CHAPTER 1

COMPLEX NUMBERS AND COMPLEX FUNCTIONS

§1.1 Complex Numbers

1.1.1 Field of complex numbers

The reader is familiar with the field of real numbers **R**. In history, difficulty was encounted when people attempted to solve the simplest quadratic equation $x^2+1=0$ which has no solution in **R** obviously. Then mathematicians imagined that there is a number of new kind $i=\sqrt{-1}$ to be its root and so its other root is $-i$. Thus, the equations $x^2+1=0$ would have two roots $\pm i$. If i is allowed to make algebraic operations with real numbers as a common letter and the usual calculating rules are assumed still to be effective but i^2 may be replaced by -1, then a new system of numbers (complex number system) yields. Thereby, as well-known, any quadratic equation $ax^2+bx+c=0$ always possesses two roots. However, in a long time, people suspected whether such numbers really exist and called them "imaginary numbers", that means, numbers in imagination, until their practical meaning was discovered and many problems, impossibly to be solved before, were easily solved by using them. Then, they were generally recognized afterwards. Moreover, it was discovered that they have wild applications in various branches of science and technology.

We would illustrate the logical definition of the complex number field which is generated by **R** together with a formal number i, called the **imaginary unit.** We allow that i may make addition, subtraction and multiplication operations with real numbers as well as with itself: let $i^2 = -1$ $(1 \cdot i, (-1) \cdot i$ and $0 \cdot i$ are denoted by $i, -i$ and 0 respectively) and assume that these operations obey the associative law, the commutative law and the distributive law. These rules of operations are consistent with each other (lead no contradiction). Thus, such a number may be generally written as $z = a + bi$ $(a, b \in \mathbf{R})$, called a **complex number.** The number $\bar{z} = a - bi$ is called the **conjugate (complex number)** of z. Evidently, $\bar{\bar{z}} = z$. Write $z = a + bi = 0$ when and only when $a = b = 0$ and so $z = a + bi \neq 0$ means that at least one of a and b is not zero. For addition, subtraction and multiplication of complex numbers, we have $(a, b, c, d \in \mathbf{R})$

$$(a + bi) \pm (c + d)i = (a \pm c) + (b \pm d)i,$$
$$(a + bi)(c + di) = (ac - bd) + (ad + bc)i;$$

division of complex numbers may be perfomed naturally as follows:

$$\frac{c + di}{a + bi} = \frac{(c + di)(a - bi)}{(a + bi)(a - bi)} = \frac{ac + bd}{a^2 + b^2} + \frac{ad - bc}{a^2 + b^2}i \quad (a + bi \neq 0).$$

In other words, we may realize the division process by multiplying both the divident and the divisor by the conjugate of the divisior. In short, the arithmetic operations for complex numbers may be performed as the same as the real number with $i^2 = -1$ to be reminded.

The set of all the complex numbers with rules of operations as shown above forms a field, called the **complex number field**, denoted by **C**.

For a complex number $z = a + bi, a$ is called its **real part**, denoted by Rez, and b, its **imaginary part**, denoted by Imz. Two complex numbers are said to be equal, iff both their real and imaginary parts are identical. When $b = 0, z = a + 0i = a$ is a real number; when $a = 0, z = 0 + bi = bi$ is called a pure imaginary number. We have the evident identities:

$$\mathrm{Re}z = \frac{z + \bar{z}}{2}, \ \mathrm{Im}z = \frac{z - \bar{z}}{2i}.$$

z is real when Im$z = 0$ or $z = \bar{z}$; z is pure imaginary when Re$z = 0$ or $z = -\bar{z}$.

1.1.2 Geometric representation of complex numbers

As we know in plane analytic gemoetry, after a rectangular coordinate system xOy is taken, any point P in the plane may be expressed by an ordered pair (a, b) of real numbers, called the coordinates of P (Fig.1-1).

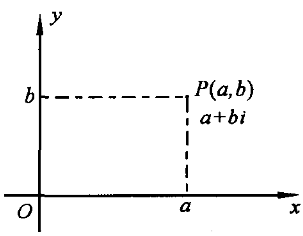

Fig. 1-1

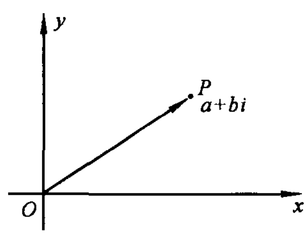

Fig. 1-2

It is obvious that we may express the location of P by the complex number $a + bi$. Actually, we may establish a 1-1 correspondence between the complex number $a + bi$ and the point $P(a, b)$. We then call the plane with a fixed rectangular coordinate system the **complex plane**, denoted by **C** again and $a + bi$ is called the **complex coordinate** or the complex representation of the point P. The complex coordinates of points on the x-axis are real so that the x-axis is also called the **real axis**, while those of points on the y-axis are pure imaginary so that the y-axis is also called the **imaginary axis**.

Complex numbers may be also used to represent planar vectors. As we have seen that a complex number $a + bi$ determines the point P (Fig. 1-2), it also determines the vector $\overrightarrow{OP}$; conversely, a vector $\overrightarrow{OP}$ determines the complex number $a + bi$ or the complex coordinate of P. We simply write $\overrightarrow{OP} = a + bi$. Of course, the complex number 0 corresponds to the null vector. Such vector representation of complex numbers is very useful. For instance, the addition or subtraction for vectors are equivalent to that for complex numbers. Precisely, when $\overrightarrow{OP} = a + bi, \overrightarrow{OQ} = c + di$ and $\overrightarrow{OR} = \overrightarrow{OP} + \overrightarrow{OQ}$, then, according to the parallelogram rule for addition of vectors, we have actually

$$\overrightarrow{OP} + \overrightarrow{OQ} = \overrightarrow{OR} = (a + c) + (b + d)i$$

(Fig. 1-3), which may be checked geometrically at once.

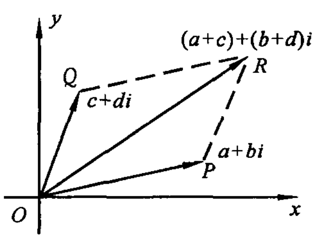

Fig. 1-3

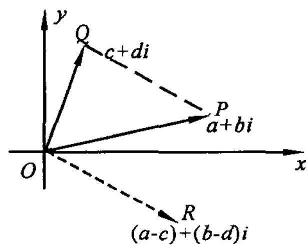

Fig. 1-4

We note that

$$\overrightarrow{OP} - \overrightarrow{OQ} = \overrightarrow{QP}$$

(Fig. 1-4). On the other hand, when the intial point P of $\overrightarrow{PQ}$, as a free vector, is moved to the origin O, it would correspond to $(a - c) + (b - d)i = (a + bi) - (c + di)$, so that

$$\overrightarrow{OP} - \overrightarrow{OQ} = \overrightarrow{QP} = (a + bi) - (c + di).$$

The multiplication of a vector $\overrightarrow{OP} = a+bi$ by a real number λ is identical to the product of $a + bi$ and λ, i.e.,

$$\lambda\,\overrightarrow{OP} = \lambda(a + bi) = \lambda a + \lambda bi.$$

However, the scalar product and the vector product of two vectors have nothing to do with the multipliation of complex numbers.

