Experiment 1

Fundamental of Calculus

The purpose of this experiment is to show some basic functions in Mathematica. Using these functions one can verify or achieve some fundamental conclusions in calculus by observation.

Mathematica is a symbol computational system. It helps to complete the numerical value computation, symbol computation and construction etc. in fundamental and advanced mathematics.

1.1 Functions and Their Graphs

The format of the Mathematica command to construct the function $y = f(x)$ in $[a, b]$ is:

```
Plot[f, {x, a, b}]
```

The format of the command to construct the parameter equation $x = f(t)$, $y = g(t)$, $t \in [a, b]$ is:

```
ParametricPlot[{f, g},{t, a, b}]
```

Exercise 1. (1) Construct the function $y = x^3 - 6x + 2$ and its derivative $y' = 3x^2 - 6$ in the same coordinate system. The Mathematica command is as follows:

```
Plot[x^3-6x+2,3x^2-6,{x,-5,5}]
```

Note. That is, we want the graphs in the interval $[-5, 5]$. Other intervals can also be specific, and other functions $f(x)$ and their derivatives $f'(x)$.

Observe the following phenomena:

(i) The increasing and decreasing condition of the graph of y when $y' > 0$ or $y' < 0$ respectively. When $y' = 0$, is y at its extremum?

(ii) The convex or concave condition of y when y' goes up or down. When y' is at its extremum, will the inflexion of the graph of y appear?

(iii) Find, by observation, an approximation a of the root of the equation $y = 0$. Say $a = 2$. Then, use the following command to get a more accurate result near 2.

```
FindRoot[x^3-6x+2,{x,2}]
```

The principle of finding the root is:

Regard the function $y = f(x)$ near $x = a$ as the linear function

$$y = f(a) + f'(a)(x - a)\,,$$

where $f'(a) = 3a^2 - 6$ is the derivative of y at $x = a$. Believing that

$$a_1 = a - \frac{f(a)}{f'(a)}\,,$$

the solution of the equation $f(a)+f'(a)(x-a) = 0$, is a better approximation than a, we replace a by a_1 to get a better approximation. This method is called **Newton's Tangent Method**. The process of finding a_1 from a in this example can be realized with the following Mathematica command:

```
g[a_]:=a-(a^3-6a+2)/(3a^2-6)
```

Note that we write `a_` to represent the independent variable `a`, waiting for a given value. First we apply the function g on the initial approximation 2 of the root to get a better approximation $g(2)$, and then apply g again we get $g(g(2))$, and then $g(g(g(2))), \ldots$. Continue in this way until we get a number a_0 that makes $g(a_0) = a_0$, i.e. $f(a_0) = 0$, then a_0 is just the root we are looking for. In Mathematica, applying g on 2 for several times (for example, four times) and listing all the results can be realized with the following command:

```
NestList[g,2,4]
```

(iv) Find by observation an approximation a of x at which y gets its minimum. Then use the following command to get a more accurate value of the minimum point near a:

```
FindMinimum[x^3-6x+2,{x,a}]
```

Choose another function $y = f(x)$ instead of $y = x^3 - 6x + 2$, and study the previous problem again.

(2) Construct the function $y = \sin \frac{1}{x}$ in the interval $[-1, 1]$, $[-0.1, 0.1]$, $[-0.01, 0.01]$ successively and observe the shape of the graph near $x = 0$.

Construct the point set $t = \{(1/k, \sin k) \mid 1 \leq k \leq 5000\}$ by using the following command in Mathematica.

```
T=Table[{1/k,Sin[k]},{k,1,5000}];
                (* Define the point set t.)
P=ListPlot[T] (* Draw all the points in the point set t.)
```

Note. The sentence behind * in the parentheses is the explanation of the function of the command.

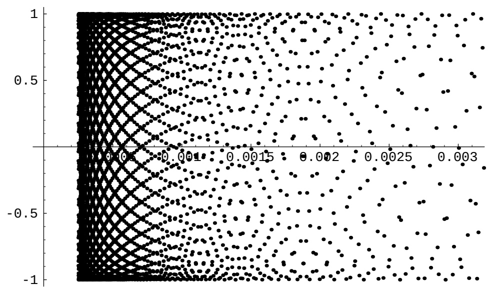

Fig. 1.1.

Observe what kind of curves are hidden in this point set. Try to construct one of these curves. Try to use the following command. What will happen? Change the value 44 in the sentence `d=44` to other values. What will happen? Why is `d=44` so good to produce smooth curves?

```
d=44;
T1=Table[{1/k,Sin[k]},{k,3,5000,d}];
T2=Table[{1/k,Sin[k]},{k,6,5000,d}];
```

```
P1=ListPlot[T1,PlotJoined->True];
P2=ListPlot[T2,PlotJoined->True];
Show[P,P1,P2]
```

Notes to the above commands.

