, it is possible to enhance IAC further. For NIG, IAC is faster for options near expiry, and slower for options 3-5 days to maturity or more. The IAC-method can be applied for option pricing under Variance Gamma Processes as well, and in this situation, IAC is especially effective.

IAC has the following additional advantage: it allows for a fairly effective error estimates, which is impossible with FFT.

1.6.6 *Extensions*

In Chapter 13, we consider a discrete-time model, corresponding to perpetual American options, that is, Bermudan options. From the analytic point of view, there is not much difference with the corresponding continuous time models, but the model becomes (infinitely) more flexible.

In Chapter 14, we consider the simplest extension to the case of Feller processes: NIG-like Feller processes constructed by Ole E. Barndorff-Nielsen and Levendorskiǐ (2001). This is the first step for developing non-Gaussian analogues of interest rate models.

1.6.7 *Basics of PDO theory*

In the last two chapters, we provide the essentials of the theory of PDO used in the book. This is useful for a reader interested in the systematic exposition of technical tools used throughout the book, though we tried to explain the most essential things in the main body of the book. There is another aim of the chapter: a specialist with the PDE-background tends to regard the Black-Scholes equation as a primitive object of Mathematical Finance, and the exposition in Wilmott *et al.* (1993) is a model example of this approach. The basic theory presented in Chapter 15 allows the reader to calculate prices of contingent claims formally, without mentioning stochastic integrals, thereby providing the background for PDE-approach in non-Gaussian situations.

1.7 Commentary

The systematic exposition of the Gaussian theory of Mathematical Finance, at different levels, can be found in many monographs. Duffie (1996) is a very good book on the level intermediate between Economics and Mathematics; Hull (2000) is, probably, the best professional book on derivatives; Karatzas and Shreve (1997) and Shiryaev (1999) are excellent rigorous mathematical texts (in addition, Shiryaev (1999) discusses various fundamental aspects of non-Gaussian Mathematical Finance and empirical facts). The last two books and Musiela and Rutkowski (1997), which reviews hundreds of papers and results, focus mainly on martingale methods, whereas Wilmott *et al.* (1993, 1995) and Kwok (1998) use the PDE-approach and discuss relevant numerical methods. The monograph Dixit and Pindyck (1996) is an excellent exposition of the Gaussian Real Options theory for Economists, in the PDE-framework (mainly, 1D-case).

Non-Gaussian models (stable Lévy processes) have been introduced to Finance by Mandelbrot (1963); see also Fama (1965) and the collection of papers Mandelbrot (1997). On truncated Lévy distributions and their application to empirical studies of Financial markets, see Mantegna and Stanley (2000) and Bouchaud and Potters (2000). The last book contains theoretical results on non-Gaussian pricing and hedging of European options (under processes of Koponen's family as well), futures and forwards, and heuristic approximate methods for portfolio optimization and some other problems.