The non-negative (real) number $\sqrt{a^2 + b^2}$ is called the **modulus** or **absolute value** of $z = a + bi$, denoted by $|z| = |a + bi|$. By the parallelogram rule, it is easy to establish the important triangular inequalities, i.e., for any complex numbers z_1 and z_2,

$$|z_1 + z_2| \leq |z_1| + |z_2|, \ |z_1 - z_2| \geq ||z_1| - |z_2||.$$

We have seen that a complex number $a + bi$ may be regarded as a vector $\overrightarrow{OP}$ which may be determined by its magnitude $|\overrightarrow{OP}| = \rho = \sqrt{a^2 + b^2}$ and its direction: the inclination angle θ, the angle from the positive real axis to the vector $\overrightarrow{OP}$ (θ is unique up to an integral multiple of 2π),[1] or, what is the same, by its polar coordinates with the origin O as the pole and the x-axis as the polar axis (Fig. 1-5). Thus, the complex number $a + bi \neq 0$, may be determined by ρ and θ, where $\rho = \sqrt{a^2 + b^2}$ is its modulus and θ, called its **argument**, denoted by $\mathrm{Arg}(a + bi)$. That is, for $z = a + bi$,

$$|z| = |a + bi| = \sqrt{a^2 + b^2},$$

$$\mathrm{Arg}z = \mathrm{Arg}(a + bi) = \theta + 2k\pi \ (k = 0, \pm 1, \pm 2, \cdots),$$

called its expression in polar coordinates. Arg$z$, as a function of z, is multi-valued as k may take arbitrary integers.

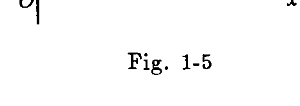

Fig. 1-5

Certainly, if $z = 0$, then $|z| = 0$ and Arg0 is meaningless.

We mention that $|z_1 - z_2|$ is exactly the distance between z_1 and z_2 and $\mathrm{Arg}(z_2 - z_1)$ is the inclination of the vector from z_1 to z_2 in case $z_1 \neq z_2$.

By the well-known relation between the rectangular coordinates and the polar coordinates:

$$x = \rho \cos\theta, y = \rho \sin\theta,$$

[1] The null vector has no definite direction.

we have immediately

$$x + yi = \rho(\cos\theta + i\sin\theta),$$

called the **trigonometric expression** of $x + yi$.

By such expressions of complex numbers, we may get simple rules for calculating products, quotients, powers and radicals of complex numbers. Thus, if

$$x_1 + y_1 i = \rho_1(\cos\theta_1 + i\sin\theta_1),$$
$$x_2 + y_2 i = \rho_2(\cos\theta_2 + i\sin\theta_2),$$

then

$$(x_1 + iy_1)(x_2 + iy_2) = \rho_1\rho_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)],$$
$$\frac{x_1 + iy_1}{x_2 + iy_2} = \frac{\rho_1}{\rho_2}[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)], \ x_2 + iy_2 \neq 0;$$

and, if $x + yi = \rho(\cos\theta + i\sin\theta)$, then

$$(x + yi)^n = \rho^n(\cos n\theta + i\sin n\theta),$$

n being an integer, and by regarding the radical as the inverse of power,

$$(x + yi)^{\frac{1}{n}} = \sqrt[n]{\rho}(\cos\frac{\theta + 2k\pi}{n} + i\sin\frac{\theta + 2k\pi}{n}), \ k = 0, 1, \cdots, n-1,$$

where $\sqrt[n]{\rho}$ (≥ 0) is the arithmetic n-th root of ρ,[2] which, is the well-known expression of the n-th roots of a complex number.

From the above discussion, we see that complex numbers actually have practical significance and are not imaginary and mysterious. The terminology "imaginary number" is retained merely by historical cause.

Real numbers are ordered in the natural way on the number axis. Complex numbers are not ordered in the complex plane. The reader should pay much attention to this distinction between **R** and **C**.

Since we mainly deal with complex numbers in this book, the numbers appeared are always understood by complex numbers unless the contrary is stated.

[2] Sometimes we write $\sqrt[n]{+\rho}$ to emphysize the arithmetic root. For example, we have $\sqrt{+1} = 1$ and $\sqrt{1} = \sqrt{+1}(\cos\frac{2k\pi}{2} + i\sin\frac{2k\pi}{2}) = \pm 1$. For briefness, the symbol "+" is usually omitted.

Problems for Meditation

1. Explain the meaning of the set equalities:

$$\mathrm{Arg}(z_1 z_2) = \mathrm{Arg}z_1 + \mathrm{Arg}z_2, \mathrm{Arg}(z_1/z_2) = \mathrm{Arg}z_1 - \mathrm{Arg}z_2,$$

where $z_1, z_2 \neq 0$.

2. Illustrate that the set equality

$$\mathrm{Arg}z^2 = 2\mathrm{Arg}z$$

is incorrect.

1.1.3 Stereographic projection, complex sphere, point at infinity and extended (complex) plane

In history, the method of stereographic projection was introduced in preparing maps of the world, which establishes a 1-1 correspondence between points on a sphere and points in a plane as follows.

Consider a unit sphere S with its center at the origin. Denote the xOy plane ("equator") by π and the point (0, 0, 1) ("north pole") by N. For any point $P(\neq N) \in S$, we draw a half-ray issured from N and passing through P, which intersects π at a point Q, called the **stereographic projection** of P (Fig. 1-6).

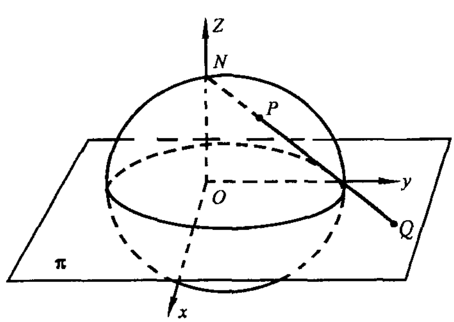

Fig. 1-6

Take π as the complex plane **C**. Let z be the complex coordinate of Q. Thus, to any point P (except N) $\in S$, there corresponds a complex number $z \in \mathbf{C}$; and vice versa. Such correspondence is bicontinuous in the usual topology. As $P \in S$ tends to N, the corresponding point Q varies without finite limit but tends to infinity, written as $z \to \infty$.

If we regard the infinity as an ideal "point" on π, called the **point at infinity** on π (denoted simply by ∞), then we may think that it corresponds to $N \in S$, in the stereographic projection. The complex plane $\mathbf{C}$ together with the point at infinity is called the **extended (complex) plane**, denoted by $\mathbf{C}_\infty$. Thus, the stereographic projection establishes a 1-1 correspordence between S and $\mathbf{C}_\infty$. Moreover, a neighborhood of N on S, for instance, a small circular region on S with center at N, would correspond to a circular exterior region (including the point ∞) on $\mathbf{C}_\infty$ (with center at O and of large radius). If we consider this exterior region or any such region (including ∞) bounded by a closed contour as a neighborhood of the point ∞, then the stereographic projection becomes a bicontinuous correspondence, i.e., a topological mapping, between S and $\mathbf{C}_\infty$. Thus, S is called the **complex sphere** (or **Riemann sphere**).

Finally, we mention that, the conceptions such as open sets, closed sets, (open) regions, closed regions, neighborhoods, etc., familiar to the reader in mathematical analysis, remain the same in $\mathbf{C}$ or $\mathbf{C}_\infty$, and the theorems, such as the finite covering theorem, are valid also.

Problems for Meditation

1. The point at infinity introduced in the text is an improper point and ∞ may be regarded as a generalized complex number. Could we define its modulus, argument, real part and imaginary part? What is the difference between the point at infinity and the infinite quantity in mathematical analysis?

