Introduction

Group theory plays a very important role in physics and chemistry, and its importance continues to grow seemingly endlessly. The representation theory of both finite and compact Lie groups is treated extensively in numerous books and articles. However, they basically follow the same fundamental theory. This theory, which we call the standard group representation theory, seems to be perfect from the mathematical point of view. Nevertheless, it is not totally satisfying from a practical or physical point of view. Many sophisticated physicists, who were quite at home in their own fields, seemed to be afraid of group theory and expressed their dissatisfaction with standard representation theory (Sokolov 1956, Salam 1963, Lipkin 1966, Slater 1975). During the 1930's–50's, outrage, disgust, characterizing group theory as a plague or calling it "*Gruppenpest*" were not atypical reactions by physicists to the use of group theory in physics. The leading American physicist J. Slater commented (p. 60 in his autobiography):

"*The authors of the 'Gruppenpest' wrote papers which are incomprehensible to those like me who had not studied group theory, in which they applied these theoretical results to the study of many electron problem. The practical consequences appeared to be negligible, but everyone felt that to be in the mainstream one had to learn group theory. Yet there were no good texts for which one could learn group theory. It was a frustrating experience, worthy the name of a pest. I had what I can only describe as a feeling of outrage at the turn which the subject had taken....*

As soon as this [Slater's] paper became known, it was obvious that a great many other physicists were disgusted as I had been with the group theoretical approach to the problem. As I heard later, there were remarks made such as 'Slater has slain the Gruppenpest'. I believe that no other piece of work I have done was so universally popular."

The Nobel Laureate A. Salam said in his opening speech for "Seminars on Theoretical Physics" held in Trieste in 1962:

"*In 1951, I had the good fortune of listening to Professor Racah's lecture on Lie groups at Princeton. After attending these lectures, I thought, "This is really too hard. I cannot learn all this ... All this is too damned hard and unphysical.*"

Despite those comments being made several decades ago, and despite group theory being now a more common part in the education of physicists, significant difficulties with the standard representation theory of groups have not been overcome. There lacks the universal applicability and acceptability associated with, for example, calculus.

The first serious drawback of the standard group representation theory is that it is *unphysical*. By this I mean it was developed by mathematicians for mathematical purposes and without physical application in mind. The group was introduced into mathematics as early as 1810, and the theory of group representation was developed during the 1920's, before quantum mechanics was formulated; in this respect it is unlike calculus, which was invented about the same time as Newton's laws were discovered. Second, there is no general method for treating various kinds of group representation problems. Any given technique applies only to a particular problem and for a particular group, or class of groups. Not only do many of the methods for dealing with point groups, permutation groups, space groups, and Lie groups all differ drastically from each other, but the methods for finding the characters, irreducible basis (IRB), irreducible matrices and Clebsch–Gordan (CG) coefficients also vary from one to the other. Therefore, in many cases, these methods are more of an art than a science. Third, in physical applications, we often need to construct an IRB $\psi_m^{(\nu)}$ symmetry adapted to a given group chain $G \supset G(s)$. The standard method is to use the projection operator

$$P_{mk}^{(\nu)} = \frac{h_\nu}{g} \sum_R D_{mk}^{(\nu)}(R)^* R,$$

where $D^{(\nu)}(R)$ are irreducible matrices in the $G \supset G(s)$ IRB, which in turn depend on the $G \supset G(s)$ IRB $\psi_m^{(\nu)}$ through the relation

$$D_{mk}^{(\nu)}(R) = \langle \psi_m^{(\nu)} | R | \psi_k^{(\nu)} \rangle.$$

Now the trouble is that the $G \supset G(s)$ IRB is not known yet. Thus we are at an impasse when both the matrices and IRB are unknown.

As pointed out by Salam (1963), a battle has raged between the amateurs and professional group theorists. The amateurs have maintained that everything one needs from the theory of groups can be discovered by the light of nature provided one knows how to multiply two matrices. As an amateur myself, in this book I have introduced a new (certainly, non–professional) approach to group representation theory and it is quite interesting to note that the foundation of the new approach is precisely the theory of the complete set of commuting operators (CSCO) initiated by Dirac, the prince of amateurs in the field of group theory.

