Preface

This book provides an elementary introduction to classical analysis on normed spaces with special attention to nonlinear topics such as fixed points, calculus and ordinary differential equations. In this second edition, a new approach to vector measures on δ-rings based on dominated convergence is introduced. Matrix-representations of groups are also included because mean-values of almost periodic functions behave similar to translation-invariant integrals and are available in infinite dimensional Banach spaces. This book is for beginners who want to get through the basic material as soon as possible and then do their own research immediately. It assumes only general knowledge in finite dimensional linear algebra, simple calculus, elementary complex analysis and in the last part also elementary group theory. The treatment is essentially self-contained except chapter 27 which may be skipped without discontinuity. With sufficient details, even an undergraduate with mathematical maturity should have no trouble to work through it alone. Various chapters can be integrated into parts of a Master Degree Program by course work organized by any regional university. Restricted to $\mathbb{R}^n$ rather than normed spaces, selected chapters can be used for a course in advanced calculus. We also hope that Engineers and Physicists would find this book to be a handy reference in classical analysis especially our approach to vector measures. High school teachers may be interested to enrich their programs by including the generalization of triangles and tetrahedra as treated in our chapter 4. Some special features are highlighted below.

Banach-Hilbert Spaces

• Sequences can be interpreted as samples taken per unit time. It seems to be more intuitive to use them as description of topological properties.
• Simplicial Complexes are treated in details for potential school projects if restricted to $\mathbb{R}^2$, $\mathbb{R}^3$.
• Transition from $\mathbb{R}_1^n$ to finite dimensional spaces is analytic, §5-1.9,12.
• Explicit formula for no retraction is given in §5-2.4.
• Infinite dimensional topological results are developed without homology.
• Higher derivatives in addition to first derivatives are represented by matrices.
• Higher Chain-Products Formulas §§10-6.2,3 are expressed more naturally in polynomials rather than in symmetric multilinear maps.
• Local solution interval to initial valued problem is independent of initial data.
• Dependence on initial conditions are in global setting. As an informal illustration, suppose a commodity can last one year in a laboratory. Local theory

says that its mass production should work for at least a few seconds but our global theory ensures a period of at least 300 days.

- Tensor products of vectors and linear maps are defined separately but we prove that they are consistent in §15-5.11.
- To the best of our knowledge, tensor products of operators on Hilbert spaces §§15-7.9 to 16 are our contribution.

Vector Measures

- Existing instructors do not have to learn new trick in order to demonstrate their leadership in helping a new generation of scientists to equip with better tools in *vector* measures-integrals.
- Our proposed approach can replace most existing courses in scalar measure theory because the treatment is self-contained without assuming Egorov's Theorem or semivariations from scalar-theory.
- Complex vector lattices are used as framework for measures and means.
- Breakable vector lattices ensure that order bounded linear forms are linear combinations of positive linear forms. This unifies several proofs in §§17-3.4, 24-6.9, 27-3.2, 31-3.2. Some results of real vector lattices are extended to complex breakable vector lattices.
- Semirings are the starting points of *all* our measures. Finite variation is characterized in terms of order §17-3.6 and absolute convergence, §17-4.5.
- Measures are defined on δ-rings so that they need not be bounded. Sets in δ-rings are called *decent sets*. They correspond to bounded Borel sets in $\mathbb{R}^n$.
- Measures are of finite variation in order to use breakable vector lattices.
- Functions of finite variation are related to Stieltjes measures.
- Simple proof of certain complex charge to be of finite variation is given in §18-1.4.
- Restriction to finite-valued outer measures allows the approximation of extension to decent sets by values on sets in semirings, §18-3.3,4,6. As a result, integrals of decent functions are defined.
- A set with a δ-ring is called a δ-space. Measurable sets are defined by localization and are independent of all measures.
- Measurable functions in general are finite-valued and are defined everywhere but μ-measurable functions depend heavily on a particular measure μ, §22-3.4.
- Approximation by simple functions has the additional property of increasing modulus, §19-4.4.
- Explanation §19-5.1,3 why measurable (vector) maps should not be defined trivially as sequential limits of simple maps as in most literatures so that continuous maps are measurable.