2. State the finite covering theorem in $\mathbf{C}_\infty$.

Exercises 1.1

1. Calculate the real part and the imaginary part of each of the complex numbers: $\frac{1}{i}, \frac{1+i}{1-i}, \left(\frac{1+i}{1-i}\right)^2, (1+\sqrt{2}i)^3$.

Ans. $0, -1; 0, 1; -1, 0; -5, \sqrt{2}$.

2. Calculate the modulus and the argument of each of the complex numbers: $1+i, \frac{1-i}{2}, -i, 2-i$.

Ans. $\sqrt{2}, \frac{\pi}{4}+2k\pi; \frac{1}{\sqrt{2}}, -\frac{\pi}{4}+2k\pi; 1, -\frac{\pi}{2}+2k\pi; \sqrt{5}, -\arctan\frac{1}{2}+2k\pi$.

3. Find the real part and the imaginary part of z^n when (1) $z=x+iy$; (2) $z=\rho(\cos\theta+i\sin\theta)$ (n is a natural number).

Ans. (1) If $n=2m$ is even,

$$\mathrm{Re}z^n=\sum_{k=0}^{m}(-1)^k C_{2k}^{2m}x^{2m-2k}y^{2k},\ \mathrm{Im}z^n=\sum_{k=0}^{m-1}(-1)^k C_{2k+1}^{2m}x^{2m-(2k+1)}y^{2k+1};$$

if $n=2m+1$ is odd,

$$\mathrm{Re}z^n=\sum_{k=0}^{m}(-1)^k C_{2k}^{2m+1}x^{2m+1-2k}y^{2k},\ \mathrm{Im}z^n=\sum_{k=0}^{m}(-1)^k C_{2k+1}^{2m+1}x^{2m-2k}y^{2k+1};$$

(2) $\rho^n\cos n\theta, \rho^n\sin n\theta$.

4. Find the n-th radicals of $1+i$ and $-i$.

Ans.$(1+i)^{\frac{1}{n}} = \sqrt[2n]{2}(\cos\frac{\pi+8k\pi}{4n}+i\sin\frac{\pi+8k\pi}{4n}), k=0,1,\cdots,n-1$; $(-i)^{\frac{1}{n}}=\cos\frac{-\pi+4k\pi}{2n}+i\sin\frac{-\pi+4k\pi}{2n},\ k=0,1,\cdots,n-1$.

5. Prove that, for $z=x+iy$,

$$|x|\ (\text{or } |y|)\le|z|\le|x|+|y|$$

and

$$|z|\ge\frac{|x|+|y|}{\sqrt{2}}.$$

6. Prove that, for arbitrary complex numbers z_1 and z_2,

$$|z_1\pm z_2|^2=|z_1|^2+|z_2|^2\pm 2\mathrm{Re}(z_1\bar{z}_2).$$

7. Prove that, for any z_1 and z_2,

$$|z_1+z_2|^2+|z_1-z_2|^2=2(|z_1|^2+|z_2|^2),$$

and illustrate its geometric meaning.

8. Give the necessary and sufficient condition for the equal sign holds in $|z_1+z_2|\le|z_1|+|z_2|$ or $|z_1-z_2|\ge||z_1|-|z_2||$.

9. Give the geometric interpretation of $z_1+z_2+z_3=0$ and generalize to the case $z_1+z_2+\cdots+z_n=0$ ($n\ge 4$).

10. Prove that

$$\left|\frac{az+b}{\bar{b}z+\bar{a}}\right|=1$$

when $|z| = 1$.

11. Write out the equation of an arbitrary straight line in complex form.

Ans. $\alpha\bar{z} + \bar{\alpha}z + c = 0$ (c is real, $\alpha \neq 0$ is complex).

12. Prove that $z\bar{z} + a\bar{z} + \bar{a}z + b = 0$ is a circle, where b is real and $|a|^2 > b$, and point out the location of its center and the magnitude of its radius. What would happen when $|a|^2 = b$ or $< b$?

Ans. Center: $-a$, radius: $\sqrt{|a|^2 - b}$; a single point or an empty set respectively.

13. Show that, under the stereographic projection, the following relations between the point $P = (x_1, x_2, x_3) \in S$ and its image $z = x + iy$ hold:

$$x_1 = \frac{z + \bar{z}}{|z|^2 + 1}, x_2 = \frac{z - \bar{z}}{i(|z|^2 + 1)}, x_3 = \frac{|z|^2 - 1}{|z|^2 + 1}$$

and

$$z = \frac{x_1 + ix_2}{1 - x_3}.$$

14. Show that, under the stereographic projection, a circle on S is projected to a circle on $\mathbf{C}$. Under what condition the projection of cirles on S are straight lines?

§1.2 Functions of a Complex Variable

1.2.1 Concept of functions of a complex variable

We are familiar with a function of a real variable

$$f : D(\subset \mathbf{R}) \to \mathbf{R}.$$

Naturally, we define a **function of a complex variable** (or, briefly, a **complex function**)

$$f : D(\subset \mathbf{C}) \to \mathbf{C},$$

that is, mapping from a set D of $\mathbf{C}$ into $\mathbf{C}$, also denoted as $w = f(z), z \in D$. D is called the defining domain of f, and $f(D)$, the value domain or image of f. If we regard w belonging to another complex plane, then $f : D \to \mathbf{C}$ or $z \to w = f(z)$ may be considered as a mapping (transformation) from the point set D in the z-plane into the w-plane.

Sometimes, we would deal with complex functions of a real variable, $f : D(\subset \mathbf{R}) \to \mathbf{C}$ or real functions of a complex variable, $f : D(\subset \mathbf{C}) \to \mathbf{R}$, both of which are special cases of general complex functions and so it is not necessary to describe them in particular.

Since complex variables may be expressed by their real and imaginary parts: $z = x + iy, w = u + iv$, the funciton $w = f(z)$ or

$$u + iv = f(x + iy) \tag{1.1}$$

is actually a pair of real functions of two (real) variables:

$$u = u(x, y), \ \ v = v(x, y), \tag{1.2}$$

where $u(x, y) = \mathrm{Re} f(x + iy)$ and $v(x, y)$=Im$f(x, iy)$. Hence, in principle, the theory of complex functions is essentially the theory of real functions of two variables and so it seems that there is no necessity to study the former particularly. However, as we shall see later, the compact form (1.1) for a complex function possesses considerable advantage, which generates a significant branch of mathematics——theory of functions of a complex variable or theory of analytic functions (comples analysis), the goal of this book.

Many concepts relating to real functions of a real variable not involved in the order of real numbers may be easily extended to complex functions, e.g., odd functions, even functions, periodic functions, bounded (unbounded) functions, etc., but the concepts like least upper bound, greatest lower bound and monotone have no meaning for complex functions.

Certainly, there would occur multi-valued complex functions which would be discussed later in detail.

1.2.2 Limit and continuity of complex functions

We are familiar with the ε-δ definition of limit for real functions, in which the notion of absolute value and the related inequalities are used. For complex numbers, we have defined the concept modulus (absolute value) and derived similar inequalities. Therefore, it is natural to define the limit of a complex function analogously: a complex function $w = f(z)$ has the limit A as $z \to z_0$, written as $\lim\limits_{z \to z_0} f(z) = A$, if for any $\varepsilon > 0$, there exists

$\delta > 0$ such that $|f(z) - A| < \varepsilon$ when $0 < |z - a| < \delta$. Its geometric interpretation is similar to the case of real functions with neighborhoods in the complex plane in place of neighborhoods on the (real) number axis. That is, $\lim_{z \to z_0} f(z) = A$ means that, for any circular ε-neighborhood of A in the w-plane, there exists a detached circular δ-neighborhood of z_0 in the z-plane such that $f(z)$ lies in the former when z lies in the latter.