The special features of the approach are as follows.

1. *Simplicity and Accessibility*: The new representation theory for groups is essentially an extension of representation theory in quantum mechanics. Group representation theory is intimately related to quantum mechanics just as calculus is to classical mechanics. Thus it should be easily acceptable to physicists. For a group G, three kinds of CSCO are introduced, the CSCO–I, –II, and –III, roughly speaking the CSCO for the class space, the irreducible space, and the group space, respectively. They are the analogies of $\mathbf{J}^2$, $(\mathbf{J}^2, J_z)$, $(\mathbf{J}^2, J_z, J_3)$, respectively, for the rotation group in three–dimensional space.

2. *Universality and Versatility*: Based on the CSCO–I, –II, and –III, all compact (discrete or continuous) groups are treated in a unified way. Therefore once one knows the representation theory of the rotation group, in principle one knows the representation theory of all other compact groups. The new approach is constructive in nature, leading to a new method, the so–called eigenfunction method (EFM) for determining group representation. The problems of determining firstly the primitive characters and isoscalar factors, secondly the $G \supset G(s)$ IRB and CG coefficients, and thirdly the irreducible matrices, are all reduced to a single recipe: Seek the eigenvectors (or eigenfunctions) of the CSCO–I, –II, and –III of G, respectively. The EFM proves to be powerful and versatile in treating point groups, permutation groups, unitary groups, graded unitary groups and space groups, and for both vector and projective representations. The EFM for a discrete group (both for finite and infinite types) is simpler than conventional methods and is flexible enough to obtain the irreducible basis adapted to any given group chain $G \supset G(s)$ without need of any knowledge of the irreducible matrices, or conversely, to obtain all the irreducible matrices in any given $G \supset G(s)$ classification without any knowledge of the irreducible basis.

3. *Applicability*: Since the ultimate step of the method is the diagonalization of the representative matrices of a certain type of CSCO, the procedure can be easily translated into a computer program. Several standard codes are already available. Furthermore, using the eigenvalues of the CSCO as irrep labels enables us to find *algebraic solutions* for the point groups, just as one has analytic solutions for rotation group.

The book is self–contained and suitable for self–study, and is a combination of a textbook and a monograph. By ignoring some proofs and some paragraphs or passages marked with asterisks, the book becomes an easily readable textbook. The theory is developed starting with concrete examples and leading up to more abstract conclusions, as well as from the special to the general, supplemented with abundant illustrative examples. The emphasis is on the EFM technique rather than on strict rigor. Some theorems are cited without proof, since the theorems are easily understandable and their proofs can be found in many group theory textbooks. A knowledge of elementary group theory is not necessary to read this book, but a knowledge of the linear algebra and representation theory as well as angular momentum theory in quantum mechanics is assumed.

The various important coefficients, such as the Clebsch–Gordan, Racah, subduction and induction coefficients, the isoscalar factors and fractional parentage coefficients are discussed in detail for point groups, permutation groups, unitary groups and space groups. Tables for several useful coefficients are given. Some new dualities between the permutation group and unitary groups are disclosed and are fully exploited for computing many coefficients of unitary groups in terms of those of the permutation groups. The theory on roots and weights in Lie groups is also reformulated in the spirit of representation theory of quantum mechanics. The applications of group theory in quantum mechanics are discussed with emphasis on application to many-body systems. The connection between the new and standard approaches is discussed. There should be no difficulty for a reader of the present book to understand the conclusions derived in other textbooks or in the literature, although the derivations given here may be totally different.

Tables and figures are indexed according to their section numbers. For example, Table(Fig.) x.y-n denotes the n–th table (figure) in Sec. x.y. If there is only a single table or figure in the section, then the suffix "-1" will be omitted. References are indicated by the names of the first author, or first two authors, followed by the year of publication; if this is still not sufficient, then an index $[x]$ will precede the year.

Chapter 1

Elements of Group Theory

In this chapter we present an introduction to the basic elements of group theory. Many books are available covering the material presented herein, and we therefore state most results without references. We have found the texts of Hamermesh (1962) and Elliott & Dawber (1979) particularly useful.

