Chapter 1

Metric Spaces

1-1 Standard Finite Dimensional Vector Spaces

1-1.1. Most treatments of functional analysis are applicable to both real and complex cases. In order to unify our notation, let $\mathbb{K}$ denote either the real field $\mathbb{R}$ or the complex field $\mathbb{C}$. Write $i^2 = -1$. The conjugate of a complex number z will be denoted by z^-, the real part by Re(z) and the imaginary part by Im(z) respectively. Scalar-valued maps are normally called *functions* in this book.

1-1.2. Let E be a vector space. A function $x \to \|x\|$ from E into $\mathbb{R}$ is called a *norm* on E if for all $x, y \in E$, we have

(a) $\|x\| \geq 0$, positive ;

(b) $\|x\| = 0$ iff $x = 0$, non-degenerate ;

(c) $\|x + y\| \leq \|x\| + \|y\|$, triangular inequality ;

(d) $\|\lambda x\| = |\lambda| \; \|x\|$, for every $\lambda \in \mathbb{K}$, scalar multiplication.

A vector space together with a given norm is called a *normed space*. Norms generalize the concept of absolute values of numbers.

1-1.3. **Example** The vector space $\mathbb{K}^n$ consists of columns of n numbers in $\mathbb{K}$ but for convenience we shall frequently write them as rows: $x = (x_1, x_2, \cdots, x_n)$ where x_j is the j-th coordinate of x. The same notation will be applied to other letters without further specification. For each $x \in \mathbb{K}^n$, let

$$\|x\|_1 = |x_1| + |x_2| + \cdots + |x_n|$$

and

$$\|x\|_\infty = \max\{|x_1|, |x_2|, \cdots, |x_n|\}.$$

It can be easily verified that they are norms on $\mathbb{K}^n$. We shall write $\mathbb{K}_1^n$ and $\mathbb{K}_\infty^n$ to indicate the normed spaces with specific norms in use. Note that we implicitly assume $n \geq 1$.

1-1.4. **Schwartz's Inequality** For all $x, y \in \mathbb{K}^n$, let $< x, y > = \sum_{j=1}^n x_j y_j^-$. Then we have $| < x, y > | \leq \|x\| \; \|y\|$. The number $< x, y >$ is called the *inner product* of x, y. Chapter 13 on Hilbert spaces should be read concurrently.

Proof. Consider the special case when all coordinates x_j, y_j are positive. Since

$$\|x\|^2 + 2t < x, y > +t^2\|y\|^2 = \sum_{j=1}^n (x_j + ty_j)^2 \geq 0$$

for all real number t, the discriminant of the above positive definite quadratic form in a real variable t must be negative, i.e. $< x, y >^2 - \|x\|^2\|y\|^2 \leq 0$ which gives the result. The general case is obtained from the following simple calculation: $| < x, y > | \leq \sum_{j=1}^n |x_j|\ |y_j| \leq \|x\|\ \|y\|$. $\square$

1-1.5. **Example** The expression

$$\|x\| = \sqrt{|x_1|^2 + |x_2|^2 + \cdots + |x_n|^2} = \sqrt{< x, x >}, \quad \forall\, x \in \mathbb{K}^n$$

defines a norm on $\mathbb{K}^n$ which is called the *usual norm* or the Euclidean norm. According to coordinate geometry, $\|x\|$ is the distance from the origin to x. We shall write $\mathbb{K}_2^n$ to indicate this norm. Whenever no norm is mentioned explicitly, the usual norm is assumed.

Proof. We shall prove the triangular inequality only and leave the other verification as an exercise. Observe that

$$(\|x + y\|\ )^2 = \sum_{j=1}^n (x_j + y_j)(x_j + y_j)^- = \sum_{j=1}^n \{x_j x_j^- + x_j y_j^- + x_j^- y_j + y_j y_j^-\}$$
$$= \sum_{j=1}^n \{x_j x_j^- + 2\mathrm{Re}(x_j y_j^-) + y_j y_j^-\} \leq \sum_{j=1}^n \{|x_j|^2 + |y_j|^2 + 2|x_j|\ |y_j^-|\}$$
$$\leq \|x\|^2 + \|y\|^2 + 2\|x\|\ \|y\| = (\|x\| + \|y\|)^2.$$

Taking square root gives $\|x + y\| \leq \|x\| + \|y\|$. $\square$

1-1.6. **Exercise** Let E be a normed space. The set $\{x \in E : \|x\| = 1\}$ is called the *unit sphere* of E. Sketch the unit spheres of the normed spaces $\mathbb{R}_1^2$, $\mathbb{R}_2^2$ and $\mathbb{R}_\infty^2$.

1-1.7. **Exercise** Let E, F be normed spaces. For every vector (x, y) in the product vector space $E \times F$, let

$$\|(x, y)\|_1 = \|x\| + \|y\|,$$

$$\|(x, y)\|_\infty = \max\{\|x\|, \|y\|\}$$

and

$$\|(x, y)\|_2 = \sqrt{\|x\|^2 + \|y\|^2}.$$

Prove that these are all norms on $E \times F$. The product set $X \times Y$ together with one of the above norms is called a *product normed space*.

1-1.8. **Exercise** Prove the inequality : $\big|\ \|x\| - \|y\|\ \big| \leq \|x - y\|$ for all x, y in a normed space.

1-2 Convergent Sequences in Metric Spaces

1-2.1. In analysis, we are interested in the concept of approximation and convergence which will be described in term of distance between two points in spaces. A metric space does not require any algebraic structure. However apart from the discrete metric spaces, practically all metric spaces in this book can be regarded as subsets of normed spaces.

