Preface

By now, the drawbacks of the Gaussian modelling in Financial Markets and Investment under Uncertainty are well-known. In particular, Gaussian models cannot produce so-called fat tails of observed probability densities, which leads to under-pricing of financial risks. One can hardly make a mistake by saying that the under-pricing of the risk was the main reason for the Long Term Capital Management disaster or recent failures of rating agencies to warn investors of a series of the defaults of the investment-graded firms.

The purpose of the book is to introduce an analytically tractable and computationally effective class of non-Gaussian models for shocks (Regular Lévy Processes of Exponential type (RLPE)), and related analytical methods similar to the initial Merton-Black-Scholes approach, which we call the Merton-Black-Scholes theory (MBS-theory). The potential range of applications of the non-Gaussian variant of the MBS-theory is huge, and the list of results we have obtained so far does not exhaust all the possibilities.

As applications to Financial Mathematics, we solve pricing problems for several types of perpetual American options, barrier options, touch-and-out options and some other options, provide analogues of several approximate methods for pricing of American options in the finite horizon case, and deduce explicit analytical formulas for the locally risk-minimizing hedging. We suggest fast computational procedures for pricing of European options; they can be used for hedging and pricing of American and barrier options as well.

As applications to Corporate Governance, we consider problems of endogenous default, pricing of bonds of corporations, yield spreads of junk

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bonds, optimal leverage, optimal timing of investment under uncertainty and optimal choice of the installed capital, and the capital accumulation under non-Gaussian uncertainty. In particular, the correct form of the Marshallian law is suggested, and numerical results are produced to show how large the difference between prescriptions of Gaussian models and non-Gaussian ones can be. We also consider discrete time analogues of perpetual American options and the problem of the optimal choice of capital, and outline several possible directions in which the methods of the book can be developed further.

We tried to choose applications interesting for Financial engineers, specialists in Financial Economics, Real Options, and Partial Differential Equations (PDE) (especially in Pseudodifferential Operators (PDO)), and we hope that specialists in Stochastic Processes will benefit from the usage of PDO-technique in non-Gaussian situations. We also tried to make the book accessible for graduate students in relevant areas and mathematicians without prior knowledge of Finance and Economics.

As in the MBS-theory, we reduce problems of pricing of options, other derivative securities and corporate bonds, optimal timing of investment etc. to boundary value problems for the (generalized) Black-Scholes equation. In the MBS-theory, when stochastic processes are assumed Gaussian, the Black-Scholes equation is a differential one, while in the non-Gaussian case, it is a non-local pseudodifferential (or integro-differential) equation, the difference of properties being the same as the one between the infinitesimal generators of Gaussian Feller processes and non-Gaussian ones. We use Dynkin's formula, and basic results from the potential theory for the Markov processes, which is simpler than the modern sophisticated approach based on the theory of semimartingales. We hope that the reader will find refreshing a book on Mathematical Finance and Investment under Uncertainty, which makes no use of the Itô formula at all.

After a boundary value problem for the generalized Black-Scholes equation is developed, we use the PDO-technique to find the solution. The analytical part is unavoidably more difficult than the PDE-part in the Gaussian MBS-theory but we manage to use the most simple part of the theory of PDO by restricting ourselves to the case of RLPE: the generators of RLPE are PDO with constant symbols having fairly favourable properties. This is the reason why we solve the boundary value problems for the generalized Black-Scholes equation in the case when the shocks are modelled as RLPE but not as more general Feller processes. We may say that RLPE is the

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simplest class of processes with stationary independent increments if the Brownian motion (BM) is not available. The PDO-technique used in the book can be applied for a wide class of strong Feller processes generalizing the class RLPE in the same spirit as Gaussian diffusions generalize BM. To illustrate this point, we construct a class of Lévy-like Feller processes, and show how to price European options under these processes.

Certainly, it was difficult to aim simultaneously at economists and mathematicians from different fields: almost each part of the book may seem trivial to one of the groups of readers whereas the other groups may find this very part illuminating. Probably, it would be better to write a separate book for each group of readers, and in fact, we had already started to write a book on Pseudodifferential Operators with Applications to Finance and Economics, when during a discussion with Ole E. Barndorff-Nielsen it was decided that a book aimed at a wider audience should be written first.