1.1. The Definition of a Group

A set of elements (or operators) $\{a, b, c, \ldots\}$ or $\{R_a : a = 1, 2, 3, \ldots\}$, or $\{R, S, T, \ldots\}$ is called a group G , if a multiplication rule is defined for any two elements so that the product ab has a definite meaning and the following four postulates are satisfied:

1. *Closure*: If a and b belong to the set, then ab also belongs to the set.
2. *Associativity*: $a(bc) = (ab)c$.
3. There exists the identity element e such that $ae = ea = a$ for any a belonging to the set.
4. There exists the inverse element, that is, for each element a, there is a corresponding element b such that $ab = ba = e$. b is called the inverse element of a and denoted by $b = a^{-1}$.

Since in general $ab \neq ba$ the order of multiplication is important. An *Abelian group* is one whose elements commute with one another, that is $[a, b] = ab - ba = 0$.

A *finite group* has a finite number of elements. For some infinite groups, called *continuous groups*, the group elements may be labelled by parameters which (or some of which) vary continuously.

We use $G = \{a\}$ to denote a group, and use $a \in G$ to denote that a is an element of G (read as a belongs to G). The *order of a finite group* G is defined as the total number of its elements and will be denoted by g or $|G|$.

For a finite group, $a^n = e$ for some positive integer n and each $a \in G$. The smallest positive n for which $a^n = e$ is called the *order of the element* a , and denoted as $|a|$. The set of elements $a, a^2, a^3, \ldots, a^n (= e)$ forms a group, called the *cyclic group* of order $n = |a|$, which is often denoted by C_n.

A set of *generators* of a group G is a set of elements $\{a, b, c \ldots\}$ of G such that every element of G is expressible as a finite product of powers of elements of $\{a, b, c, \ldots\}$.

From a mathematical point of view, the product ab of two elements a and b can be defined arbitrarily. In physics, we are mainly interested in the group of transformations, or the group of operators. In such cases, the group elements $R_a, R_b \ldots$ represent a transformations or operators, and the product $R_a R_b = R_c$ is defined as the operation resulting from first operating with R_b and then with R_a.

Definition 1.1: Two sets S and S' are said to be commutative, denoted by $[S, S']$=0, if each element of S commutes with each element of S'.

All the operations which leave a system (or a geometric object) unchanged (that is it appears not to have changed after the operation) form a group called the *symmetry group* of the system (or the object).

Obviously, the Hamiltonian H of a microscopic system commutes with the symmetry group G of the system, that is

$$[H, G] = 0 \ . \tag{1-1a}$$

According to Definition 1.1, this means that

$$[H, R_a] = 0 \quad \text{for} \ R_a \in G \ . \tag{1-1b}$$

Examples of groups:

1. All the integers under addition form an infinite discrete group.
2. The integers modulo n (that is $a = b$ if $a - b = mn, m$ being an integer) under addition form a group called the *group of integers modulo n* and denoted as Z_n.
3. The n complex numbers $\exp(2\pi mi/n), m = 0, 1, \ldots, n-1$ form the cyclic group C_n under multiplication.
   The three previous groups are Abelian, the following group is non–Abelian.
4. The rotation group R_3 in three dimensions: A system with spherical symmetry is invariant under rotations through any angle φ about any axis $\mathbf{n}(\theta', \varphi')$ passing through its center. All these operations

$$R_{\mathbf{n}(\theta',\varphi')}(\varphi), \quad 0 \le \theta' \le \pi, \quad 0 \le \varphi' \le 2\pi, \quad 0 \le \varphi \le \pi \tag{1-2a}$$

   form the three–dimensional rotation group R_3. The identity is $R_{\mathbf{n}}(0)$, and the inverse of $R_{\mathbf{n}(\theta',\varphi')}(\varphi)$ is

$$R^{-1}_{\mathbf{n}(\theta',\varphi')}(\varphi) = R_{\mathbf{n}(\pi-\theta',\pi+\varphi')}(\varphi) \ . \tag{1-2b}$$

