Introduction

1. This book essentially consists of four parts:

Chapters 1 to 12 in Normed spaces;
Chapters 13 to 15 in Hilbert spaces and tensor products;
Chapters 16 to 27 in vector measures on δ-rings and
Chapters 28 to 31 in group representations.

Within the first part, nonlinear analysis by topological method is covered in Chapters 4,5 and nonlinear analysis involving differentiation in Chapters 8 to 11. Readers with suitable background should start with any of Chapters 4, 8, 13, 16 or 28. They can also work on these chapters concurrently. *Informal* introduction is given here while the chapters tend to be concise and precise. Maps in this book generally need *not* be scalar-valued but functions normally are scalar-valued. Complex functions mean that they need not be real. Characteristic functions are denoted by ρ instead of χ so as to avoid possible confusion with x, X in hand-writing.

Banach Spaces

2. The first three chapters of this book provide the necessary background for any course in analysis. Sequences are used consistently to characterize continuity, closures, completeness, precompactness and compactness. The fact that closed bounded sets are compact is proved by upper and lower limits rather than bisecting infinite sets. One dimensional intermediate value theorem, fixed point theorem and structure of open sets §2-10.9 follow as a result of connectedness. Partition of unity is developed at the end of first chapter and will be used twice in §§5-3.3, 12-4.5. After finite dimensional normed spaces are characterized, standard criteria of compactness in infinite dimensional spaces (Ascoli's theorem) and also approximation of continuous functions (Stone-Weierstrass theorem) are given by the end of chapter 3. This would quickly cover the topological background required by advanced calculus and complex variable at the undergraduate level. It would be interesting to see the connection of functional analysis and neural network, e.g. [Cotter]. For general history of functional analysis, consult [Dunford], [Dieudonne-81] and [Musielak].

3. Affine approximations on simplicial complexes have been playing an active role in computation with computers as indicated for example by [Kearfott],

[Talman], [Todd], [Eaves], [Mara]. Because surfaces can be approximated by gluing triangles together and solids by tetrahedra, perhaps Mechanical Engineers and also experts in physical chemistry, e.g. [Bytheway] may be interested to know more about its theory. Restricting to one, two and three dimensional spaces, e.g. [Steenrod], [Shashkin], chapter 4 also offers an opportunity for high-school projects. Convex sets §4-2 will be required in the subsequent context. However if you are not interested in topological method, skip to chapter 6. Alternatively, you read only the statement of §4-10.8, and then skip to chapter 5. We treat chapter 4 thoroughly. Proofs on geometrical independence in §4-1 are rarely found in any existing textbooks. In §4-4.11, vertices are proved to be extreme points. The rest of the chapter is devoted to the construction of simplicial approximations. Good background has been laid for you to continue the study in simplicial homology theory which is beyond the scope of this book. Consult [James], [Fan-90] for history.

4. Instead of waiving our hands demanding acceptance with faith of topological invariance, Borsuk-Ulam theorem is transported from $\mathbb{R}_1^n$ in §5-1.4 to finite dimensional normed space §5-1.12 with scaling homeomorphism. Brouwer's fixed point theorem is derived from Borsuk-Ulam theorem with explicit formula in §5-2.4. Retraction theorem is proved by elementary technique which is within the capacity of undergraduates. A general fixed point theorem on convex sets is given in §5-3.5. The treatment of compact fields is simplified from [Granas-62] and only Tietze's extension theorem is used to develop homotopy extension theorem. By the end of chapter 4, we have covered most of the traditional applications of classical algebraic topology but in a more general context of infinite dimensional spaces as in [Granas-62]. The whole chapter is within the reach of undergraduates without asking them to take anything for granted. Consult for example, [Steinlein], [Fan-99] and [Jaworowski] for further information about Borsuk-Ulam theorem.

5. Standard material of linear functional analysis is developed in chapters 6 and 7. Hahn-Banach extension theorems guarantee that there are sufficient amount of continuous linear forms to separate convex sets. This is applied to derive Krein-Milman theorem. It also allows us to reduce certain cases from infinite to one dimensional spaces, e.g. §§8-3.3, 4.6, etc. Uniform boundedness theorem ensures that weakly bounded sets are norm-bounded. One of its applications is given in §8-4.5. References to show that open map and closed graph theorems cannot be generalized to bilinear maps are included. Chapter 7 covers topics such as embedding into bidual spaces, duality of quotient spaces

and subspaces, direct sums, transposes, reflexive spaces and weak convergence.