1-2.2. Let X be a non-empty set. A function $d : X \times X \to \mathbb{R}$ is called a *metric* if for all $x, y, z \in X$, we have
(a) $d(x, y) \geq 0$, positive;
(b) $d(x, y) = 0$ iff $x = y$, non-degenerate;
(c) $d(x, y) = d(y, x)$, symmetric;
(d) $d(x, z) \leq d(x, y) + d(y, z)$, triangular inequality.
The ordered pair $X[d]$ is called a *metric space*. For simplicity we shall use the symbol d for all metrics whenever there is no ambiguity. Hence we write X instead of $X[d]$.

1-2.3. **Example** Let X be a non-empty set. For all $x, y \in X$, let $d(x, y) = 1$ if $x \neq y$ and $d(x, y) = 0$ otherwise. Then d is obviously a metric on X called *discrete metric*. In this case, $X[d]$ is called a *discrete metric space*.

1-2.4. **Exercise** Let X be a subset of a normed space E. For all $x, y \in X$, let $d(x, y) = \|x - y\|$. Show that d is a metric on X. In particular, every normed space is a metric space.

1-2.5. Let X be a set. A map from the subset $\{1, 2, 3, \cdots\}$ of integers into X is called a *sequence* in X. Because we always emphasize on the image of the function we shall adopt the notation $\{x_n : n \geq 1\}$ or simply $\{x_n\}$. For definiteness and convenience, we always assume that the starting index is 1 unless it is specified otherwise.

1-2.6. Let X be a metric space. A sequence $\{x_n\}$ in X is said to *converge to a point* $b \in X$ if for every $\varepsilon > 0$, there is an integer p such that for every $n \geq p$, we have $d(x_n, b) \leq \varepsilon$. In this case, the point b is called the *limit* of $\{x_n\}$. We shall write $x_n \to b$ or $\lim x_n = b$ as $n \to \infty$. Clearly a sequence $\{x_n\}$ converges to $b \in X$ iff the sequence $\{d(x_n, b)\}$ of real numbers converges to $0 \in \mathbb{R}$.

1-2.7. **Theorem** Every convergent sequence $\{x_n\}$ has a unique limit.

Proof. Suppose to the contrary that $a \neq b$ are limits of a convergent sequence $\{x_n\}$. Then for $\varepsilon = \frac{1}{3}d(a, b) > 0$, there are integers p, q such that $d(x_n, a) \leq \varepsilon$

for all $n \geq p$ and $d(x_n, b) \leq \varepsilon$ for all $n \geq q$. Let $n = p + q$. Then the following contradiction establishes the proof:

$$3\varepsilon = d(a,b) \leq d(a,x_n) + d(x_n,b) = d(x_n,a) + d(x_n,b) \leq \varepsilon + \varepsilon = 2\varepsilon. \qquad \square$$

1-2.8. A sequence $\{y_n\}$ is called a ***subsequence*** of $\{x_n\}$ if there is a sequence of integers $n(1) < n(2) < n(3) < \cdots$ such that $y_j = x_{n(j)}$ for all j. Since all indices of sequences in this book start with 1, we have $n(j) \geq j$ for all j.

1-2.9. **Theorem** If $x_n \to b$ and if $\{y_n\}$ is a subsequence of $\{x_n\}$ then $y_n \to b$.

Proof. Let $\varepsilon > 0$ be given. Since $x_n \to b$, there is an integer p such that for all $n \geq p$ we have $d(x_n, b) \leq \varepsilon$. With the same notation of last paragraph, for every $j \geq p$, we have $n(j) \geq j \geq p$ and hence $d(y_j, b) = d(x_{n(j)}, b) \leq \varepsilon$. This proves $y_n \to b$. $\square$

1-2.10. **Exercise** Various intervals denoted by circular and square brackets are described by the following examples : $[2,3) = \{x \in \mathbb{R} : 2 \leq x < 3\}$, $(3,4] = \{x \in \mathbb{R} : 3 < x \leq 4\}$, $[4,3] = \{x \in \mathbb{R} : 4 \leq x \leq 3\} = \emptyset$ and $(3,3) = \{x \in \mathbb{R} : 3 < x < 3\} = \emptyset$. Is the sequence $\{\frac{1}{n}\}$ convergent in the metric spaces $\mathbb{R}$, $[0,1]$ and $(0,1]$ respectively?

1-2.11. **Exercise** Let $x_n = (\frac{1}{n}, \sqrt{n+1} - \sqrt{n})$ and $y_n = (3, (-1)^n)$. Prove or disprove that they are convergent in the normed spaces $\mathbb{R}^2_1$, $\mathbb{R}^2_2$ and $\mathbb{R}^2_\infty$ respectively.

1-2.12. **Exercise** Prove that if $x_n = a \in X$ for all n, then $x_n \to a$.

1-2.13. **Exercise** Prove that in a discrete metric space X, if $x_n \to a$ then there is an integer p such that for all $n \geq p$, we have $x_n = a$.

1-3 Continuous Maps

1-3.1. Let X, Y be metric spaces and let $f : X \to Y$ be a given map. Then f is said to be *continuous at a point* $b \in X$ if for every sequence $x_n \to b$ in X, we have $f(x_n) \to f(b)$ in Y. Plotting the sequence $\{x_n\}$ and its image $\{f(x_n)\}$ should give an intuitive idea that when x is near b, $f(x)$ must be near $f(b)$. The map f is said to be *continuous on* X if it is continuous at every point of X. In calculus, typical examples of continuous functions include polynomials, exponential functions and trigonometric functions. At the end of this chapter, we shall prove that every metric space has plenty of continuous functions.

1-3.2. **Theorem** Let X, Y, Z be metric spaces. Suppose that $f : X \to Y$ and $g : Y \to Z$ are given maps.
(a) If f is continuous at $b \in X$ and if g is continuous at $f(b)$ then the composite map gf is continuous at $b \in X$.
(b) If f is continuous on X and if g is continuous on Y then the composite map gf is continuous on X.