By examining carefully the properties relating to limits in mathematical analysis, it is found that many of them may be transferred to complex functions, such as the arithmetic operations of limits, Cauchy criterion, etc. Of course, some of them, e.g., the bounded monotonic principle, have no similarities in complex analysis.

Moreover, we should note that, because ∞ is regarded as a single point in $\mathbf{C}_\infty$, so $\lim_{z \to z_0} f(z) = \infty$, and $\lim_{z \to \infty} f(z) = A$ or ∞ may be defined in terms of moduli and inequalities or equivalently by neighborhoods (including those of ∞) also, and that, in general, $+\infty$ or $-\infty$ is meaningless unless z or $f(z)$ taking real values only in certain special cases.

Similarly, we may define the continuity of complex funcitons: f is said to be continuous at $z = z_0$ if

$$\lim_{z \to z_0} f(z) = f(z_0), \tag{1.3}$$

f is continuous in a region D if f is continuous at each point of D, f is continuous on a bounded closed region $\bar{D} = D + \partial D$ (∂D is the boundary of D) if f is continuous in D and for each boundary point $z_0 \in \partial D$, it is required that (1.3) holds only when $z \in \bar{D}$ in the limiting process. Naturally, we may introduce the concept of uniform continuity by ε-δ method, which will be omitted here.

Many properties of continuous real functions, such as arithematic operations, continuity of compound of continuous functions, bounded and uniform continuity for functions continuous on a bounded closed region, etc., hold for complex continuous functions, and those relating to medium values and extrema theorems have no similarities there.

It is easily seen that polynomials in z (with complex coefficients) is continuous in the entire plane $\mathbf{C}$ and so do the rational functions except at the zero-points of their denominators.

1.2.3 Concept of homotopy and connectedness of regions

We know that a (planar) continuous curve L may be expressed by parametric equations

$$x = x(t), y = y(t) \quad (\alpha \leq t \leq \beta),$$

where $x(t)$ and $y(t)$ are contiunous real functions, and so, in complex form,

$$z = L(t) = x(t) + iy(t) \quad (\alpha \leq t \leq \beta),$$

where $L(t)$ is a continuous complex function of real variable t. $L(\alpha)$ and $L(\beta)$ are the end-points of L. If $L(t_1) = L(t_2) = \cdots = L(t_p)$ $(p \geq 2)$ for certain distinct valnes $t_1, t_2, \cdots, t_p$ on $[\alpha, \beta]$, at least one of which is different from α or β, then point $L(t_1)$ is called a multiple point of L. A continuous curve without multiple points is called a **simple curve** or **Jordan curve.** A simple curve with coincident (different) end-points, i.e., $L(\alpha) = L(\beta)$ $(L(\alpha) \neq L(\beta))$ is called a **simple closed contour (open arc)**, or a **Jordan closed contour (open arc).** If, besides, $L'(t) = \alpha'(t) + i\beta'(t)$ exists, continuous and $\neq 0$ on the closed interval $[\alpha, \beta]$, then L is called a **simple smooth curve**, or simply, a **smooth curve**. A smooth curve with coincident end-points together with coincident directed tangents is called a **smooth closed contour**. A simple (continuous) curve consists of a finite number of consecutive smooth arcs is called an **arcwise smooth curve.** The concepts of simple open arcs, smooth or arcwise smooth open arcs are easily and naturally extended to the case where $L(t) \to \infty$ as $t \to \alpha$ or(and) $t \to \beta$. We would particularly mention such cases and say that the mentioned curves are extended to infinity.

Take the increasing direction of t as the positive sense of L. $L(\alpha)$ and $L(\beta)$ are called its initial point and terminal point of L respectively. They are identical when L is closed. For a closed contour, we usually take the counter-clockwise sense as its positive orientation, denoted by L^+ or simply L. L with opposite orientation is denoted by L^-, having complex equation $z = L(-t)$ $(-\beta \leq t \leq -\alpha)$.

Let L_0 and $L_1 : [0,1] \to D$ be two (oriented) simple curve in a region D with the same initial and terminal points: $L_0(0) = L_1(0), L_0(1) = L_1(1)$. If there exists a continuous function $\psi : [0,1] \times [0,1] \to D$ such that

$$(1) \quad \psi(t,0) = L_0(t), \psi(t,1) = L_1(t), \;\; 0 \leq t \leq 1; \tag{1.4}$$

$$(2) \quad \psi(0,s) = L_0(0) = L_1(0), \psi(1,s) = L_0(1) = L_1(1), \tag{1.5}$$

then L_0 and L_1 are said to be **homotopic** (to each other) in D, denoted by $L_0 \sim L_1(D)$, or simply $L_0 \sim L_1(D)$, and ψ is called a **homotopic deformation (displacement)** from L_0 to L_1.

For an arbitrarily fixed $s \in [0,1], \psi_s$: $[0,1] \to D$ is a continuous curve in D with the same initial point $L_0(0)$ and the same terminal point $L_0(1)$. When s varies from 0 to 1, we get a family of continuous curves $\{\psi_s\}$ in D. Thus, intuitively, homotopy means that there exists a family of continuous curves $\{\psi_s\}$, through which L_0 may be continuously deformed to L_1 within the region D (Fig. 1-7(a), (b)).

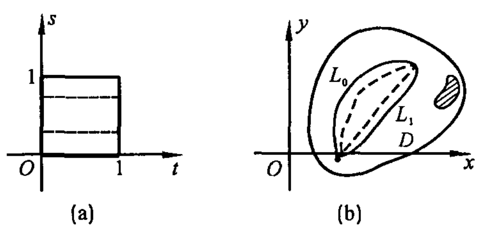

Fig. 1-7

For example, any two continuous curves L_0 and L_1 in a circular disk D or in the entire plane with the same initial and terminal points are always homotopic to each other, since by the convexity of such region we may take $\psi : [0,1] \times [0,1] \to D$ as $\psi(t,s) = sL_1(t) + (1-s)L_0(t)$ as the homotopic deformation.

If L_0 and $L_1 : [0,1] \to D$ are two simple closed contours in a region D and there exists a continuous function $\psi : [0,1] \times [0,1] \to D$ satisfying (1.4) together with

$$\psi(0,s) = \psi(1,s), 0 \leq s \leq 1, \tag{1.6}$$

then L_0 and L_1 are called homotopic to each other in D.

If $L_1(t)$=const, i.e., L_1 is a single point, then we say that L_1 is a null contour. For a curve L_0 homootpic to a null contour (in D), we write $L_0 \sim 0(D)$.

If L_0 and $L_1 : [0,1] \to D$ are two smooth closed contours, homotopic to each other in D with ψ, ψ'_t and ψ'_s continuous on $R = [0,1] \times [0,1]$, then we say that L_0 and L_1 are **smoothly homotopic** in D. We mention (without proof) that, in this case, "homotopic" is equivalent to "smoothly homotopic".