Having in mind a diverse audience, we tried to write a book so that it was simple in the beginning and more technical in further chapters. The main part of the book, especially chapters on Investment under Uncertainty and Endogenous Default, are written (almost) as an economic publication though economists may find some parts of the exposition too terse. We wrote a detailed Introduction in order to explain informally the main ingredients of our approach, so that the reader can read any part of the book she is interested in conjunction with the Introduction and Chapter 3 only; for the reader who is interested in all technical details, in Chapter 2 we list main definitions and results of the theory of Lévy processes, which we use in the book, and provide a scheme for the reduction of the pricing problem to a boundary value problem for the generalized Black-Scholes equation; and we finish the book with rigorous proofs of the most technical statements and with a systematic list of the results of the theory of PDO, which are used in the book.

During our work on various topics covered in the book, we benefited from illuminating comments and suggestions by I. Bouchouev, J. Cvitanić, A. Dixit, D. Duffie, J. M. Harrison, I. Karatzas, G. Peskir, A. N. Shiryaev, and Ken-Iti Sato; long discussions with E. Eberlein and O. E. Barndorff-Nielsen were especially useful.

We are thankful to our son, Dmitriy Boyarchenko, who read the manuscript, finding errors, suggesting improvements in exposition, and correcting our grammar.

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0.0.1 *General notation*

We denote the real (complex) n-dimensional space by $\mathbf{R}^n$ ($\mathbf{C}^n$); $\mathbf{R}^n_+$ denotes the cone of real vectors with non-negative components, and $\mathbf{R}^n_{++}$ stands for the cone of vectors with positive components. For $x, y \in \mathbf{R}^n$, $x \geq y$ means $x - y \in \mathbf{R}^n_+$, and $x > y$ means $x - y \in \mathbf{R}^n_{++}$. $\mathbf{Z}_+$ stands for the set of non-negative integers.

Unless otherwise stated, $\langle x, y \rangle$ denotes the standard scalar product of vectors $x, y \in \mathbf{R}^n$, and $|\cdot|$ denotes the standard norm in $\mathbf{R}^n$ (or in $\mathbf{C}^n$, depending on the context). The Lebesgue measure on $\mathbf{R}^n$ is denoted by dx, and the notation for partial derivatives used in the book is $\partial_j = \frac{\partial}{\partial x_j}$, $D_j = -i\partial_j$, where $i = \sqrt{-1}$.

For $x \in \mathbf{R}^n$ and a *multi-index* $\alpha = (\alpha_1, \ldots, \alpha_n) \in (\mathbf{Z}_+)^n$, set $|\alpha| = \alpha_1 + \ldots + \alpha_n$,

$$x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}, \ \partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}, \ D^\alpha = D_1^{\alpha_1} \cdots D_n^{\alpha_n}.$$

If a is a function on $\mathbf{R}^n$, and α is a multi-index, then $a^{(\alpha)} = \partial^\alpha a$; and if a is a function on $\mathbf{R}^n_x \times \mathbf{R}^n_\xi$, then $a^{(\alpha)}_{(\beta)}(x, \xi) = \partial^\alpha_\xi D^\beta_x a(x, \xi)$.

For $U \subset \mathbf{R}^n$, $\mathbf{1}_U$ denotes the indicator function of U: $\mathbf{1}_U(x) = 1$ if $x \in U$ and 0 otherwise.

For real a, b, set $a \wedge b := \min\{a, b\}$, $a \vee b := \max\{a, b\}$, and $a_+ = a \vee 0$, $a_- = a - a_+$.

If B is viewed as a subset of a set U, then $B^c := U \setminus B$ denotes the complement of B in U.

For a subset B of a topological space, $\bar{B}$ denotes the closure of B, and B^o the interior.

$\mathcal{S}(\mathbf{R}^n)$ denotes the space of infinitely smooth functions vanishing at the infinity faster any power of $|x|$, together with all derivatives, and $C_0(\mathbf{R}^n)$ is the space of continuous functions vanishing at the infinity.

If B_1, B_2 are normed spaces, and $A : B_1 \to B_2$ is a bounded linear operator then $||A||_{B_1 \to B_2}$ denotes the operator norm of A.

For a positive integer m, $C^m_0(\mathbf{R}^n)$ (resp., $C^m(\mathbf{R}^n)$) denotes the space of m times continuously differentiable functions vanishing at infinity (resp., with each derivative up to order m uniformly bounded); the notation $L_p(\mathbf{R}^n)$, $p \in [1, +\infty]$, is also standard.

Depending on the context, $\mathcal{F}$ denotes the Fourier transform or a filtration on the filtered probability space Ω. The Fourier transform of a function f is denoted by $\hat{f}$.

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