Proof. Let $x_n \to b$ in X. Since f is continuous at b, we have $f(x_n) \to f(b)$. Since g is continuous at $f(b)$, we get $g[f(x_n)] \to g[f(b)]$, i.e. $(gf)(x_n) \to (gf)(b)$. This proves (a). Part (b) becomes an easy exercise. □

1-3.3. **Exercise** Show that every map from a discrete metric space into a metric space is continuous.

1-3.4. Let X, Y be metric spaces. For all (x, y) and (a, b) in the product set $X \times Y$, let

$$d_1[(x,y),(a,b)] = d(x,a) + d(y,b);$$

$$d_2[(x,y),(a,b)] = \sqrt{d(x,a)^2 + d(y,b)^2}\ ;$$

and

$$d_\infty[(x,y),(a,b)] = \max\{d(x,a), d(y,b)\}.$$

It is routine to verify that d_1, d_2, d_∞ are metrics on $X \times Y$. The product set $X \times Y$ together with one of the metrics d_1, d_2, d_∞ is called a *product metric space*.

1-3.5. **Theorem** Let (x_n, y_n) and (a, b) be points in the product metric space $X \times Y$. The sequence $\{(x_n, y_n)\}$ converges to (a, b) in $X \times Y$ iff $x_n \to a$ in X and $y_n \to b$ in Y.

Proof. We shall prove part the case d_1 but leave the cases d_2, d_∞ as exercises. Suppose $(x_n, y_n) \to (a, b)$ in $X \times Y$. Then we have

$$0 \le d(x_n, a) \le d_1[(x_n, y_n), (a, b)] \to 0$$

as $n \to \infty$. Hence $x_n \to a$ in X. Similarly $y_n \to b$ in Y. Conversely, suppose $x_n \to a$ in X and $y_n \to b$ in Y. Then

$$d_1[(x_n, y_n), (a, b)] = d(x_n, a) + d(y_n, b) \to 0$$

as $n \to \infty$. Therefore $(x_n, y_n) \to (a, b)$ in $X \times Y$. □

1-3.6. **Exercise** Let X, Y, Z be metric spaces. Show that the projection $\pi : X \times Y \to X$ given by $\pi(x, y) = x$ is continuous. Prove that a map $f : Z \to X \times Y$ is continuous iff both coordinate maps $\pi f : Z \to X$ and

$\varphi f : Z \to Y$ are continuous where $\varphi : X \times Y \to Y$ is the projection onto the second coordinate.

1-3.7. **Exercise** Prove that every convergent sequence $\{x_n\}$ in a normed space is bounded, i.e. there is $M > 0$ such that $\|x_n\| \le M$ for all n. Note that bounded sets in metric spaces will be defined later in §2-2.1. Also see §2-2.9.

1-3.8. **Theorem** (a) The addition on a normed space E is a continuous map from the product space $E \times E$ into E.
(b) The scalar multiplication is a continuous map from the product space $\mathbb{K} \times E$ into E.

Proof. (a) Let $(x_n, y_n) \to (a, b)$ be a convergent sequence in $E \times E$. Then $x_n \to a$ and $y_n \to b$. Observe that

$$\|(x_n + y_n) - (a + b)\| = \|(x_n - a) + (y_n - b)\| \le \|x_n - a\| + \|y_n - b\| \to 0,$$

as $n \to \infty$. Therefore the addition is continuous.
(b) Let $\lambda_n \to \alpha$ in $\mathbb{K}$ and $x_n \to a$ in E be convergent sequences. Then $\{\lambda_n\}$ is bounded in $\mathbb{K}$. There is $M > 0$ such that all $|\lambda_n| \le M$. Now observe that

$$\|\lambda_n x_n - \alpha a\| = \|\lambda_n (x_n - a) + (\lambda_n - \alpha) a\|$$
$$\le |\lambda_n| \; \|x_n - a\| + |\lambda_n - \alpha| \; \|a\| \le M\|x_n - a\| + |\lambda_n - \alpha| \; \|a\| \to 0$$

as $\to \infty$. Therefore the scalar multiplication is continuous. □

1-3.9. **Exercise** Prove that the function $x \to \|x\|$ is continuous on E.

1-3.10. **Theorem** Let X, Y be metric spaces and $f : X \to Y$ be a given map. Then f is continuous at a point $b \in X$ iff for every $\varepsilon > 0$, there is $\delta > 0$ such that whenever $x \in X$ satisfies $d(x, b) \le \delta$, we have $d(f(x), f(b)) \le \varepsilon$.

Proof. ($\Rightarrow$) Let f be continuous at $b \in X$. Suppose to the contrary that $\exists\, \varepsilon > 0, \forall\, \delta > 0, \exists\, x \in X,\ d(x, b) \le \delta$, and $d(f(x), f(b)) > \varepsilon$. Taking $\delta = 1/n$, there is $x_n \in X$ such that $d(x_n, b) \le 1/n$ and $d(f(x_n), f(b)) > \varepsilon$. Therefore $x_n \to b$ but $f(x_n) \not\to f(b)$. Consequently, f cannot be continuous at b.
($\Leftarrow$) Assume $x_n \to b$ in X. Let $\varepsilon > 0$ be given. Find $\delta > 0$ such that $d(x, b) \le \delta$ implies $d(f(x), f(b)) \le \varepsilon$. Choose p so that $\forall\; n \ge p, d(x_n, b) \le \delta$, that is $d(f(x_n), f(b)) \le \varepsilon$. Therefore $f(x_n) \to f(b)$. Consequently f is continuous at b.