We classify all the smooth closed contours in D such that homotopic contours will be put into one class, which is possible since the homotopic relation satisfies the reflexive law, symmetric law and transitive law. The class containing a curve L will be denoted by $[L]$ and the class containing the null curve, by $[0]$. If there is only the single class $[0]$ in a region D, i.e., any closed contour in D is homotopic to 0, then D is called a **simply connected region**; otherwise, a **multiply connected region.**

For example, in the complex plane, the interior region D bounded by a simple closed contour is simply connected since the interior region G bounded by any simple closed contour L lying entirely in D may obviously shrink into a single point in D. Similarly, the complex plane $\mathbf{C}$ itself is also simply connected.

In plane $\mathbf{C}$, the region D bounded between two non-intersecting simple closed contours L_1 and L_2 (L_2 lying entirely in the interior region bounded by L_1) is multiply connected (Fig. 1-8), since , besides the closed contours in D homotopic to 0, a closed contour L in D surrounding L_2 could not shrink to a single point in D, i.e., not homotopic to $0 : [L] \neq [0]$. Moreover, any simple closed contour L' in D either surrounds L_2 or not. Such a region is called **doubly connected.**

Similarly, the exterior region D in $\mathbf{C}$ bounded by a simple closed contour L is doubly connected (Fig. 1-9), since a closed contour L' in D surrounding L could not shrink continuously in D to a single point. The same is the region of the whole plane $\mathbf{C}$ with a point detached.

In general, roughly speaking, a finite region bounded by a simple closed contour with n holes (or point-holes, or cracks) in its interior is said to be $(n+1)$-**tuply connected**, or simply, $(n+1)$-**connected**; in such case, there are exactly n simple closed contours (each of which surrounds a hole) not homotopic to each other and not to zero. The number $n+1$ is called the **connectedness** of the region. The same is the infinite region of the whole plane $\mathbf{C}$ with n holes.

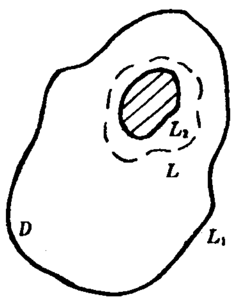

Fig. 1-8

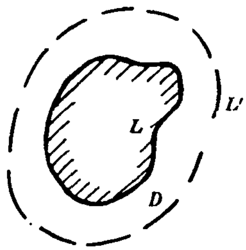

Fig. 1-9

The concepts of homotopy and connectedness may be extended to a region on the sphere S, provided that the closed continuous contours and their homotopic deformations are on S. For instance, the whole sphere S is simply connected. However, we should note that the region on the sphere bounded by a single closed contour or with a single point detached is also simply connected.

Since the extended complex plane $\mathbf{C}_\infty$ and the sphere S have the same topological structure, the entire plane $\mathbf{C}_\infty$ as well as the region in $\mathbf{C}_\infty$ bounded by a sinple closed contour (internally or externally) or $\mathbf{C}_\infty$ with a simgle point detached is simply connected. Let us intuitively verify the latter. Let D be the region $\mathbf{C}_\infty$ with a point z_0 detached and L be a simple closed contour in D. If L does not surround z_0, then evidently $L \sim 0$. If L surrounds z_0 (Fig.1-10), then L may "shrink" (actunlly "expand") continuously to the point at infinity and so $L \sim 0$ also. If L passes through ∞, the situation is the same. Thus, the considered region is simply connected in $\mathbf{C}_\infty$.

Therefore, we should always bear in mind that, for an unbounded region D with a simple closed contour as its boundary to be simply connected or not, we must clarify that our discussion is made with respect to $\mathbf{C}_\infty$ or $\mathbf{C}$. However, for a bounded region D, the simple connectedness in $\mathbf{C}$ or $\mathbf{C}_\infty$ is the same.

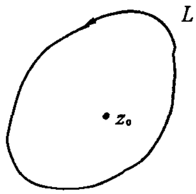

Fig. 1-10

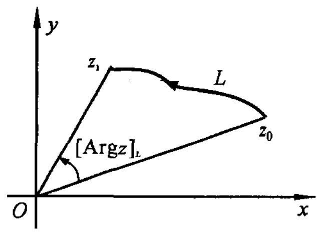

Fig. 1-11

1.2.4 Argument function

As we shall see later, many elementary complex functions are multi-valued, the multiplicity of which is due to the **argument funciton** Arg$z$. Therefore it is beneficial to clarify the latter in advance.

As we know, the argument of any complex number z $(\neq 0)$ possesses infinitely many values. So the argument function w =Arg$z$ is a multi-valued function, the defining domain of which is $\mathbf{C} - \{0\}$.

Let L be an oriented simple continuous arc (or closed contour) in $\mathbf{C} - \{0\}$ with its initial point z_0 and terminal point z_1. As z describes along L from z_0 to z_1, the angle of rotation of the vector $\overrightarrow{Oz}$ is called the increment of Arg$z$ along L, or simply the **increment of argument**, denoted by $[\mathrm{Arg}z]_L$ (Fig. 1-11).[3] For example, for the three arcs shown in Fig. 1-12 (a), (b), (c) with the same initial point $1 - i$ and the same terminal point $1 + i$, we

[3] For an arc L initiating from (or terminating at) the origin O, we regard the argument of $z = 0$ with respect to L is the inclination of the directed tangent of L at O, provided it exists. Then $[\mathrm{Arg}z]_L$ may be defined as well.

have

$$[\mathrm{Arg}z]_{L_1} = \frac{\pi}{2}, \ [\mathrm{Arg}z]_{L_2} = -\frac{3\pi}{2}, \ [\mathrm{Arg}z]_{L_3} = -\frac{3\pi}{2}.$$

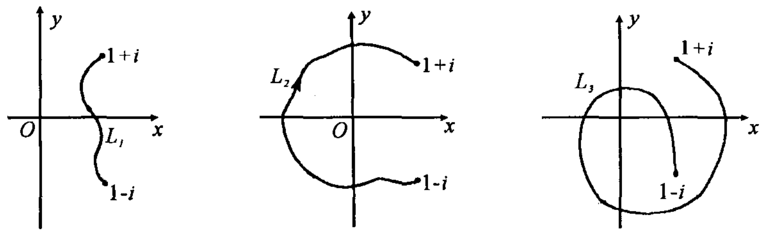

Fig. 1-12

Generally speaking, the increments of argument are different for different arcs even if their initial points and terminal points are the same, which may differ by multiples of 2π. We would know under what condition they are identical. The answer is: for L_0 and L_1 with the same initial point and the same terminal point, not passing by O, $[\mathrm{Arg}z]_{L_0} = [\mathrm{Arg}z]_{L_1}$ when and only when $L_0 \sim L_1$ in $\mathbf{C} - \{0\}$. For, in this case, L_0 may be deformed continuously to L_1 without passing through the origin and so the value of $[\mathrm{Arg}z]_{L_0}$ would likely vary continuously to $[\mathrm{Arg}z]_{L_1}$ so that it could not make a jump $2k\pi$ ($k \neq 0$), that means, it would keep the initial value unchanged. If L_1 is a null contour, it is evident that $[\mathrm{Arg}]_{L_1} = 0$. Since any simple closed contour L in $\mathbf{C} - \{0\}$ not surrounding the origin O may shrink to a single point, i.e., $L \sim 0$, we have:

(1) for any simple closed contour $L \subset \mathbf{C} - \{0\}$,

$$[\mathrm{Arg}z]_{L^+} = \begin{cases} 0, & \text{when } z = 0 \text{ lies in the exterior of } L, \\ \pm 2\pi, & \text{when } z = 0 \text{ lies in the interior of } L; \end{cases} \tag{1.7}$$

moreover,

(2) for any L, closed or open,

$$[\mathrm{Arg}z]_{L^+} = -[\mathrm{Arg}z]_{L^-}; \tag{1.8}$$

(3) if $L = L_1 + L_2$ with the terminal point of L_1 coincident to the initial point of L_2 (we say that L_1 and L_2 are consecutive arcs), then

$$[\mathrm{Arg}z]_L = [\mathrm{Arg}z]_{L_1} + [\mathrm{Arg}z]_{L_2}. \tag{1.9}$$

Let L be a continuous arc in $\mathbf{C} - \{0\}$ with the initial point z_0 and the terminal point z. Take a fixed valne of Arg$z$ at z_0, denoted by argz_0, called the initial value of Arg$z$ at z_0. Then $\arg z_0 + [\mathrm{Arg}z]_L$ is called the terminal value of Arg$z$ at z_0, denoted by arg$z$:

$$\arg z = \arg z_0 + [\mathrm{Arg}z]_L; \tag{1.10}$$

that means, when the independent variable varies along L from z_0 to z, the initial value argz_0 varies continuously to arg$z$ which is dependent on the initial value argz_0 and the increment of argument $[\mathrm{Arg}z]_L$. By definition, $[\mathrm{Arg}z]_L$ relies on the initial point z_0 and the terminal point z of L as well as the path L itself but independent of the initial value argz_0.

Multi-valued functions are not convenient in applications and so it is hoped that arg$z$ may be separated into certain single-valued continuous functions. Such a process is called **uniformization.**

We see that, by (1.10), even if the initial point z_0 and the initial value argz_0 are fixed, for any $z \in \mathbf{C} - \{0\}$, arg$z$ is not unique since $[\mathrm{Arg}z]_L$ depends upon the path L. That means, arg$z$ could not be regarded as a single-valued function in $\mathbf{C} - \{0\}$ in general. It is natural to ask whether this is possible when we confine ourselves to certain smaller region instead of $\mathbf{C} - \{0\}$. The key to the solution of the problem lies in that we should look for a region in which the increment of argument along an arc L relies on the initial point and the terminal point of L but independent of the shape of L itself. By (1.7), we know that this is possible when any simple closed contour in the considered region could not surround the origin O. Thus, for instance, let us cut the complex plane $\mathbf{C}$ along the positive real axis so that it becomes a simply connected region D with the cut as its boundary. Each position on the cut should be regarded as two distinct points on its boundary, one on the upper bank (side) and the other, on the lower bank, or, what is the same, the upper and the lower banks of the cut should be considered as two different boundary lines of the region D (Fig. 1-13). Take any simple closed contour L in D (or in $\bar{D}$, i.e., L may contact to the upper or(and) lower banks of the cut not passing through it) with parametric equation

$$L(t) = \rho(t)[\cos\theta(t) + i\sin\theta(t)],\ 0 \leq t \leq 1,$$

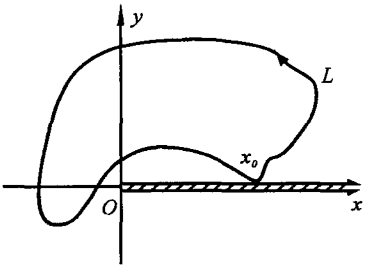

Fig. 1-13

where $\rho(0) = \rho(1) > 0, \theta(0) = \theta(1)$, and define homotopic displacement $\psi(t, s)$ by

$$\psi(t,s) = [-s\rho(0) + (1-s)\rho(t)]\{\cos[(1-s)\theta(t)] + i\sin[(1-s)\theta(t)\},$$

which is continuous on $[0,1] \times [0,1]$ and $\in D$,

$$\psi(t,0) = L(t), \psi(t,1) = -\rho(0), \psi(0,s) = \psi(1,s),$$

and so $L \sim 0(D)$ which implies $[\mathrm{Arg}z]_L = 0$. Thereby, for any simple arc L, $[\mathrm{Arg}z]_L$ is independent of the path L. Fix an initial point z_0 in D and take a definite value $\arg z_0$, then $\arg z = \arg z_0 + [\mathrm{Arg}]_L$ (L with the initial point z_0 and a variable terminal point z) is a single-valued continuous function in D. If we take the initial value $\arg z_0 + 2\pi$ instead, then we obtain another single-valued continuous function $\arg_1 z = \arg z + 2\pi = \arg z_0 + 2\pi + [\mathrm{Arg}z]_L$. In general, if we take the initial value $\arg z_0 + 2k\pi$ (k: integer), then we get a single-valued function $\arg_k z = \arg z + 2k\pi$ (we may also write $\arg z = \arg_0 z$). Thus, we have separated $\mathrm{Arg}z$ in D into infinitely many single-valued continuous functions:

$$\arg_k z = \arg z + 2k\pi, z \in D, k \in J\ \ (J : \text{the set of integers}).$$

Each of them is called a (single-valued continuous) **branch** of Arg$z$ in D. If we take a definite point $x_0 > 0$ on the upper bank of the positive real axis as the initial point and take arg$x_0 = 0$ as the initial value, then we get a single-valued branch arg$z$ ($0 \leq \arg z \leq 2\pi$); if z is restricted in the open region D, then $0 < \arg z < 2\pi$.[4] Thus, we may write

$$\mathrm{Arg}z = \arg z + 2k\pi,\ \ 0 \leq \arg z \leq 2\pi, k \in J. \tag{1.11}$$

Similarly, if we cut the plane $\mathbf{C}$ along the negative real axis and take the argument of a certain point ($\neq 0$) on its upper bank to be π, then we may write

$$\mathrm{Arg}z = \arg z + 2k\pi, -\pi \leq \arg z \leq \pi, k \in J. \tag{1.12}$$

In general, let the plane be cut by a simple arc connecting 0 and ∞, then in the cut region, Arg$z$ may be also separated into infinitely many single-valued branches. Since the point 0 and ∞ possess special positions, they are called the **branch points** of Arg$z$.

In short, a single-valued branch of Arg$z$ is determined by the cut used as well as its initial value at a certain point; different branches would be obtained even if the cut is fixed as different initial values at that point are taken. For instance, if the cut is taken as the positive real axis and regard arg1=0 or 2π for $z = 1$ on the upper bank, then the corresponding value of arg$i$ is $\frac{\pi}{2}$ or $\frac{5\pi}{2}$ respectively. When the cuts used are different, different branches are also obtained even if the initial value is the same at certain point. For example, if we take the positive or negative real axis as the cut and take $\arg i = \frac{\pi}{2}$ as the initial value, then we would have $\arg(-i) = \frac{3\pi}{2}$ or $-\frac{\pi}{2}$ respectively.