6. Vector-valued maps of a scalar variable are introduced in chapter 8. We support the simple and elegant way to develop integration theory starting with step maps. As soon as the fundamental theorem of calculus is proved, we can evaluate integrals as antiderivatives and this is what we use most of the time. At this point, we expect the readers to have basic knowledge of complex analysis including criteria of holomorphic functions and Cauchy's integral formula. These results are treated in the context of Banach spaces. Laurent series expansion including Taylor series as special case, is done from scratch. Liouville's theorem follows from the characterization of polynomials from entire maps. Resolvent map is an important example of holomorphic vector maps. It is used to show that the spectrum of an operator on complex Banach space is non-empty. Chapter 8 finishes with holomorphic maps of an operator defined by Cauchy's integral formula together with a practical formula for functions of square matrices without Jordan forms. See e.g. [Taylor-71] for history and [Fan-96], [Sharma] for recent development.

7. Chapters 9, 10 are devoted to advanced calculus. Most undergraduate textbooks restrict themselves to scalar-valued functions of several variables. Based on our numerical examples §§9-3.8, 5.6, 7, 10-5.6, 15-4.13, you may be interested to standardize the notations of higher derivatives of maps and also polynomials from $\mathbb{R}^n$ to $\mathbb{R}^m$ in terms of matrices. Transition from scalar to vector variable for differential theory is given in §9-1.2. Integral and uniform mean-value theorems of §§9-2.7,8 later become Taylor's formula and its corollary §§10-5.2,3. Inverse and implicit mapping theorems are proved by contraction which is also an important tool in numerical analysis due to its ability to estimate the error. Local properties of differentiable maps §9-6 are restricted to finite dimensional maps because of the simplicity of using determinants although they have been extended to infinite dimensional spaces. The theorem on Lagrange multiplier is modified from [Sagan] without the assumption on the rank of certain matrix. It is well-known that a special case of §10-5.4 is an important tool in differentiable manifolds, e.g. to prove the Morse's lemma. The higher chain formula §10-6.2 is expressed in the natural setting of polynomials rather than multilinear maps in unnecessary generality. Our proof involves only simple combinatorial method. Consult e.g. [Ma-01] for inverse mapping theorem on locally convex spaces.

8. Chapter 11 deals with initial value problem $x' = f(t, x)$. Existence of common solution interval for all initial conditions near a given point is proved

in §11-1.6 which is used to derive *global* continuity of initial condition in §11-5.4 where the *size of solution interval* is guaranteed by an interval (α, β). Its importance is illustrated by an example §11-1.10. As a result of continuity, the domain Ω_f in §11-5.5 of the flow associated with the vector field f is open and the flow φ is smooth §11-5.10. Since we do not carry the topic any further, we do not introduce the concept of flow in the context. Our boundary theorem §11-2.6 holds in infinite dimensional spaces. Linear differential equations are studied in §11-3 and §11-4. Two estimates of the solution to a linear equation are given in §11-3.4. Fulmer's method of finding the exponential function of a matrix is by method of differential equations. Finally, topological method is used to generalize Peano's theorem to infinite dimensional spaces. Consult e.g. [Lobanov] for locally convex spaces. Just like compact fields, we prefer to suitable rather than maximum generality in an undergraduate course. We believe that at this point, students are well equipped to study differential topology which is beyond the scope of this book. Most of this chapter work for complex Banach spaces even we restrict ourselves to the real case.

9. Chapter 12 deals with compact linear operators. Fredholm alternative §12-2.9 is stated in a form identical to a characterization of non-singular matrices in linear algebra. Readers should look up some references in order to have a feeling of the importance of compact linear operators because it is probably the simplest tractable infinite dimensional linear operators. For example, their spectra are null sequences, §12-3.4. The proof of a lemma for the existence of hyperinvariant subspace is slightly shortened by the order of the operators K, T in §12-4.5. For an extension of Lomonosov's techniques to non-compact operators, see [Simonivc].

Hilbert Spaces

10. Systematic exposition of Hilbert spaces is given in two subsequent chapters. They should be read concurrently with the corresponding topics in Banach spaces. The link among operators, sesquilinear forms and quadratic forms is established first. The second half-chapter studies various types of operators based on the star operation. Geometric properties of subspaces are characterized in terms of algebraic equations of projectors. It is a pity that my teaching duty was governed by official course outlines otherwise I would start with C^*-algebras.