1-3.11. **Exercise** Let X, Y, Z be metric spaces and let (a, b) be a point in the product space $X \times Y$. Prove that a map $f : X \times Y \to Z$ is continuous at (a, b) iff for every $\varepsilon > 0$ there is $\delta > 0$ such that for every $d(x, a) \le \delta$ in X and $d(y, b) \le \delta$ in Y we have $d[f(x, y), f(a, b)] \le \varepsilon$.

1-3.12. **Exercise** Let E be a normed space with more than one point and let d be the metric associated with the norm. Prove that d cannot be the discrete metric.

1-4 Open Sets

1-4.1. An alternative way to describe the concept of nearness without norms or distances in more general context is to use open sets. Full development along this line is called *general topology* which is beyond our scope. Only essential properties of open sets will be introduced.

1-4.2. Let X be a metric space. Suppose $a \in X$ and $r > 0$. Then the set $\mathbb{B}(a, r) = \{x \in X : d(x, a) < r\}$ is called the *open ball* with center a and radius r. Similarly, the closed ball is defined as the set $\overline{\mathbb{B}}(a, r) = \{x \in X : d(x, a) \leq r\}$. By a ball, we always mean an open ball. We may drop r such as $\mathbb{B}(a)$ if the radius is not critical in the context.

1-4.3. **Lemma** If $a \in \mathbb{B}(x, \alpha) \cap \mathbb{B}(y, \beta)$, then there is $\delta > 0$ such that

$$\mathbb{B}(a, \delta) \subset \mathbb{B}(x, \alpha) \cap \mathbb{B}(y, \beta).$$

Proof. Let $\delta = \min\{\alpha - d(x, a), \beta - d(y, a)\}$. Take any $z \in \mathbb{B}(a, \delta)$. Then we have $d(z, a) < \delta \leq \alpha - d(x, a)$. Hence $d(z, x) \leq d(z, a) + d(a, x) < \alpha$, i.e. $z \in \mathbb{B}(x, \alpha)$. Similarly, $z \in \mathbb{B}(y, \beta)$. This completes the proof. Beginners should sketch the picture of balls with their radii. □

1-4.4. A subset M of X is said to be *open* if for every $x \in M$, there is some ball $\mathbb{B}(x)$ contained in M. As a result of last theorem when $x = y$ and $\alpha = \beta$, every open ball is open.

1-4.5. **Theorem** (a) Both $\emptyset$ and X are open.
(b) If M, N are open then $M \cap N$ is open.
(c) If $\{M_i : i \in I\}$ is a family of open sets then the union $\bigcup_{i \in I} M_i$ is open.

Proof. Take any $x \in M \cap N$. There are balls A, B with the same center x such that $A \subset M$ and $B \subset N$. There is another ball C with center x such that $C \subset A \cap B$. Hence $C \subset M \cap N$. Since $x \in M \cap N$ is arbitrary, $M \cap N$ is open. This proves (b). The rest is left as an exercise. □

1-4.6. **Exercise** Prove that a sequence $\{x_n\}$ in X converges to $b \in X$ iff for every open set V containing b, there is an integer p such that for all $n \geq p$, we have $x_n \in V$.

1-4.7. Let M be a subset of a metric space X. Then a point $x \in X$ is called an *interior point* of M if there is a ball $\mathbb{B}(x)$ contained in M. The set of all interior points of M is called the *interior* of M. It is denoted by M^o.

1-4.8. **Theorem** (a) If A is an open subset of M, then we have $A \subset M^o$.
(b) M^o is the largest open subset of M.
(c) M is open iff $M = M^o$.
(d) $M^{oo} = M^o$.

Proof. (a) Suppose A is an open subset of M. Take any $x \in A$. Since A is open, there is a ball $\mathbb{B}(x) \subset A$. By $A \subset M$, we have $\mathbb{B}(x) \subset M$. Therefore, x is an interior point of M, i.e. $x \in M^o$. This proves $A \subset M^o$.
(b) Clearly M^o is a subset of M by definition. Take any $x \in M^o$. There is a ball $\mathbb{B}(x) \subset M$. Since $\mathbb{B}(x)$ is an open subset of M, it follows from (a) that $\mathbb{B}(x) \subset M^o$. Because $x \in M^o$ is arbitrary, the set M^o is open. It follows from (a) that M^o is the largest one.
(c,d) These are left as exercises. □

1-4.9. **Exercise** Describe the interiors of the sets $\{(x, y) \in \mathbb{R}^2 : y = x^2\}$ and $\{(x, y) \in \mathbb{R}^2 : y \geq x^2\}$ respectively.

1-4.10. **Exercise** Show that every subset of a discrete metric space is open.

1-4.11. **Exercise** Prove that a finite product of open sets is open.

1-4.12. **Exercise** Prove that the only non-empty open vector subspace of a normed space E is E itself.

1-5 Closures of Sets

1-5.1. Let M be a subset of a metric space X. Then a point y is called a *closure point* of M if there is a sequence $\{x_n\}$ in M which converges to y. The set of all closure points of M is called the *closure* of M. It is denoted by $\overline{M}$.

1-5.2. **Theorem** A point $y \in X$ is a closure point of M iff every ball $\mathbb{B}(y)$ contains a point of M.

Proof. Assume y is a closure point of M. Then there is sequence $\{x_n\}$ in M which converges to y. Let $\mathbb{B}(y, r)$ be a given ball with center y. There is an integer p such that for all $n \geq p$, we have $d(x_n, y) \leq r/2$. In this case, the ball $\mathbb{B}(y, r)$ contain the point x_p of M. Conversely, suppose every ball $\mathbb{B}(y, \frac{1}{n})$

contains a point of M, say x_n. Then $\{x_n\}$ is a sequence in M which converges to y. □

1-5.3. **Exercise** Prove that a point $y \in X$ is a closure point of M iff every open set containing y also contains a point of M.