Let us return to the region D in Fig. 1-13 and denote it now by D_0. The upper bank and the lower bank of the cut used (the positive real axis) are denoted by l_0^+ and l_0^- respectively, and a definite branch $\arg_0 z$ of Arg$z$ on $\bar{D}_0$ is taken ($0 \leq \arg_0 z \leq 2\pi$). Let us take another copy of the cut plane

[4] Many authors call this single-valued branch the principal branch of Arg$z$, denoted by arg$z$. However, the "principal value" is not always the value (principally) used in many cases.

and denote the corresponding region by D_1, the upper bank and the lower bank of the cut by l_1^+ and l_1^- respectively, and take a branch $\arg_1 z$ of Arg$z$ on $\bar{D}_1$ ($2\pi \le \arg_1 z \le 4\pi$). We imagine that the latter is overlapped on the above of the former. For any position x (> 0) on the positive real axis, the argument of its lower bank on $\bar{D}_0$ is $\arg_0 x = 2\pi$ and that of its upper bank on $\bar{D}_1$ is $\arg_1 x = 2\pi$ too. Then, if we paste the lower bank of the cut of D_0 with the upper bank of the cut of D_1 (the same position x pasted together) to form an overlapped region D, then we obtain a single-valued continuous function $\arg z$ on $\bar{D}$ ($0 \le \arg z \le 4\pi$), the boundary of which consists of the upper bank of the cut of D_0 and the lower bank of the cut of D_1.

In general, if we make the k-th copy of the same cut plane, say D_k ($k = 0, \pm 1, \pm 2, \cdots$), take the single-valued branch $\arg_k z$ of Arg$z$ on $\bar{D}_k$:

$$2k\pi \le \arg_k z \le 2(k+1)\pi,$$

and paste the lower bank of the cut l_k^- of D_k with the upper bank of the cut l_{k+1}^+ of D_{k+1}, then we would arrive at an ideal "region" D with infinitely many overlapped sheets, and in the mean while, Arg$z$ may be regarded as a single-valued continuous funciton in D. Fig. 1-14 illustrates some sheets of D.

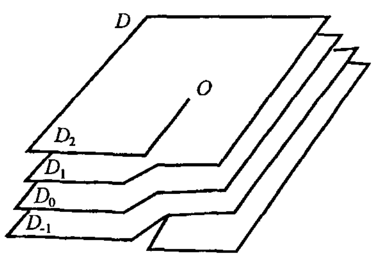

Fig. 1-14

The idea for the uniformization of the argument function in this way is very significant. This is the prototype of an important branch in modern theory of functions, the Riemann surface, which will be introduced later.

The foregoing discussion on Arg$z$ is easily extended to Arg$(z - z_0)$ (z_0

being a fixed number) by taking z_0 in place of the origin, in which case its branch points are z_0 and ∞.

Problems for Meditation

1. When a definite branch arg$z$ is considered, does the domain of variation of the argument of z always not exceed 2π?

2. Take a definite single-valued branch arg$z$. Illustrate by examples that the following equalities do not hold in general:

$$\arg(z_1 z_2) = \arg z_1 + \arg z_2, \quad \arg(z_1/z_2) = \arg z_1 - \arg z_2.$$

Exercises 1.2

1. Prove that $\lim\limits_{z \to z_0} f(z) = A$, where $A = a + ib, z_0 = x_0 + iy_0$ and $f(z) = f(x+iy) = u(x,y) + iv(x,y)$, iff

$$\lim_{\substack{x \to x_0 \\ y \to y_0}} u(x,y) = a, \quad \lim_{\substack{x \to x_0 \\ y \to y_0}} v(x,y) = b.$$

2. Prove that $f(z) = u(x,y) + iv(x,y)$ $(z = x + iy)$ is continuous at z iff both the real functions $u(x,y)$ and $v(x,y)$ are continuous at (x,y).

3. Discuss the continuity of the argument function $w = \arg z, 0 \le \arg z < 2\pi$, in C.

Ans. It is continuous everywhere except the positive real axis (including the origin).

4. Is the function $f_1(z) = e^{-1/|z|}$ uniformly continuous in $0 < |z| < 1$? How about $f_2(z) = e^{1/|z|}$?

Ans. Yes for $f_1(z)$; no for $f_2(z)$.

5. Describe the graphs of the following equations:

(1) $|z-5| = 6$; (2) $|z+i| = |z-i|$;

(3) $\mathrm{Re}(i\bar{z}) = -z$; (4) $|z+3| + |z+1| = 4$;

(5) $\arg(z-i) = \frac{\pi}{4}$; (6) $\mathrm{Im}\frac{1}{z} = 1$.

Ans. (1) the circle with center at $z = 5$ and of radius 6; (2) the x–axis;(3) $y = -3$; (4) the ellipse with foci $(-3,0)$ and $(-1,0)$, and major axis 4; (5) the half-ray issued from i (not including i) with angle of inclination $45°$; (6) the circle with center at $(0, \frac{1}{2})$ and of radius $\frac{1}{2}$, with the origin detached.

6. Point out the regions determined by the following inequalities and mention that they are open or closed, simply or multiply connected, bounded or unbounded, and how about in $\mathbf{C}_\infty$:

(1) $|z-1| > 1$; (2) $2 < |z-1| < 3$; (3) $|\frac{1}{z}| > 3$;
(4) $-1 \leq \mathrm{Re}z \leq 2$; (5) $\mathrm{Im}z \leq 0$; (6) $|\arg z| < \frac{\pi}{3}$;
(7) $\mathrm{Re}\frac{1}{z} < 1$; (8) $\mathrm{Im}z^2 \geq 1$.

Ans. (7) interior region bounded by the circle with center at $(\frac{1}{2},0)$ and of radins $\frac{1}{2}$; (8) the region not containing the origin bounded by the hyperbola $xy = \frac{1}{2}$.

7. What are the image curves in the w-plane of the curves in the z-plane:

(1) $x^2+y^2=4$; (2) $y=x$;
(3) $x=1$; (4) $(x+1)^2+y^2=1$;

mapped by the function $w=\frac{1}{z}$?

Ans. (1) $u^2+v^2=(\frac{1}{2})^2$; (2) $u+v=0$; (3) $(u-\frac{1}{2})^2+v^2=\frac{1}{4}$; (4) $u=\frac{1}{2}$.

8. Prove that, for any simple closed contour L in a star region D (i.e., there exists a point $\sigma \in D$ such that the line-segment connecting any point in D with σ lies entirely in D), $L \sim 0(D)$.

9. Could the function $\mathrm{Arg}z$ be uniformized in the annus (circular ring region) $1 < |z| < 3$ or $0 < |z| < \rho$ $(\rho > 0)$? Why?

Ans. Yes; no.

10. Construct four regions with cuts such that in each of which $\mathrm{Arg}(z-i)$ may be uniformized.

11. Assume $L : |z-3i| + |z-i| = 4$. Evaluate $[\arg(z-2i)]_{L^-}$, $[\arg(z+i)]_{L^+}$ and $[\arg z]_{L^+}$.

Ans. -2π, 0 and π respectively.

12. Cut the plane $\mathbf{C}$ by the spiral $\rho = \theta$ $(0 \leq \theta < +\infty)$ and take $\arg 3\pi = 0$ for a branch $\arg z$. Find the values of $\arg(5\pi + \frac{\pi}{2})i$ and $\arg(-\frac{\pi}{2}i)$.

Ans. $\frac{5}{2}\pi$; $-\frac{5}{2}\pi$.

§1.3 Complex Sequences and Complex Series

1.3.1 Sequences and series with complex terms

Just as the theory of sequences and series in real analysis, sequences and series with complex terms play significant role in complex analysis.