   Since θ', φ' and φ are continuous variables, and two rotations do not in general commute, R_3 is a continuous non–Abelian group.
5. The rotation group R_2 in two dimensions: A linear molecule such as CO is invariant under rotations through any angle about the axis z passing through the line connecting the centers of the atoms. The symmetry operations are

$$R_z(\varphi), \quad 0 \le \varphi \le 2\pi \ . \tag{1-2c}$$

   Together they form the two–dimensional rotation group R_2. It is clear that

$$R_z(\varphi_1)R_z(\varphi_2) = R_z(\varphi_2)R_z(\varphi_1) = R_z(\varphi_1 + \varphi_2) \ ;$$

   so that R_2 is a continuous Abelian group.
6. Space inversion group G_I or $\mathcal{C}_i$ consists of two elements: the identity e and the space inversion I which takes the point $P(x, y, z)$ to $P'(-x, -y, -z)$.
7. Space reflection group $\mathcal{C}_s$ consists of two elements: the identity e and the space reflection σ_z, a reflection plane in the xy plane (with z as its normal) which takes the point $P(x, y, z)$, to $P'(x, y, -z)$.
8. The group $\mathcal{C}_{3v}$: The ammonia molecule NH_3 (Figs. 1.1-1a and 1.1-1b) has six symmetry operations,

$$e, C_3, C_3^2, \sigma_1, \sigma_2, \sigma_3 \ , \tag{1-3}$$

   where $C_3 = R_z(120°), C_3^2 = R_z(240°)$, and σ_i are reflection planes containing the z–axis and the vertices i, as shown in Fig. 1.1-1b. The six operations in Eq. (1-3) form the symmetry group of NH_3, denoted as $\mathcal{C}_{3v}$.

   Using the interchange of the vertices 1, 2, 3 under the operations (1-3), we can obtain the multiplication table for $\mathcal{C}_{3v}$. Note that the anti–clockwise rotation is taken to be positive,

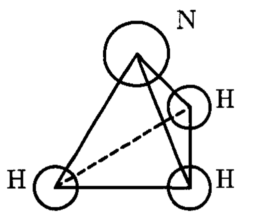

Fig. .1.1-1a. The ammonia molecule NH_3 with C_{3v} symmetry.

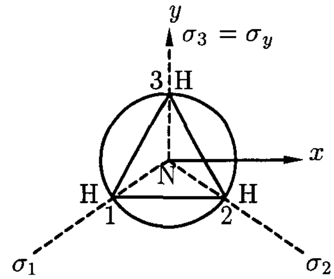

Fig. 1.1-1b. The ammonia molecule NH_3 seen from the top

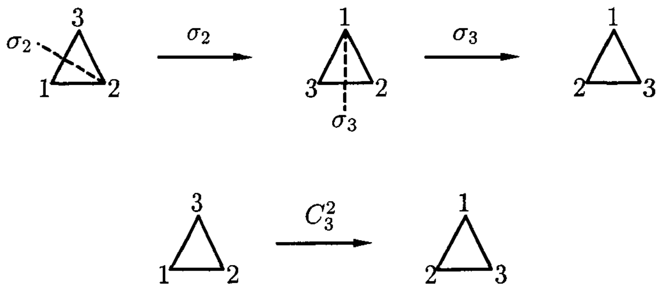

Fig. 1.1-2. Reflections and rotations on a triangle.

and the reflection planes σ_i are fixed in space (that is they do not change with the vertices). Consider for example the sequence shown in Fig. 1.1-2.

Hence we see that $\sigma_3\sigma_2 = C_3^2$. The multiplication relations for the elements of C_{3v} is listed in Table 1.1. Such a table is called a group table.

10. An arbitrary quaternion can be written as

$$q = q^0\lambda_0 + q^1\lambda_1 + q^2\lambda_2 + q^3\lambda_3$$

The q^i are real and λ_i obey

$$\begin{aligned} &\lambda_0\lambda_i = \lambda_i,\ \lambda_i^2 = -\lambda_0,\ i = 1,2,3; \\ &\lambda_i\lambda_j = -\lambda_j\lambda_i,\ \lambda_i\lambda_j = \lambda_k,\ i,j,k \text{ cyclic.} \end{aligned} \tag{1-4a}$$

(when $q_3 = q_4 = 0$, a quaternion is reduced to a complex number). The quaternions $\pm\lambda_0, \cdots, \pm\lambda_3$ form a group under multiplication($e = \lambda_0$), called the *quaternion group* Q .

