values of $u(x) = \frac{\log_{10}(1+x)}{x}$ for $x = 0.1, 0.01, 0.001, \ldots, 10^{-7}$ etc.

```
Do[Print[Log[10,1+10^(-m)]/(10^(-m))],{m,1,7}]
```

(*In Mathematica, `Log[a,N]` represents the logarithm of N based on a, so `Log[10,N]` represents the common logarithm $\lg N$. When we write `Log[N]` with the base omit, it means that the base is e and `Log[N]` represents the natural logarithm. The loop command `Do[<circulation body>,{n,1,7}]` represents operating the loop body orderly, for n from 1 to 7. In this example the loop body is `Print[<the expression>]`, whose function is to print the value of `<the expression>`.)

Observe whether $u(x)$ tends to a certain limit u when x is close to 0. This limit u is just the derivative of the common logarithm $y = \lg x$ at $x = 1$. One may see that it is not a simple number. One may agree that if this limit is equal to 1, it would be more pleasant. This can be easily done: divide the function $\lg x$ by u, replace it by $(\lg x)/u$, then its derivative at 1 will be 1. However,

$$\frac{\lg x}{u} = \frac{\lg x}{\lg(10^u)} = \log_{10^u} x$$

is just the logarithm based on 10^u.

(2) Calculate the value of 10^u. Is it a value you are familiar with?

(3) Calculate the values $v(x) = \frac{\ln(1+x)}{x}$ for $x = 10^{-n}$, with $n = 1, 2, \ldots, 7$. The Mathematica command is

```
Do[Print[Log[1+10^(-m)]/(10^(-m))],{m,1,7}]
```

Observe whether $v(x)$ tends to a certain limit when x trends to 0. What is the value of this limit?

1.3 Integral and Natural Logarithm

For a positive real number a, study the area of $S(a)$ which is bounded by the graph of the inverse function $y = \frac{1}{x}$, the x-axis and the straight lines $x = 1$ and $x = a$.

This area is just the value of the definite integral $\int_1^a \frac{1}{x} dx$. Notice that we define the straight line $x = 1$ to be the reference of calculating the area. The area to the right of this line is positive, and to the left is negative. That is to say, when $a > 1$, $S(a) > 0$; when $0 < a < 1$, $S(a) < 0$; when

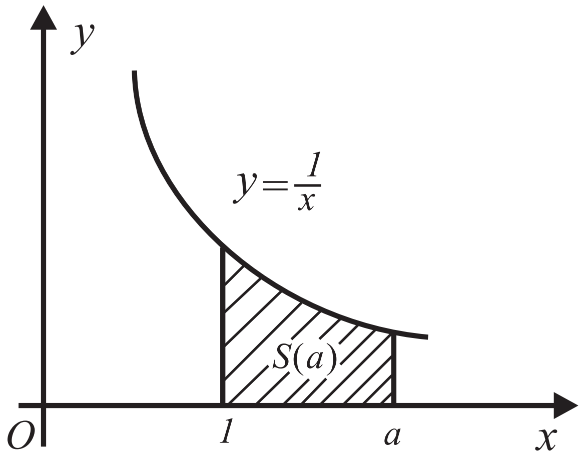

Fig. 1.2.

$a = 1$, $S(a) = 0$. In the case of $a = 2$, divide the interval $[1, 2]$ into n equal partitions and the length of every partition is $\frac{1}{n}$. The dividing points are $x_k = 1 + \frac{k}{n}$ $(1 \le k \le n-1)$. The area $S(2)$ to be calculated is divided into n parts accordingly. The kth part S_k is the area bounded by the straight lines $x = x_{k-1}$ and $x = x_k$. The upper boundary of S_k is the part of the curve $y = \frac{1}{x}$ over the interval $[x_{k-1}, x_k]$. Its largest height is $\frac{1}{x_{k-1}}$, while the smallest height is $\frac{1}{x_k}$. So

$$\frac{1}{n}\frac{1}{x_{k-1}} = \frac{1}{n+k-1} > S_k > \frac{1}{n}\frac{1}{x_k} = \frac{1}{n+k}\,.$$

Take the sum of all these S_k we obtain

$$\Sigma_n = \sum_{k=1}^{n} \frac{1}{n+k-1} > S(2) > \sigma_n = \sum_{k=1}^{n} \frac{1}{n+k}$$

Σ_n, σ_n are called the "large sum" and "small sum" respectively, and the accurate area $S(2)$ is between them.