$\{z_n\}_1^\infty$ is called a (complex) sequence when $z_n, n = 1, 2, \cdots$, are complex numbers. Because the modulus (absolute value) of a complex number and the triangular inequality are similar with those of real numbers, so the definition of $\{z_n\}$ or z_n convergent to A, i.e., $z_n \to A$ or $\lim_{n\to\infty} z_n = A$, or divergent to ∞, i.e., $z_n \to \infty$ or $\lim_{n\to\infty} z_n = \infty$, as well as many propositions such as Cauchy criterion for convergence, arithmetic operations, etc., is analogous to the case in real analysis. Sometimes we also say z_n "converges" to ∞ for $z_n \to \infty$, since ∞ may be regared as a point in $\mathbf{C}_\infty$.

In connection with real sequences, we have the following principle for complex sequences: if $z_n = a_n + ib_n (a_n, b_n$ being real), then z_n is convergent iff both a_n and b_n are convergent; moreover, $z_n \to A = a + ib$ when $a_n \to a$ and $b_n \to b$.

$\sum\limits_{n=1}^{\infty} z_n$, or simply Σz_n, is called a complex series. As in real analysis, its convergence is defined by the convergence of its partial sum $S_n = \sum\limits_{k=1}^{n} z_k$; its sum and remainder (when it is convergent) are defined as the same in real analysis. It is evident that $\Sigma(a_n + ib_n)$ is convergent iff both Σa_n and Σb_n are convergent.

We may also define the absolute convergence of Σz_n by the convergence of $\Sigma |z_n|$ and establish that an absolutely convergent series is necessarily convergent itself, which is easily seen by considering the real series formed by the real parts and the imaginary parts of its terms respectively.

Besides addition or subtraction may be performed term by term for convergent series, the order of the terms of an absolutely convergent series may be arbitrarily rearranged without changing its sum, and multiplication may be realized for two absolutely convergent series, i.e., if both Σz_n and $\Sigma z'_n$ are absolutely convergent to S and S' respectively, then the series $\Sigma z_j z'_k$(with terms arranged in any order) is also absolutely convergent and has SS' as its sum. These and many other properties of complex series are similar to the corresponding ones of real series and may be established analogously or proved by transferring them to real cases.

1.3.2 Sequences and series of complex functions

For a sequence of complex functions $\{f_n(z)\}_1^\infty$ or a series of complex functions $\Sigma f_n(z)$, where $f_n(z), n = 1, 2, \cdots$, are functions defined on certain set in $\mathbf{C}$, e.g., a region D, we have the terminologies such as *convergence* or divergence, point or domain of convergence, limit function or sum function, etc., especially the concept of uniform convergence. For the latter, we have also the Cauchy criterion and the following well-known propositions:

$1°$ If $f_n(z), n = 1, 2, \cdots$, are continuous in a region D and $\{f_n(z)\}$ (or $\Sigma f_n(z)$) is uniformly convergent in D, then its limit (or sum) function is also continuous in D.

$2°$ If $|f_n(z)| \le M_n, n = 1, 2, \cdots, z \in D$, where M_n's are constants, and ΣM_n is convergent, then $\Sigma f_n(z)$ is uniformly convergent in D.

Moreover, the condition in $1°$ may be weaken: keeping the assumption of the continuity of $f_n(z)$ $(n = 1, 2, \cdots)$, we may get the same conclusion as in $1°$ even if $f_n(z)$ (or $\Sigma f_n(z)$) is uniformly convergent on any compact (i. e. , bounded and closed) subregion $\bar{D}_1 \subset D$, called **internal uniform convergence in D** . The proof of these properties are similar to those in real case, which will be left to the reader.

The other (analytic) properties of uniformly convergent series of complex functions (including differentiation or integration term by term) and discussions on complex power series will be given in chapter 4.

Exercises 1.3

1. Do the limits of the following sequences exist? If yes, find them:

(1) $i^n + \frac{i}{n}$; (2) $\frac{n!}{n^n} i^n$; (3) $(z/\bar{z})^n$.

Ans. (1) no; (2) yes, 0; (3) no.

2. Are the following series convergent? absolutely convergent?

(1) $\Sigma \frac{i^n}{n}$; (2) $\Sigma \frac{i^n}{n!}$; (3) $\Sigma (1+i)^n$.

Ans. (1) conditionally covergent; (2) absolutely convergent; (3) divergent.

3. Are $\{|z_n|\}$ and $\{\arg z_n\} (0 < \arg z_n \le 2\pi)$ convergent if $\{z_n\}$ is convergent? Is it true that, conversely, if both $\{|z_n|\}$ and $\{\arg z_n\}$ are convergent?

4. Prove that $\Sigma \frac{z^n}{n!}$ is internally uniformly convergent in the complex plane $\mathbf{C}$.

Exercises of Chapter 1

1. Prove that the necessary and sufficient condition for the three points z_1, z_2, z_3 to be colinear is: there exist real numbers $\lambda_1, \lambda_2, \lambda_3$, not all zero, fulfilling $\lambda_1 + \lambda_2 + \lambda_3 = 0$, such that

$$\lambda_1 z_1 + \lambda_2 z_2 + \lambda_3 z_3 = 0.$$

2. Define the **cross-ratio** of four complex numbers z_1, z_2, z_3, z_4 by

$$(z_1, z_2, z_3, z_4) = \frac{z_3 - z_1}{z_3 - z_2} : \frac{z_4 - z_1}{z_4 - z_2}.$$

Show that they are con-circular iff $\mathrm{Im}(z_1, z_2, z_3, z_4) = 0$.

3. Let $d(z_1, z_2)$ be the distance between the points P_1 and P_2 on the complex sphere, corresponding to z_1 and z_2 respectively under the stereographic projection. Prove that

$$(1) \quad d(z_1, z_2) = \frac{2|z_1 - z_2|}{\sqrt{|z_1|^2 + 1}\sqrt{|z_2|^2 + 1}};$$

$$(2) \quad d(z, \infty) = \frac{2}{\sqrt{|z|^2 + 1}}, \quad z \in \mathbf{C},$$

4. Give the precise definition of $\lim\limits_{z \to \infty} f(z) = A$. Let $f(z) = u(x, y) + iv(x, y)$ ($z = x + iy$) and $A = \alpha + i\beta$. Does it mean both $\lim\limits_{\substack{x \to \infty \\ y \to \infty}} u(x, y) = \alpha$ and $\lim\limits_{\substack{x \to \infty \\ y \to \infty}} v(x, y) = \beta$? How about the converse?

5. What is the graph of $|\frac{z-a}{z-b}| = K$ ($a \neq b, K > 0$ are constants)?

6. Is it true: $\mathrm{Arg}\frac{1}{z} = -\mathrm{Arg}z$? After a single-valued branch is taken, is it true: $\arg\frac{1}{z} = -\arg z$? Similarly discuss about $\mathrm{Arg}\,\bar{z} = -\mathrm{Arg}z$ and $\arg\bar{z} = -\arg z$.

7. Assume that Σa_n is convergent and $|\arg a_n| \le \alpha$ ($< \frac{\pi}{2}$). Prove that Σa_n is absolutely convergent.

8. If $z_n \to A$ ($\neq \infty$) and $\{p_n\}_1^\infty$ is an arbitrary sequence of positive numbers, prove that $z_n' = \frac{p_1 z_1 + \cdots + p_n z_n}{p_1 + \cdots + p_n} \to A$ as $n \to \infty$. Does the conclusion remain true if $z_n \to \infty$?

9. Prove that $\Sigma f_n(z)$ to be internally uniformly convergent in a region D is equivalent to the series

(1) to be uniformly convergent in any open circular disk in D with a positive distant to ∂D, or

(2) to be uniformly convergent on any bounded closed subset of D.