1-5.4. A subset M of a metric space X is said to be *closed* if it contains all its closure points, i.e. $\overline{M} \subset M$.

1-5.5. **Theorem** (a) $\overline{M}$ is a closed set containing M.
(b) If H is any closed set containing M then $\overline{M} \subset H$. Therefore $\overline{M}$ is the smallest closed set containing M.
(c) M is closed iff $M = \overline{M}$.
(d) $\overline{\overline{M}} = \overline{M}$.

Proof. (a) Let x be a closure point of $\overline{M}$. Consider any ball $\mathbb{B}(x)$. There is $y \in \mathbb{B}(x) \cap \overline{M}$. Since the ball $\mathbb{B}(x)$ is open, there is another ball $\mathbb{B}(y) \subset \mathbb{B}(x)$. Since y is a closure point of M, there is $z \in \mathbb{B}(y) \cap M$. Thus, $z \in \mathbb{B}(x) \cap M$. Therefore that x is a closure point of M, i.e. $x \in \overline{M}$. This proves that $\overline{M}$ is a closed set. By considering the constant sequences, clearly we get $M \subset \overline{M}$.
(b) Take any $y \in \overline{M}$. There is sequence $\{x_n\}$ in M convergent to y. From $M \subset H$, $\{x_n\}$ is also a sequence in H convergent to y, i.e. $y \in \overline{H}$. Since H is closed, we have $y \in H$. This proves (b).
Parts (c,d) are left as exercises. □

1-5.6. **Theorem** A subset M of a metric space X is closed iff $X \setminus M$ is open.

Proof. Assume that M is closed. Suppose to the contrary $X \setminus M$ is not open. There is $x \in X \setminus M$ such that for every $r > 0$, $\mathbb{B}(x, r) \not\subset X \setminus M$, i.e. $\mathbb{B}(x, r) \cap M \neq \emptyset$. Thus $x \in \overline{M}$. Since M is closed, we have $x \in M$ which contradicts the choice of x. Therefore $X \setminus M$ is open. Conversely, assume that $X \setminus M$ is open but M is not closed. Then there is $x \in \overline{M} \setminus M$. Hence X belongs to the open set $X \setminus M$. There is a ball $\mathbb{B}(x) \subset X \setminus M$, i.e. $\mathbb{B}(x) \cap M = \emptyset$. On the other hand, since x is a closure point of M, we have $\mathbb{B}(x) \cap M \neq \emptyset$. This contradiction establishes the proof. □

1-5.7. **Corollary** Both the empty set and the whole space are closed sets. Finite unions of closed sets are closed. Arbitrary intersections of closed sets are closed.

1-5.8. **Exercise** Prove $\overline{A \times B} = \overline{A} \times \overline{B}$ for all subsets A, B of metric spaces X, Y respectively. Prove that finite products of closed sets are closed.

1-5.9. **Exercise** Find the closure of an open ball $\mathbb{B}(x, 1)$ in a discrete metric space. What is the closed ball $\overline{\mathbb{B}}(x, 1)$?

1-5.10. **Exercise** Prove that every finite subset of a metric space is closed.

1-5.11. **Exercise** Prove that the closure of a vector subspace of a normed space is a vector subspace.

1-5.12. **Exercise** Describe the closures of the sets $\{(x, y) \in \mathbb{R}^2 : y = x^2\}$ and $\{(x, y) \in \mathbb{R}^2 : y > x^2\}$ respectively.

1-5.13. **Exercise** Show that the set $\{(\cos t, \sin t, t) : t \in \mathbb{R}\}$ is closed in $\mathbb{R}^3$.

1-5.14. **Exercise** Find the closure of the set $\{(\cos \frac{1}{t}, \sin \frac{1}{t}, t) : t > 0\}$ in $\mathbb{R}^3$.

1-5.15. **Example** For every subset A of a metric space X, we have

$$\overline{A} = \bigcap_{n=1}^{\infty} \bigcup_{a \in A} \mathbb{B}(a, 1/n) = \bigcap_{n=1}^{\infty} \bigcup_{a \in A} \overline{\mathbb{B}}(a, 1/n).$$

Proof. Let $x \in \overline{A}$. Choose $a_j \in A$ with $a_j \to x$ as $j \to \infty$. For every n, there is j such that $d(x, a_j) < 1/n$, i.e. $x \in \mathbb{B}(a_j, 1/n)$. We have proved that $\overline{A} \subset \bigcap_{n=1}^{\infty} \bigcup_{a \in A} \mathbb{B}(a, 1/n)$. Next, let $x \in \bigcap_{n=1}^{\infty} \bigcup_{a \in A} \overline{\mathbb{B}}(a, 1/n)$. For every n, there is $a_n \in A$ such that $x \in \overline{\mathbb{B}}(a_n, 1/n)$, i.e. $d(a_n, x) \leq 1/n$. Hence $a_n \to x$ with $a_n \in A$, i.e. $x \in \overline{A}$. □

1-6 Characterization of Continuity

1-6.1. An important way to prove a set to be open or closed is by inverse images of continuous maps. A natural question at this stage is whether every metric space has a continuous function. The answer will be provided by the distance function. Glue Theorem will offer a nice way to piece continuous maps together.

1-6.2. **Theorem** Let X, Y be metric spaces and $f : X \to Y$ be a given map. Then the following statements are equivalent.

(a) f is continuous on X.

(b) The inverse image of every closed set in Y is closed in X.

(c) The inverse image of every open set in Y is open in X.

(d) For every subset A of X, we have $f(\overline{A}) \subset \overline{f(A)}$.