(i) In the definition of the set `T1`, {`k,3,5000,d`} means that k ranges over a sequence $3, 3+d, 3+2d, \ldots, 3+md, \ldots$ with each term obtained by adding the same constant d to the last term, and all the terms not exceeding 5000. Similarly, in the definition of `T2` the expression {k`,6,5000,d`} means k ranges over $6, 6+d, 6+2d, \ldots \leq 5000$.

(ii) The function of the option "`PlotJoined-->True`" in the definition of the graphs `P1,P2` is to connect the successive points into a smooth curve.

(iii) The command "`Show[P,P1,P2]`" is used to display the graphs `P,P1,P2` (which was defined before) again in the same coordinate system.

(3) Construct the function $y = \frac{\sin x}{x}$ in the interval $[-0.1, 0.1]$ and observe the shape of the graph near $x = 0$. When x tends to 0, what value is y approaching to?

Exercise 2. Series and infinite products

(1) Construct the function $y = \sin x$ and the polynomial functions $y = x - \frac{x^3}{3!}$, $y = x - \frac{x^3}{3!} + \frac{x^5}{5!}, \ldots$ consisting of the first several terms in the Taylor expansion of $\sin x$. Observe the condition of these graphs of polynomials approaching the graph of $y = \sin x$.

(2) Let $n = 3, 5, 50, 550$ successively, and construct the function

$$y = \sin(x) + \frac{1}{3}\sin(3x) + \frac{1}{5}\sin(5x) + \cdots + \frac{1}{2n-1}\sin((2n-1)x)$$

in the interval $[-2\pi, 2\pi]$. When n tends to infinity, what function does this function tend to?

(3) Let $n = 5, 10, 100$ successively, and construct the function $f(x) = \sin(x)$ and

$$p(x) = x\left(1 - \frac{x^2}{\pi^2}\right)\left(1 - \frac{x^2}{4\pi^2}\right)\cdots\left(1 - \frac{x^2}{n^2\pi^2}\right)$$

in $[-2\pi, 2\pi]$. Observe the phenomenon of the graph of $p(x)$ approaching the $\sin x$ graph. Notice that the roots $\pm k\pi$ $(k = 0, 1, 2, \ldots, n)$ of the function $p(x)$ are also the roots of $f(x)$, and these two functions have the same derivative at $x = 0$. In any finite interval, the function $p(x)$ approaches $\sin x$ when n tends to infinity.

The commands needed to complete the above three assignments in Exercise 2 can be as follows.

```
s[x_,n_]:=NSum[(-1)^(k-1)x^(2k-1)/((2k-1)!),{k,1,n}];
figsin=Plot[Sin[x],{x,-2Pi,2Pi},
                        PlotStyle->{RGBColor[1,0,0]}];
figtylor2=Plot[s[x,2],{x,-3,3}];
                        PlotStyle->{RGBColor[1,0,1]}];
figtylor34=Plot[{s[x,3],s[x,4]},{x,-4,4];
figtylor5=Plot[{s[x,4],s[x,5]},{x,-5,5},
                        PlotStyle->{RGBColor[0,0,1]}];
Show[figsin,figtylor2,figtylor34,figtylor5]
```

* **Notes.** (i) The command NSum[<the expression>,{k,1,n}] has the same meaning as the mathematical symbol "$\sum_{k=1}^{n}$<the expression>", which means to calculate the sum of the terms obtained from <the expression> by taking $k = 1, 2, \ldots, n$.

(ii) The option "PlotStyle->{RGBColor[1,0,0]}" means to draw the graph red. The three independent variables in the function RGBColor[r, g, b] range from 0 to 1, representing the intensity of the three colors red, green and blue. So, RGBColor[1,0,0] means to draw the graph red, while RGBColor[0,0,1] means blue, RGBColor[1,0,1] means purple, etc. Since we draw many different curves in the same graph, it is good to use different colors to distinguish the curves.

```
f[x_,n_]:=Sum[Sin[k*x],{k,1,n,2}];
Plot[f[x,9],{x,-2Pi,2Pi}]

p[x_,n_]:=x*Product[1-x^2/((k*Pi)^2),{k,1,n}];
Plot[{Sin[x],p[x,5]},{x,-2Pi,2Pi}]
```

The f[x,9] in the commands above can be replaced by f[x,19] or f[x,50] or even f[x,550], while p[x,5] can be replaced by p[x,10],p[x,15].

1.2 The Number e

The logarithm we learned in high school is based on 10, which is called the **common logarithm** and denoted by $\lg N$. But the real commonly used