When n increases, the large sum Σ_n decreases and the small sum σ_n increases. With n tends to infinity both of them tend toward the same limit $S(2)$. One can easily see that $\Sigma_n - \sigma_n < \frac{1}{2n}$. So if we take Σ_n or σ_n as approximations of $S(2)$, the error will be less than $\frac{1}{2n}$, and the error of their average $(\Sigma_n + \sigma_n)/2$ will be even less. In fact, this average value is just the approximation of $S(2)$ obtained by the "trapezoid formula".

Exercise 6. (1) For $n = 10^m$ with $m = 3, 4, 5, 6$, use the `NSum` command to calculate the "large sum", the "small sum" and their average.

```
Do[n=10^m; s=NSum[1/(n+k-1),{k,1,n}];
   t=NSum[1/(n+k),{k,1,n}]; Print[{s,t,(s+t)/2}],
             {m,3,6}]
```

Observe what happens when n increases? Does the "large sum" decrease and the "small sum" increase? Do they get closer and closer to the same limit? Compare this limit with the value of integral $\int_1^2 \frac{1}{x} dx$ obtained by the following command

```
NIntegrate[1/x,{x,1,2}]
```

(2) Use the command

```
S[x_]:= NIntegrate[1/t,{t,1,x}]
```

to define the function $S(x) = \int_1^x \frac{1}{t} dt$. And then, use the command

```
Plot [S[x],{x,0.1,5}]
```

to construct this function in the interval $[0.1, 5]$. (Of course one can try other intervals.) Observe the shape of the graph and guess what function it looks like?

(3) You may guess by observation that the graph in (2) looks like the graph of a logarithm function. Try to find the base b of this logarithm. It should satisfy $S(b) = 1$. One can obtain an approximation 3 of b by observation. Use the `FindRoot` command

```
FindRoot[S[x]-1,{x,3}]
```

to obtain a more accurate value result of the root b of the equation $S(x) - 1 = 0$. You will see that b is just the base e of the natural logarithm. Construct the function $S(x)$ and $\ln x$ in the same coordinate system in different colors, you will see they coincide very well.

Of course, the result $S(x) = \int_1^x \frac{1}{t} dt$ can be easily obtained from indefinite integral. But what we have done above is to obtain this conclusion using the numerical and graphic experiment.

Mathematica has the commands for calculating indefinite integral $\int f(x)dx$ and definite integral $\int_a^b f(x)dx$, which are

```
Integrate[f,x]
Integrate[f,{x,a,b}]
```

respectively. Compare them with the command

```
NIntegrate[f,{x,a,b}]
```

for calculating definite integral by numerical method.

1.4 Harmonic Series

The series that composed of the inverses of the natural numbers n is called the **Harmonic Series**. We denote by $H(n)$ the sum of its first n items $1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$, which is the function of an integer variable n.

Exercise 7. (1) Construct the set of the points with coordinates $(n, H(n))$ $(n = 1, 2, \ldots, 100)$ by using the following command in Mathematica:

```
H[n_]:=NSum[1/k,{k,1,n}];    (*Define the function H(n).)
T=Table[{n,H[n]},{n,1,100}]; (*Define the point set t.)
pic1=ListPoint[T]      (*Construct the points of T and
                 keep it in pic1 so that we can draw it again.)
```

Observe the graph. The points look like being along the graph of a logarithm function. To check whether this is true, construct the natural logarithm $y = \ln x$ $(1 \le x \le 100)$ in the same coordinate system as `pic1`. And compare this logarithm curve with the curve along the points in `pic1`.

```
pic2=Plot[Log[x],{x,1,100},PlotStyle->{RGBColor[0,0,1]}];
Show[pic1,pic2]
```

Does this logarithm curve coincide with the curve along the point set `pic1`? No. But these two curves seem to be "parallel". Namely, the difference between $H(n)$ and $\ln n$ seems to be a constant for large values of n. Calculate $c = H(100) - \ln 100$, and construct in red the graph `pic3` of the function $y = \ln x + c$. Show `pic1,pic2,pic3` in the same coordinate system, (see Fig. 1.3).