9. If a set of matrices constitutes a group under matrix multiplication, then it is called a *matrix group.* For example, one realization of the quaternions is

$$\lambda_0 = \{1,1\}_{\text{diag}},\ \lambda_1 = -i\sigma_x,\ \lambda_2 = -i\sigma_y,\ \lambda_3 = -i\sigma_z,$$

where σ_x, σ_y and σ_z are the Pauli matrices,

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} . \tag{1-4b}$$

The eight matrices $\{\pm e, \pm i\sigma_x, \pm i\sigma_y, \pm i\sigma_z\}$ from a matrix group

Ex. 1.1. Show that the elements in the same class have the same order.

Ex. 1.2. Prove that if two elements a and b commute, then a^{-1} and b also commute.

Table 1.1. The group table of C_{3v}.

ab (a \ b)	e	σ_3	σ_2	σ_1	C_3	C_3^2
e	e	σ_3	σ_2	σ_1	C_3	C_3^2
σ_3	σ_3	e	C_3^2	C_3	σ_1	σ_2
σ_2	σ_2	C_3	e	C_3^2	σ_3	σ_1
σ_1	σ_1	C_3^2	C_3	e	σ_2	σ_3
C_3	C_3	σ_2	σ_1	σ_3	C_3^2	e
C_3^2	C_3^2	σ_1	σ_3	σ_2	e	C_3

Ex. 1.3. Check the multiplication relations in Table 1.1.

Ex. 1.4. Construct the multiplication table for the quaternion group (1-4a).(Hint: a 4×4 table is sufficient.)

Ex. 1.5. Construct the group table for the group C_{4v} which consists of the following eight elements: $\{e, C_4, C_4^2, C_4^3, \sigma_1, \sigma_2, \sigma_3, \sigma_4\}$, where $C_4 = R_z(90°)$, and σ_i are reflection planes shown in Fig. 1.1-3.

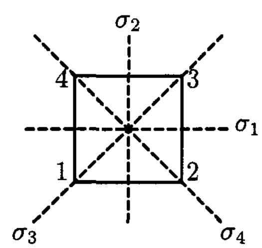

Fig. 1.1-3. The group C_{4v}.

1.2. The Permutation Group S_n

1.2.1 The definition of S_n

The $n!$ permutations

$$\begin{pmatrix} 1, & 2, & \dots & n \\ p_1, & p_2, & \dots & p_n \end{pmatrix} \equiv \begin{pmatrix} i \\ p_i \end{pmatrix} \tag{1-5}$$

form a group $S_n(1, 2, \dots, n) \equiv S_n$, called the *permutation group* or the *symmetric group*. Equation (1-5) denotes a permutation of the indices i to p_i. The product $R_1 R_2$ of two permutations is defined as the resultant permutation of first permuting with R_2 and then with R_1. For example,

$$R_1 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}, \quad R_2 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix},$$

$$R_1 R_2 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix},$$

$$R_2 R_1 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}.$$

One can show that for general permutations R_1 and R_2

$[R_1, R_2] = 0$, if R_1 and R_2 do not involve the permutation of the same index,

$[R_1, R_2] \neq 0$, if R_1 and R_2 involve the permutation of the same index. Obviously we have that

$$\begin{pmatrix} 1 & 2 & 3 \\ p_1 & p_2 & p_3 \end{pmatrix} = \begin{pmatrix} 2 & 1 & 3 \\ p_2 & p_1 & p_3 \end{pmatrix} = \begin{pmatrix} 3 & 2 & 1 \\ p_3 & p_2 & p_1 \end{pmatrix} = \dots$$

and that the inverse of $\begin{pmatrix} i \\ p_i \end{pmatrix}$ is $\begin{pmatrix} p_i \\ i \end{pmatrix}$. Consider for example the permutation group S_3 with the following 3!=6 permutations:

$$R_1 = e = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, R_2 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}, R_3 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix},$$

$$R_4 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}, R_5 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}, R_6 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}. \tag{1-6}$$