11. The spectrum of an operator on Hilbert space is studied in chapter 14. For a self-adjoint operator, the connection between its quadratic form, its spectrum

and its norm is given by §14-2.12. Diagonal operators generalize diagonal matrices of which the properties are completely determined by the diagonal coefficients. It turns out that every compact normal operator is diagonable. As an important consequence §14-5.14, every compact operator on a Hilbert space can be approximated by finite dimensional operators in norm. This is not true in Banach spaces, e.g. [Enflo-73], [Szankowski]. The chapter ends with a functional calculus of self-adjoint operators and polar decomposition. Consult [Gohberg-00] for traces and determinants on Hilbert spaces and Konig-75 on Banach spaces.

12. Our tensor products of vector spaces are constructed within the framework of product and dual spaces without the naive concept of formal sums. The notation $f \otimes g$ is justified in §15-4.1. In contrast to the current system, we adopt the reverse lexical order in §15-4.3 because of the matrix representations of multilinear maps, e.g. §15-4.9, 12. Tensor products of linear maps are defined in a way completely different from tensor product of vectors. We prove that they are consistent under the natural injection in §15-5.11. It is obvious that tensor products of normal operators are normal. Motivated by finite dimensional spaces, the converse is given in §15-7 for several classes of normal operators which to the best of our knowledge were our contribution.

Vector Measures on δ-Rings

13. My interest in measure theory was inspired by [Apostol, pp207-212], [Zaanen-59, p42], [Kolmogorov], [Loomis, 12C] in early sixties and my teaching assignment from mid-seventies until 1986. After a long gap, I began to work in vector measures started with [Dinculeanu-67] and [Diestel-77] during 2000 and 2001. In addition to our own new results, most steps in the following treatment are modifications with *improvement* of known facts but our final overall version seems to be unique in this area. We introduce breakable vector lattices to unify proofs, inner regularity to define measures from decent sets. Then we extend to integrable sets and finally define vector integrals by dominated convergence property. We do not need Egorov's Theorem, semivariations, Vitali-Hans-Saks Theorem, Nikodym's Theorem in order to develop the vector version of measures. We also explain why measurable vector maps should not simply be defined as sequential limits of simple maps if we want continuous maps on locally compact spaces to be measurable. Through our innovative approach at no extra cost, existing staffs can easily help a new generation of scientists to have better tools of integration used practically everyday in our lives. If

measure theory on σ-rings is an abstract generalization of Lebesgue measure then measures on δ-rings are the counter parts of Stieltjes measures. Finite-valued signed measures on σ-rings must be bounded but measures on δ-rings need not be bounded. See [Hawkins], [Chae] for history of early development. All our measures are of finite variation. The following is a more detailed introduction.

14. Conjugation rather than the popular complexification is introduced in Chapter 16. Our definition of complex vector lattices is easier to verify and sufficient to provide service to this book although most of the concrete examples also satisfy the popular axioms. We also introduce breakable vector lattices as generalization of *real* vector lattices, §16-3.12. Its existence is justified by the order duals §16-5.5 and the unification of proofs in §§17-3.4, 24-6.9, 27-3.2, 31-3.2.

15. Semi-intervals and semi-rectangles should be used as guiding examples of semirings in Chapter 17. Every finite family of semi-rectangles can be decomposed into disjoint unions of semi-rectangles. This is formalized analytically into Semiring Formula, §17-1.9 and developed into Step Mapping Theorem, §17-2.3. Vector charges and vector integrals are introduced in §17-2. Algebraic method §17-2.10,11 identifying charges and linear forms simplifies the proof of geometrical results, §17-2.12. We characterize finite variation in terms of order-boundedness §17-3.6 and also absolute convergence, §17-4.5 by modifying [Munster]. Admissible bilinear map ensures the continuity connecting vector measures and integrands. Countable additivity of charges is related to monotone convergence in set-level §17-5.7 and at function-level, §17-5.10. All measures in this book are of finite variation. Their variations are defined in §17-3.4 and proved to be measure in §17-6.5. Being of finite variation of maps on $\mathbb{R}$ is defined in terms of charges so that we can apply measure theory to get the old results later. For an approach without the abstract theory of this chapter, see [Apostol; Thm 8-14, also Ch. 9]. For application to stochastic processes, see for example [Dinculeanu-00s].

16. A δ-space is a set with a δ-ring. Sets in the δ-ring are called decent sets for convenience. In Chapter 18, we give a simple proof that countably additive scalar charges are of finite variation. Uniqueness of extension of vector measures from semirings to generating δ-rings is proved in §18-1.11. The actual extension of positive measures is by standard method of outer measures. In order to have rich algebraic operations among complex measures, it is necessary

to start off §18-2.3 with a finite-valued function on $\mathcal{S}$. Classical results from [Kolmogorov] are put into abstract setting.

