Mathematica has the commands for calculating indefinite integral $\int f(x)dx$ and definite integral $\int_a^b f(x)dx$, which are

```
Integrate[f,x]
Integrate[f,{x,a,b}]
```

respectively. Compare them with the command

```
NIntegrate[f,{x,a,b}]
```

for calculating definite integral by numerical method.

1.4 Harmonic Series

The series that composed of the inverses of the natural numbers n is called the **Harmonic Series**. We denote by $H(n)$ the sum of its first n items $1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$, which is the function of an integer variable n.

Exercise 7. (1) Construct the set of the points with coordinates $(n, H(n))$ $(n = 1, 2, \ldots, 100)$ by using the following command in Mathematica:

```
H[n_]:=NSum[1/k,{k,1,n}];    (*Define the function H(n).)
T=Table[{n,H[n]},{n,1,100}]; (*Define the point set t.)
pic1=ListPoint[T]      (*Construct the points of T and
                 keep it in pic1 so that we can draw it again.)
```

Observe the graph. The points look like being along the graph of a logarithm function. To check whether this is true, construct the natural logarithm $y = \ln x$ $(1 \le x \le 100)$ in the same coordinate system as `pic1`. And compare this logarithm curve with the curve along the points in `pic1`.

```
pic2=Plot[Log[x],{x,1,100},PlotStyle->{RGBColor[0,0,1]}];
Show[pic1,pic2]
```

Does this logarithm curve coincide with the curve along the point set `pic1`? No. But these two curves seem to be "parallel". Namely, the difference between $H(n)$ and $\ln n$ seems to be a constant for large values of n. Calculate $c = H(100) - \ln 100$, and construct in red the graph `pic3` of the function $y = \ln x + c$. Show `pic1,pic2,pic3` in the same coordinate system, (see Fig. 1.3).

```
c=H[100]-Log[100];
pic3=Plot[Log[x]+c,{x,1,100},PloyStyle->{RGBColor[1,0,0]}];
Show[pic1,pic2,pic3]
```

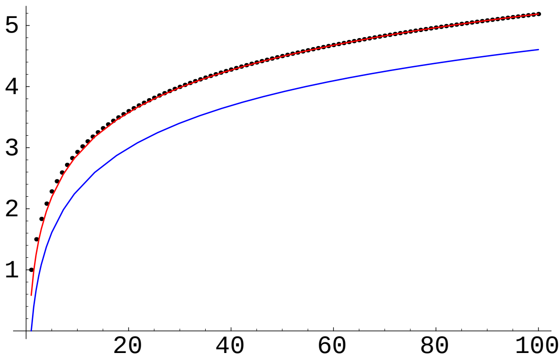

Fig. 1.3.

Observe the phenomenon that the black points in `pic1` is almost superimposed by the red curve `pic3`.

(2) We guess after observing graphs of $H(n)$ and $\ln x$ that the differences $H(n) - \ln n$ would approach a limit when n increases infinitely. To verify this conjecture we calculate $C(n) = H(n) - \ln n$ and $c(n) = H(n) - \ln(n+1)$ for $n = 10^m$, $m = 1, 2, \ldots, 7$.

```
Do[n=10^m; Print[{H[n]-Log[n],H[n]-Log[n+1]}],{m,1,7}]
```

Observe the phenomena that `H[n]-Log[n]` is decreasing, `H[n]-Log[n+1]` is increasing, and both approach the same limit.

The limit

$$C = \lim_{n\to\infty}\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \ln n\right)$$

we just obtained is called the **Euler's constant**.

Observe the two graphs in Fig. 1.4.

Obviously

$$C(n) = H(n) - \ln n > c(n) = H(n) - \ln(n+1)\,.$$

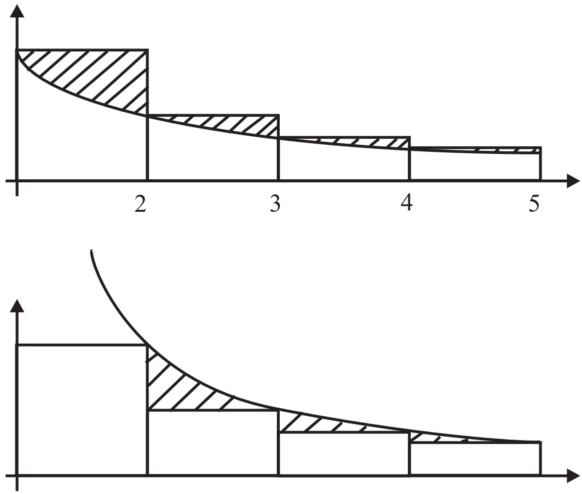

Fig. 1.4.

In the upper graph we see that $c(n)$ is just the sum of the areas of the first n shadowed triangles. Of course it increases as n increases. In the lower graph we see that $C(n)$ can be obtained by subtracting the areas of the first n shadowed triangles from the area 1 of the unit square, which decreases obviously as n increases. Because $C(n) - c(n) = \ln\left(1 + \frac{1}{n}\right)$ tends to 0 as n approaches infinity, $C(n)$ and $c(n)$ tend to reach the same limit C.