Proof. ($a \Rightarrow b$) Let M be a closed set in Y and let $a \in X$ be a closure point of $f^{-1}(M)$. Then there are $x_n \in f^{-1}(M)$ satisfying $x_n \to a$. Since f is continuous at a, we have $f(x_n) \to f(a)$ in Y. Now $f(x_n) \in M$ implies that

$f(a)$ is a closure point of the closed set M. Hence $f(a) \in M$, i.e. $a \in f^{-1}(M)$. Therefore $f^{-1}(M)$ is closed.

($b \Rightarrow c$) It follows immediately by taking complements.

($c \Rightarrow a$) Let $a \in X$ and $\varepsilon > 0$ be given. Since the inverse image of the open set $\mathbb{B}(f(a), \varepsilon)$ is an open set containing the point a, there is a ball $\mathbb{B}(a, 2\delta) \subset f^{-1}\mathbb{B}(f(a), \varepsilon)$. Now suppose $d(x, a) \leq \delta$. Then $x \in \mathbb{B}(a, 2\delta)$. Hence $x \in f^{-1}[\mathbb{B}(f(a), \varepsilon)]$, i.e. $f(x) \in \mathbb{B}(f(a), \varepsilon)$, or, $d(f(x), f(a)) \leq \varepsilon$. Therefore f is continuous at every point of X, i.e. continuous on X.

($a \Rightarrow d$) Let $a \in \overline{A}$. There are $x_n \in A$ convergent to a. Since f is continuous, we have $f(x_n) \to f(a)$. Hence $f(a) \in \overline{f(A)}$. Therefore $f(\overline{A}) \subset \overline{f(A)}$.

($d \Rightarrow b$) Let M be a closed set in Y. Define $A = f^{-1}(M)$. Then we have

$$f(\overline{A}) \subset \overline{f(A)} \subset \overline{ff^{-1}(M)} \subset \overline{M} \subset M$$

i.e. $\overline{f^{-1}(M)} \subset \overline{A} \subset f^{-1}(M)$. Therefore $f^{-1}(M)$ is closed. □

1-6.3. **Exercise** Show that the set $\{(x, y) \in \mathbb{R}^2 : ye^{-x} \sin(x + y) > x \cos xy\}$ is open in $\mathbb{R}^2$ and the set $\{(x, y) \in \mathbb{R}^2 : ye^{-x} \sin(x + y) \geq x \cos xy\}$ is closed in $\mathbb{R}^2$.

1-6.4. **Theorem** Let $X[d]$ be a metric space. Then the distance function $d : X \times X \to \mathbb{R}$ is continuous on the product space. In particular, $d(a, x)$ is a continuous function in x.

Proof. It follows immediately from $|d(x, y) - d(a, b)| \leq d(x, a) + d(y, b)$. □

1-6.5. **Exercise** Show that the sphere $\{x \in X : d(a, x) = r\}$, the closed ball $\overline{\mathbb{B}}(a, r)$, and the set $\{x \in X : d(a, x) \geq r\}$ are closed. Along the same line, prove that the open ball $\mathbb{B}(a, r)$ and the set $\{x \in X : d(a, x) > r\}$ are open.

1-6.6. Let $X[d]$ be a metric space and H a subset of X. Then the restriction $d|_H$ of the metric d onto H is a metric on H. It is called the *relative metric*. The metric space $H[d|_H]$ is called a *subspace* of X. For simplicity, we shall write d instead of $d|_H$.

1-6.7. **Exercise** Let $X = \{(x, y) \in \mathbb{R}^2 : |x| \leq 2, |y| \leq 2\}$ be equipped with the relative metric from $\mathbb{R}^2$. Sketch the open ball in X with center $(1, 1)$ and radius 2.

1-6.8. Let X, Y be metric spaces and $f : X \to Y$ be a given map. Suppose H is a subset of X. Then f is said to be *continuous on* H if the restriction $f|_H$

is continuous on the metric subspace H. Clearly, if f is continuous on X then f is continuous on H. For simplicity, we write f instead of $f|_H$.

1-6.9. **Exercise** Show that the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = 0$ for $x \leq 1$ and $f(x) = 1$ for $x > 1$ is discontinuous on $\mathbb{R}$ but continuous on the subset $(0, 1) \cup (1, 2)$.

1-6.10. Let Y be a subspace of a metric space X. Write $x_n \to a$ in Y if all x_n, a are in Y and $d(x_n, a) \to 0$. A subset V of Y is said to be *closed* (respectively open) *in* Y if V is closed (respectively open) in the metric subspace Y. Note that Y is open in itself but need not be open in X.

1-6.11. **Lemma** Let $A \subset B$ be two subsets of a metric space X. If A is closed in B and if B is closed in X, then A is closed in X.

Proof. Let $x_n \in A$ and $y \in X$. Suppose $x_n \to y$ in X. Since $A \subset B$, we have $x_n \in B$. Since B is closed in X, we have $y \in B$. Now y is a closure point of A on the subspace B. Because A is closed in B, y belongs to A. Therefore A is closed in X.

1-6.12. **Exercise** State and prove a result for open sets similar to the last lemma.

1-6.13. **Glue Theorem** Let X, Y be metric spaces and $f : X \to Y$ be a given map. Suppose $X = M \cup N$ is the union of two closed subsets M, N. If f is continuous on both M, N separately, then f is continuous on X.

Proof. Let V be a closed subset of Y. Since f is continuous on M, the set $(f|_M)^{-1}(V) = M \cap f^{-1}(V)$ is closed in M. Since M is closed in X, $M \cap f^{-1}(V)$ is closed in X. Similarly, $N \cap f^{-1}(V)$ is closed in X. Therefore

$$f^{-1}(V) = [M \cap f^{-1}(V)] \cup [N \cap f^{-1}(V)]$$

is closed in X. Consequently, f is continuous on X.

1-6.14. **Exercise** Show that the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x$ for $x \leq 0$ and $f(x) = x^2$ for $x > 0$ is continuous on $\mathbb{R}$.