- Extension of sign-function to vector maps, §19-6.2.
- Weak and weak-star measurability of maps are unified in §19-6.7.
- Inner regularity defines the integrals of simple functions, §20-1.3.
- Positive measures are defined for all measurable sets, §20-2.2, while vector measures are defined only on integrable sets, §21-1.6.
- Integrals of vector maps are defined by Dominated Convergence, §21-2.8.
- L_p-spaces are in the context of vector maps and vector measures, §21-4.2,4,6.
- L_∞ is defined without measure, §21-5.1.
- Various modes of convergence are defined for vector maps.
- Algebra of measures are developed in §§20-5, 21-8.2.
- Products of vector measures do not require σ-finiteness, §22-2.2
- Product spaces are in the context of vector measures, §22-3.9,19.
- Reduction of elementary operations in linear algebra into two cases, §23-5.3.
- Absolute continuity is characterized at the level of semirings §§24-2.7,9.
- Polar form §24-4.4 reduces complex measures to positive measures.
- The condition of σ-finiteness for Radon-Nikodym theorem together with the vector version of concentration of continuous linear forms §24-6.6 removes the σ-finiteness from the duality of L_p-spaces for $1 < p < \infty$, §24-6.7.
- Cantor set and function are developed on the familiar decimal system, §25-4.
- Spectral measures are treated as a continuation of functional calculus.
- Simple classical technique is employed for monotone convergence, §26-1.7. Our approach is intuitive as shown by the proof of spectral theorem, §26-6.4.
- Spectral measures are extended from semirings to σ-algebras, §26-2.
- Regularity of measures on locally compact spaces are defined in terms of variation to accommodate vector measures.

Group Representations

- Our notation of mean-values strongly indicates the resemblance to integrals but monotone convergence theorem fails for mean-values, §§31-1.3,4. We wish to draw the attention of the community that something similar to translation-invariant integrals has been available on infinite dimensional locally convex spaces although we restrict ourselves to Banach spaces in this book.
- Further development on top of von Neumann's almost periodic functions, abbreviated as ap-functions, has close relation to group representations.
- Restriction to *matrix* representations of groups avoids unnecessary formality.
- Matrix-valued maps are used whenever possible.
- On $\mathbb{R}$, we may be interested only in continuous objects because continuous characters are of the exponential form $e^{i\theta x}$ but our groups have no topology. We get by with comfortable almost periodic functions, abbreviated

as cap-functions. We propose the study of saturated closed invariant ideals, §§30-3.2,3,5,8,9,20.

• Develop product groups based on representations, §31-1.6.

• Means on groups are defined as continuous linear forms on cap-functions. Rich properties §§31-2.4,5,6 deserve better attention from the community.

• As the dual spaces, monotone convergence theorem holds, §§31-3-10, 11.

• $\ell_1(G)$ may be the infinite dimensional counter part of $L_1(\mathbb{R}^n)$ but the establishment of $\ell_p(G)$ versus $L_p(\mathbb{R}^n)$ is an open challenge to readers.

Web-page of this book has been in service since 1997:

http://www.maths.uwa.edu.au/~twma/free/norm/norm.htm

or http://maths.uwa.edu.au/~twma/free/norm/norm.htm

Every paragraph is prefixed with a unique identification. For example, §§1-2.3, 4.5,6 means chapter 1, section 2 paragraph 3 and section 4 paragraphs 5,6. Notations are introduced or recalled at the early stage of a section and then will be used throughout the section unless further specification is mentioned. Theorems without proofs and exercises have the same meaning to us. Exercises are normally illustrative but not tricky because we believe that your time may be used more profitably to do your own research.

References are given at the end of every chapter. Each reference is identified by the family name of the first author only and, if necessary, also by the year of publication. If necessary again, the first letter of one or several words from the title will be included. This method is purely for convenience because it is easy to understand and is independent of the enumeration of references in different books and papers. In order to arouse the curiosity of the beginners, the selection of references is based on informative titles rather than the significance in the history of development. To enrich their cultural background, readers are advised to look up the titles even they may not have the time or facility to study the literatures. Our references offer starting points for further study or to do research in various areas. For example, polymeasures, multimeasures and special functions on groups are not covered in this book but limited references are included. We also include a few references on unbounded measures because our measures need not be bounded. Each reference is normally mentioned only once in the book although it may be related to several chapters. In order to cut down the number of references, if a series of papers on related topics is published by the same person, only the most recent one available in our record is normally quoted. It should give enough information if readers want to trace their earlier work.

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