17. Motivated by earlier Chinese edition of [Xia], measurable sets are defined by localization independent of measures. We try to avoid μ^*-measurable sets which are obtained from an outer measure μ^*. It follows with standard properties of measurable functions. Approximation of measurable functions by simple functions with increasing modulus prepares the ground to define integrals by dominated convergence. Measurable complex functions and vector maps take values in $\mathbb{R}$, $\mathbb{C}$ or Banach spaces but measurable real functions are allowed to take $\pm\infty$ as value in order to define integrals. Example §19-5.3 explains why we have to define measurable maps §19-5.5 by localization in order to include continuous maps on locally compact spaces, §27-1.4. The characterization §19-5.4 is obtained by modification of [Kuttler, 23.1]. Uniform approximation of measurable maps by sequences of simple maps is given in §19-6.4,5. Our lemma §19-6.7 takes care of both weak and weak-star measurability at the same time. Finally we prove that sequantial limits of measurable maps are measurable.

18. Upper functions are measurable extended-valued positive functions. The integrals of upper functions with respect to positive measures are developed in Chapter 20. Measurable functions are approximated by simple functions but our measures are defined only on decent sets. The crucial bridging step is by inner regularity, §20-1.3. Positive measures are defined on measurable sets §20-2.2 but vector measures on integrable sets only, §21-1.6. Equality almost everywhere and σ-finiteness allow us remove infinity from values of integrable functions so that their sums are defined. The important features of integration theory takes its most primitive form in §20-3. Characterization of σ-finite sets §20-4.9 in terms of decent sets will be used repeatedly. Working with two positive measures is motivated by [Taylor-65].

19. Our unique approach to vector integration with respect to a vector measure μ is carried out in Chapter 21. Since the variation $|\mu|$ is a positive measure, μ-measurable sets are defined and can be approximated by decent sets. This is the basis of extending a vector measure μ to all integrable sets. For a measurable map f, its variation $|f|$ is an upper function and the integral $\int |f| \, d|\mu|$ defines the scope of integrable maps of which their integrals $\int f d\mu$ are defined by dominated convergence §21-2.8. Important results of vector L_p-spaces for $1 \le p < \infty$ required by classical harmonic analysis is given

in §21-4. They include Dominated Convergence Theorem, Monotone Convergence Theorem, Integration Term by Term and differentiation under integral sign. Density theorem §21-4.6 incorporates increasing modulus. Vector L_∞-spaces are introduced without any measure because of the need in spectral measures in §26-4. All modes of convergence are for vector maps. This Chapter concludes with integration on subspaces and comparison with improper Riemann integrals.

20. Because our machinery begins with semirings, they are applicable to the product spaces directly in Chapter 22. Product measures does not require σ-finiteness, §22-2.2. To have nice formula $|\mu \otimes \nu| = |\mu| \otimes |\nu|$, we have to assume one of them to be scalar, §22-2.4. To proof the Fubini's Theorem, most text book takes the advantage of $0\infty = 0$ to shorten the proofs. For vector measures, we have to trace back bolts and nuts of our machine in §22-3.2. Exercise §22-3.7 is a modified version of [Bartle]. We may identify the L_p-space on the product space $X \times Y$ as the tensor product of L_p-spaces on X, Y respectively by §22-3.19.

21. Outer measures are recalled in Chapter 23 because it is handy to check if a set is null. The reduction of three elementary operations in linear algebra into two §23-5.3 can shorten many proofs including our §23-5.5. Because our measurable sets are independent of measures, it requires a little work to show that smooth images of measures sets are measurable §23-5.7. Change-variable of multiple integral is influenced by [Cohn, pp170-175].

22. Chapter 24 is devoted to Radon-Nikodym derivatives and duality of L_p-spaces. The first section on the relationship between $hd\mu$ and μ is an improvement of [Garnir-72, pp142-157]. Absolute continuity of measures is brought to the level of semirings §24-2.7,8 so that it can link to absolutely continuous maps, §24-2.9. Convergence in L_p is characterized in terms of convergence in measures and equicontinuity at the empty set. Positive and negative sets are initiated as tools but Hahn Decomposition Theorem is derived after Radon-Nikodym Theorem for simplicity. Polar form §24-4.4 reduces complex measures to positive measures. Tight restriction to σ-finite sets for Radon-Nikodym Theorem will be used later. The isometry of L_p with L'_q, §24-6.2,12 are modification of [Dinculeanu-67, pp229,234]. Continuous linear forms on vector L_p-spaces for $1 < p < \infty$ concentrates on σ-finite sets, §24-6.6. This result is a vector version obtained from [Bartle]. The duality of L_p-spaces for $1 < p < \infty$, §24-6.7 does not require σ-finiteness

any more. Sharper results for scalar measures make use of our breakable vector lattices, §24-6.10. Sufficient condition for Banach spaces to have Radon-Nikodym property is reorganized from [Diestel-77, ch 3].