1-7 Duality of Closure-Interior Operators

1-7.1. Open and closed sets are complement to each other as in §1-5.6. The following theorem extends this duality to operators.

1-7.2. **Theorem** For every subset M of a metric space X, we have $M'^{-\prime} = M^o$ and $M'^{o\prime} = M^-$ where $M' = X \setminus M$ denotes the complement of M.

Proof. Observe that $x \in M'^{-\prime}$ iff $x \in M'^-$ is false. The negation of $\forall\, r > 0, \mathbb{B}(x,r) \cap M' \neq \emptyset$ is the statement: $\exists\, r > 0, \mathbb{B}(x,r) \cap M' = \emptyset$, that is $\mathbb{B}(x,r) \subset M$. This is equivalent to $x \in M^o$. Therefore $M'^{-\prime} = M^o$. Replacing M by its complement M', we obtain the second identity. □

1-7.3. **Exercise** Let $M = (1,2] \cup \{\frac{1}{n} : n > 1\}$ be the union of a semi-interval and a sequence in the real line. How many new sets can you obtain by constructing interior, closure and complement repeatedly?

1-7.4. Let M be a subset of a metric space X. A point $x \in X$ is called a *boundary point* of M if every ball $\mathbb{B}(x)$ contains a point in M and also a point not in M. The set of all boundary points of M is called the *boundary* of M and is denoted by ∂M. A point $x \in X$ is called an *exterior point* of M if there is a ball disjoint from M. The set of all exterior points of M is called the *exterior* of M and is denoted by $ext(M)$.

1-7.5. **Exercise** Prove that $\partial M = M^- \setminus M^o = M^- \cap M'^-$. Hence show that the boundary of a set is closed. Also prove that M^o and ∂M form a partition of M^-.

1-7.6. **Exercise** Prove that $ext(M) = M^{-\prime} = M'^o$. Hence deduce that $ext(M)$ is open. Show that $ext(M)$ and M^- form a partition of the whole space X.

1-7.7. **Example** Every non-empty open interval contains a rational number and an irrational number.

Proof. Let $a < b$ be the endpoints of the given interval. Choose any integer $n > \frac{1}{b-a}$ and mark the points $0, \pm\frac{1}{n}, \pm\frac{2}{n}, \pm\frac{3}{n}, \cdots$ on the real line $\mathbb{R}$. It is obvious that the interval (a,b) has to contain a rational number of the form $\frac{m}{n}$. Repeating the same process with $0, \pm\frac{1}{n\sqrt{2}}, \pm\frac{2}{n\sqrt{2}}, \pm\frac{3}{n\sqrt{2}}, \cdots$, we prove the case for irrational. □

1-7.8. **Exercise** Find the closure, interior, boundary and exterior of the set of rational numbers in the interval $(0,1)$.

1-7.9. **Exercise** Find the closure, interior, boundary and exterior of the set of points $(\frac{1}{m}, \frac{1}{n}) \in \mathbb{R}^2$ where m, n run over all non-zero integers.

1-7.10. **Exercise** Find the closure, interior, boundary and exterior of the closed ball of $\mathbb{R}^2_2$ with center at the origin and radius $\frac{1}{2}$. Repeat the same problem when $\mathbb{R}^2_d$ is given the discrete metric.

1-8 Partition of Unity

1-8.1. Intuitively, through a partition of unity, a point x in an abstract metric space is described by a vector $(\alpha_1(x), \alpha_2(x), \cdots, \alpha_n(x))$ in $\mathbb{R}^n$ where α_j are continuous functions of x. Compactness in Chapter 2 will allow us to reduce an arbitrary open cover to a finite cover so that partition of unity can be applied. For example, see §§5-3.3, 12-4.5. We start off with the distance function.

1-8.2. Let A be a non-empty subset of a metric space X. The *distance from a point* $x \in X$ to A is defined by $d(x, A) = \inf_{a \in A} d(x, a)$.

1-8.3. **Lemma** For all $x, y \in X$, we have $|d(x, A) - d(y, A)| \leq d(x, y)$. As a result, $d(x, A)$ is a continuous function of $x \in X$. Consequently, we have sufficient amount of continuous functions on every metric space.

Proof. For each $a \in A$, we have $d(x, A) \leq d(x, a) \leq d(x, y) + d(y, a)$. Taking infimum over $a \in A$, we obtain $d(x, A) \leq d(x, y) + d(y, A)$, that is, $d(x, A) - d(y, A) \leq d(x, y)$. Interchanging x, y, we obtain the required inequality. □

1-8.4. **Lemma** $d(x, A) = 0$ iff x is a closure point of A.

Proof. Suppose $d(x, A) = 0$. For every $n \geq 1$, there is $a_n \in A$ such that $d(x, a_n) \leq \frac{1}{n}$. Hence $a_n \to x$. Therefore x is a closure point of A. Conversely, let x be a closure point of A. There is a sequence $\{a_n\}$ in A convergent to X. Hence we have $0 \leq d(x, A) \leq d(x, a_n) \to 0$ as $n \to \infty$. Therefore $d(x, A) = 0$. □

1-8.5. **Theorem** Let A, B be disjoint closed subsets of a metric space X. Then there is a continuous function $f : X \to [0, 1]$ such that $f(A) = 0$ and $f(B) = 1$.

Proof. Note that if one of A, B is empty, then a constant function would do the job. So, assume both A, B are non-empty. Firstly, we claim $d(x, A) + d(x, B) > 0, \forall\, x \in X$. In fact, suppose to the contrary that for some $x \in X, d(x, A) + d(x, B) = 0$. Then $d(x, A) = d(x, B) = 0$. Hence x is a closure point for both A, B. Since A, B are closed, x belongs to both A, B. This contradicts the fact that A, B are disjoint. Therefore the following function is well-defined:

$$f(x) = \frac{d(x, A)}{d(x, A) + d(x, B)}, \forall\, x \in X.$$

Clearly it is a required continuous function on X. □

1-8.6. **Exercise** Prove that the function $g(x) = [1 - f(x)]a + f(x)b$ where $a, b \in \mathbb{R}$ is continuous on X and satisfies $g(A) = a$, $g(B) = b$.