1.2.2. Permutations expressed in terms of cycles and transpositions

Define:

$$\begin{aligned} e &= (i), && \text{one–cycle} \quad (i \,\text{remains}\, i), \\ p_{ij} &= (ij) = \begin{pmatrix} ij \\ ji \end{pmatrix}, && \text{two–cycle, or transposition,} \\ p_{ijk} &= (ijk) = \begin{pmatrix} ijk \\ jki \end{pmatrix}, && \text{three–cycle.} \end{aligned}$$

Similarly, $(p_1, p_2 \dots p_k)$ is called a k–cycle and k is called the *length* of the cycle. A k–cycle $(p_1, p_2 \dots p_k)$ generates the cyclic group C_k.

Any permutation can be expressed as a product of cycles without common indices

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 4 & 3 & 7 & 1 & 5 & 8 & 2 & 6 \end{pmatrix} = (14)(237)(5)(68)\,. \tag{1-7}$$

Remarks:

1. In expressing a permutation as a product of cycles without common indices, the order with which we write the cycles is irrelevant.
2. One–cycles can be omitted. Hence the permutation (1-7) can be written as (14) (237) (68).
3. A cycle can be written in several forms, since it is itself invariant under cyclic permutations. For example

$$(\alpha\beta\gamma\delta) = (\beta\gamma\delta\alpha) = (\gamma\delta\alpha\beta) = (\delta\alpha\beta\gamma).$$

Any transposition can be expressed as a product of adjacent transpositions by the following recursive formula:

$$(i, i+v) = (i+1, i+v)(i, i+1)(i+1, i+v). \tag{1-8a}$$

Example 1: The group S_4. Letting $v = 2$ in (1-8a),

$$(13) = (23)(12)(23), \quad (24) = (34)(23)(34). \tag{1-8b}$$

Letting $v = 3$ in (1-8a) and using (1-8b),

$$(14) = (24)(12)(24) = (34)(23)(34)(12)(34)(23)(34). \tag{1-8c}$$

The following important relations can be verified:

$$1.\ (\alpha\beta\gamma\delta\varepsilon) = (\alpha\beta)(\beta\gamma\delta\varepsilon) = (\alpha\beta\gamma)(\gamma\delta\varepsilon) = (\alpha\beta\gamma\delta)(\delta\varepsilon) = (\alpha\beta)(\beta\gamma)(\gamma\delta)(\delta\varepsilon). \tag{1-9a}$$

$$2.\ (\alpha\beta\gamma\delta\varepsilon) = (\alpha\varepsilon)(\alpha\beta\gamma\delta) = (\alpha\delta\varepsilon)(\alpha\beta\gamma) = (\alpha\gamma\delta\varepsilon)(\alpha\beta) = (\alpha\varepsilon)(\alpha\delta)(\alpha\gamma)(\alpha\beta). \tag{1-9b}$$

$$3.\ (\alpha\beta\gamma\delta\varepsilon)^{-1} = (\alpha\varepsilon\delta\gamma\beta) = (\varepsilon\delta\gamma\beta\alpha). \tag{1-10}$$

Note that the order of the cycles in the above equations is crucial, since they involve common indices.

The *generators* of S_n are the $n-1$ adjacent transpositions $(i, i+1), i = 1, 2, \ldots, n-1$.

A permutation can be written as a product of transpositions in many ways but the number N of factors is always even or odd. Therefore for any permutation p, we can define a *permutation parity*, or *sign*, δ_p by

$$\delta_p = (-1)^N. \tag{1-11}$$

A permutation with $\delta_p = 1(-1)$ is called an even (odd) permutation. If $p = p_1 p_2$, then $\delta_p = \delta_{p_1}\delta_{p_2}$. The k–cycle $(i_1 i_2 \ldots i_k)$ has parity $(-1)^{k-1}$.

Example 2: The six elements (1-6) of S_3 can be written in terms of cycles as

$$R_1 = e, \quad R_2 = (12), \quad R_3 = (13), \quad R_4 = (23), \quad R_5 = (123), \quad R_6 = (132), \tag{1-12}$$

where e, (123) and (132) are even, while (12), (13) and (23) are odd. The group table of S_3 is shown in Table 1.2.