23. In Chapter 25, cubes rather than general measurable sets as in [Rudin-74], are used for the geometrical expression of Radon-Nikodym derivatives because they are simple, intuitive and probably general enough for applications in physics. We prefer to Vitali cover instead of Raising Sun Lemma because Vitali Covering Theorem is consistent with our approach via semi-intervals. Lemma §25-1.10 is modified from [Cohn, p181]. The general theory is applied to the specific situation of the real line §25-2 and we start with factorizing pulse functions. Our Cantor set and function are developed on the familiar decimal system, §25-4.

24. The root of spectral measure theory on Hilbert space H in Chapter 26 is the product formula §26-2.2 based on §13-9.8 that if the sum of projectors is a projector then the products of any two summands are zero. In §26-2.5,6 we explain why spectral measures cannot be derived as a special case of vector measures. Spectral measures on semirings are extended in §26-2 to decent sets by finite variation and finally to measurable sets by inner regularity. Spectral integrals are defined by dominated convergence,§26-3.2. Commutativity is transferred from measures on decent sets to spectral integrals, §26-2.12,3.10,11. Null sets are defined in terms of μ_{xy} for $x, y \in H$ rather than one single measures but the ground has been prepared in §21-5. Properties unique to spectral measures are developed in §26-4. Product spectral measures prepare the amalgamation of spectral measures of self-adjoint operators to normal operators. For a specific operator A, $f(A)$ is defined in §14-6 for every continuous function f. In the first section of this chapter, we extend to semi-continuous functions to obtain a spectral measure on semi-intervals leading to spectral representation of A. Finally, isolated eigenvalues provide an interpretation of spectral measures in terms of diagonable matrices.

25. For integration on locally compact spaces in Chapter 27, we define regularity in terms of valuations of vector measures. Positive linear forms on the spaces of continuous functions with compact support are identified with regular positive measures and through breakable vector lattices, order-bounded linear forms on with regular measures. Finally the duals of continuous functions on compact spaces are identified with regular measures.

Almost Periodic Functions and Group Representations

26. Almost periodic functions appear in older books such as [Loomis] and [Yosida] but practically vanish in most of recent texts. We want to promote them because mean-values behave like translation-invariant integrals while Haar integrals demand local compactness which is not available in infinite dimensional groups or spaces. Probably special functions or harmonic analysis [Gong] could be developed in infinite dimensional cases.

27. We motivate the readers with a simple example that the sum of two periodic functions need not be periodic and almost periodic functions, abbreviated as ap-functions, have closed relation to group representations. We follow [vonNeumann] and [Maak] closely to introduce mean-values, convolutions and eigen expansion in terms of projectors. We restrict ourselves to matrix representations because of their simplicity and richness. Matrix notation is used whenever possible. Chapters 28,29 should cover the contents of most undergraduate courses in this area. Last two chapters are what we have done on top of [vonNeumann]. For a history of ap-functions, see [Levitan].

28. In real life, we are interested only in continuous unitary representations such as $e^{i\theta x}$ in one dimensional case. We have to define the scope of ap-functions that we work comfortably. Motivated by the duality of compact groups, we introduce saturated closed invariant ideals of comfortable almost periodic functions on groups G, abbreviated as cap-functions. Chapter 30 deals with the duality between cap-functions and representations. Finally, we point out the special cases of additive groups of normed spaces and compact groups.

29. Although mean-values behave like integrals but Monotone Convergence Theorem fails §31-1.3,4. The final Chapter starts with representations of product groups. The mean space $M(G)$ is defined as the dual of $C_\infty(G)$ of cap-functions. It turns out that $M(G)$ has rich structures including convolution, variation and mean-values. With dual order, Monotone Convergence Theorem and Fatou's Lemma hold for means. Parallel to integration theory, we embed cap-functions into $M(G)$ and its closure $\ell_1(G)$ acts like the counter part of L_1-spaces. Due to the restriction of §30-3.9, we have are unable to develop something on $\ell_p(G)$. Look up our web-page for recent development.

30. We hope that the uniqueness of this book could fill in a gap among the current literatures.

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