1-8.7. **Corollary** Let A be a closed set and V an open set in X. If $A \subset V$ then there is an open set W satisfying $A \subset W \subset \overline{W} \subset V$.

Proof. Since A and $X \backslash V$ are disjoint closed sets, there is a continuous function $f : X \to [0, 1]$ such that $f(A) = 0$ and $f(X \setminus V) = 1$. Then $W = f^{-1}(-\infty, \frac{1}{2})$ is an open set containing A and $\overline{W} \subset f^{-1}(-\infty, \frac{1}{2}] \subset V$. □

1-8.8. **Lemma** Let A be a closed subset of a metric space X and let $\{V_j : 1 \leq j \leq n\}$ be an open cover of A, i.e. $A \subset \bigcup_{j=1}^{n} V_j$ and all V_j are open. Then there are closed subsets B_j of A such that $A = \bigcup_{k=1}^{n} B_k$ and $B_j \subset V_j$ for each $1 \leq j \leq n$.

Proof. It suffices to prove the case for $n = 2$. Let U, V be open sets such that $A \subset U \cap V$. Let $M = A \backslash V$ and $N = A \backslash U$. Then both M, N are closed sets and they are disjoint. There is a continuous function f on X such that $f(M) = 0$ and $f(N) = 1$. Then both $P = f^{-1}(-\infty, \frac{1}{2})$ and $Q = f^{-1}(\frac{1}{2}, \infty)$ are open sets containing M, N respectively. Therefore $E = A \setminus Q$ and $F = A \setminus P$ are closed subsets of A. Clearly we have $E \cup F = A \setminus (P \cap Q) = A$. Furthermore, observe that $E = A \setminus Q \subset A \setminus N \subset U$. Similarly we obtain $F \subset V$. This completes the proof. It would be helpful if you sketch a picture to go along with the above constructions. □

1-8.9. Let X be a metric space and $f : X \to \mathbb{K}$ a given function. Then the *support* of f is defined to be the closure of the set $\{x \in X : f(x) \neq 0\}$. It is denoted by supp($f$). Let A be a closed subset of X and $\{V_j : 1 \leq j \leq n\}$ an open cover of A. A sequence of continuous functions $\alpha_j : X \to [0, 1]$ is called a *partition of unity* on A subordinated to $\{V_j\}$ if the following conditions hold:

(a) $\sum_{j=1}^{n} \alpha_j(a) = 1$ for all $a \in A$.

(b) $\sum_{j=1}^{n} \alpha_j(x) \leq 1$ for all $x \in X$.

(c) for each j, the support of α_j is contained in V_j.

1-8.10. **Example** Let $A = (0, 4]$, $X = (0, \infty)$, $U = (0, 3)$ and $V = (1, 6)$. Clearly

$\{U, V\}$ is an open cover of the closed subset A of X. Let

$$\alpha(x) = \begin{cases} 1, & \text{if } 0 < x \leq 1, \\ 2 - x, & \text{if } 1 < x \leq 2, \\ 0, & \text{if } 2 \leq x, \end{cases}$$

and

$$\beta(x) = \begin{cases} 1 - \alpha(x), & \text{if } 0 < x \leq 4, \\ 5 - x, & \text{if } 4 \leq x \leq 5, \\ 0, & \text{if } 5 \leq x. \end{cases}$$

Find the supports of α, β. Show that $\{\alpha, \beta\}$ is a partition of unity on A subordinated to $\{U, V\}$.

1-8.11. **Theorem** Let A be a closed subset of a metric space X. Then for every open cover $\{V_j : 1 \leq j \leq n\}$ of A, there is a partition of unity on A subordinated to $\{V_n\}$.

Proof. Let $V_0 = X \setminus A$. Then $\{V_j : 0 \leq j \leq n\}$ is an open cover of the whole space X. Let B_j be closed sets in X such that $X = \bigcup_{k=0}^{n} B_k$ and for all j, $B_j \subset V_j$. Now for each j, there is an open set W_j such that $B_j \subset W_j \subset \overline{W_j} \subset V_j$ and also there is a continuous function $f_j : X \to [0, 1]$ such that $f_j(B_j) = 1$ and $f_j(X \setminus W_j) = 0$. Take any $x \in X$. Then $x \in B_j$ for some $0 \leq j \leq n$, i.e. $f_j(x) = 1$. Hence $\sum_{k=0}^{n} f_k(x) \geq 1$. Therefore the functions $\alpha_j : X \to [0, 1]$ given by $\alpha_j(x) = \dfrac{f_j(x)}{\sum_{k=0}^{n} f_k(x)}$ are well-defined and continuous on X. Observe that if $\alpha_j(x) \neq 0$, then $x \notin W_j$, i.e. $x \in W_j$. Hence $\text{supp}(\alpha_j) \subset \overline{W_j} \subset V_j$ for each $j = 0, 1, 2, \cdots, n$. In particular, if $\alpha_0(x) \neq 0$, we have $x \in V_0$, i.e. $x \notin A$. Thus $\alpha_0(A) = 0$. Now it is obvious to verify all other conditions for $\{\alpha_j : 1 \leq j \leq n\}$ to be a partition of unity on A subordinated to $\{V_j : 1 \leq j \leq n\}$. □

1-99. **References and Further Readings** : Dunford, Taylor-58, Kreyszig, Yosida and Meise.