Table 1.2 Group table of S_3. Subscripts record the permutation column/row number.

c \ b ($a = cb$)	e_1	$(12)_2$	$(13)_3$	$(23)_4$	$(123)_5$	$(132)_6$
e_1	e_1	$(12)_2$	$(13)_3$	$(23)_4$	$(123)_5$	$(132)_6$
$(12)_2$	$(12)_2$	e_1	$(132)_6$	$(123)_5$	$(23)_4$	$(13)_3$
$(13)_3$	$(13)_3$	$(123)_5$	e_1	$(132)_6$	$(12)_2$	$(23)_4$
$(23)_4$	$(23)_4$	$(132)_6$	$(123)_5$	e_1	$(13)_3$	$(12)_2$
$(123)_5$	$(123)_5$	$(13)_3$	$(23)_4$	$(12)_2$	$(132)_6$	e_1
$(132)_6$	$(132)_6$	$(23)_4$	$(12)_2$	$(13)_3$	e_1	$(123)_5$

1.3. Subgroups

If there is a subset of elements in a group G, which by itself forms a group G_s under the same multiplication rule as that of G, then G_s is said to be a *subgroup* of G, denoted by $G \supset G_s$. Every group has two trivial subgroups, called *improper subgroups*: the group consisting of the identity element alone, and the whole group itself. All other subgroups are called *proper subgroups*.

A subgroup G_s may itself contain a subgroup G_s'. Together they form a group (or subgroup) chain,

$$G \supset G_s \supset G_s' \supset \ldots .$$

Examples of subgroups:

1. $R_3 \supset R_2$. The two–dimensional rotation group is a subgroup of the three-dimensional rotation group.

2. The group $\mathcal{C}_{3v}$ has four subgroup chains, namely

$$\mathcal{C}_{3v} \supset \mathcal{C}_3, \quad \mathcal{C}_{3v} \supset \mathcal{C}_{s_i}, \quad i = 1, 2, 3$$

$$\mathcal{C}_3 = (e, C_3, C_3^2), \quad \mathcal{C}_{s_i} = (e, \sigma_i).$$

3. The group S_n has the group chain $S_n \supset S_{n-1} \supset S_{n-2} \supset \ldots \supset S_2$.
4. The group S_4 has a variety of other subgroups, for example $S_3(134)$, $S_3(234), \ldots$ as well as the following two Abelian subgroups:

$$\text{the Four–group:} \quad F = \{e, (12)(34), (13)(24), (23)(14)\} , \tag{1-13}$$

$$\text{the cyclic group:} \quad \mathcal{C}_4 = \{e, (1234), (1234)^2 (= (13)(24)), (1234)^3\} . \tag{1-14}$$

5. The *alternating group* $\mathcal{A}_n$ is the group consisting of all the even permutations of S_n.

Ex. 1.6. Prove the following theorem: A group without any nontrivial subgroup is a group of prime order.

Ex. 1.7. Using (1-9) check Table 1.2.

Ex. 1.8. Construct the group table for the four–group.

1.4. Isomorphism and Homomorphism

Two groups G and G' are said to be *isomorphic*($G \approx G'$) if their elements can be put into a one–to–one correspondence which preserves the multiplication rule, that is, corresponding to $ab = c$, we have $a'b' = c'$.

Two groups are said to be *anti–isomorphic* if their elements have a one–to–one correspondence and corresponding to $ab = c$ we have $b'a' = c'$ (instead of $a'b' = c'$).

If $ab = c$, then $b^{-1}a^{-1} = c^{-1}$. Letting $R_a = a^{-1}, R_b = b^{-1}, \ldots$ then the set $\{R_a\}$ forms a group $\tilde{G}$ which is isomorphic with G'. The difference between the group $\tilde{G}$ and G is merely a matter of nomenclature for the elements. Therefore, if G is anti–isomorphic to G', then essentially G is isomorphic to G'.

Two groups which are isomorphic are the same abstract group, though they may have totally different realizations. Therefore, we sometimes simply use $G = G'$ to indicate that G is isomorphic to G'.

Example 1: The group of integers modulo n, Z_n, is isomorphic to the cyclic group C_n.
Example 2: From Tables 1.1 and 1.2 one sees that $\mathcal{C}_{3v}$ is isomorphic to S_3:

$$\begin{array}{cccccc} e, & C_3, & C_3^2, & \sigma_1, & \sigma_2, & \sigma_3. \\ e, & (123), & (132), & (23), & (13), & (12). \end{array} \tag{1-15}$$

The isomorphism (1-15) can also be established without using the group table. First label the three vertices of a triangle in a definite way, such as ${}_1\overset{3}{\triangle}{}_2$. This is called the original triangle. Under the group operations, the vertices are interchanged among themselves. The group operator is represented by the permutation of the vertices of the original triangle. For example, under the rotation C_3.

$${}_1\overset{3}{\triangle}{}_2 \xrightarrow{C_3} {}_3\overset{2}{\triangle}{}_1 \ , \quad C_3 \leftrightarrow (123) \ .$$

Thus C_3 corresponds to the permutation (123), which signifies that after the operation C_3, the vertex 1 goes to where 2 was, 2 to where 3 was, and 3 to where 1 was.

We can use this method to obtain the isomorphism (1-15). This method also applies to other point groups [see for example Eq. (3-21) and Tables 8.2-2].

Note that the correspondence $C_3 \leftrightarrow (123)$ only refers to the original triangle. For example, from

$$ {}_1\overset{3}{\triangle}_2 \xrightarrow{\sigma_1} {}_1\overset{2}{\triangle}_3 \xrightarrow{C_3} {}_2\overset{3}{\triangle}_1 $$

one sees that the effect of the operation C_3 on the second triangle is not to move the vertex 1 to where 2 was, and so on. Summarizing, in discussing the isomorphism between C_{3v} and S_3, C_3 always corresponds to (123), while in discussing the effect of C_3 on a triangle, it always rotates the triangle through 120°.

Example 3: The subgroups $S_3(124), S_3(134)$ and $S_3(234)$ are all isomorphic to $S_3 \equiv S_3(123)$.

Cayley's Theorem: Every finite group G is isomorphic to a subgroup of the permutation group $S_{|G|}$.

Suppose that $R_a(R_1, R_2, \ldots, R_{|G|}) = (R_{a_1}, R_{a_2}, \ldots R_{a_{|G|}})$, which shows that the effect of multiplying the group elements of G from the left by an element R_a is a permutation of the group elements. Therefore the group element R_a corresponds to the permutation

$$ R_a \longleftrightarrow p_a = \begin{pmatrix} 1 & 2 & \ldots & |G| \\ a_1 & a_2 & \ldots & a_{|G|} \end{pmatrix}, \tag{1-16} $$

which is an element of the permutation group $S_{|G|}$. It can be shown that $p_1, p_2, \ldots, p_{|G|}$ form a group isomorphic to G.

Reading the group table horizontally, we can easily establish the isomorphism (1-16). For example from Table 1.2 we get the isomorphism of S_3 to a subgroup of S_6, that is

$$ \begin{aligned} &e = e = (1)(2)(3)(4)(5)(6)\ , && p_{12} = \begin{pmatrix} 123456 \\ 216543 \end{pmatrix} = (12)(36)(45)\ , \\ &p_{13} = \begin{pmatrix} 123456 \\ 351624 \end{pmatrix} = (13)(25)(46)\ , && p_{23} = \begin{pmatrix} 123456 \\ 465132 \end{pmatrix} = (14)(26)(35)\ , \\ &p_{123} = \begin{pmatrix} 123456 \\ 534261 \end{pmatrix} = (156)(234)\ , && p_{132} = \begin{pmatrix} 123456 \\ 642315 \end{pmatrix} = (165)(243)\ . \end{aligned} \tag{1-16'} $$

Suppose that to each element a of a group G, there is an element a' in a group G'; however, there may be several elements in G mapped to the same a' in G', and if corresponding to $ab = c$, we have $a'b' = c'$, then we say that G is *homomorphic* to G', denoted as $G \to G'$. If $G \to G'$, the set of elements of G which is mapped onto the identity of G' is called the *kernel of the homomorphism* (Bacry 1977). Isomorphism is a special case of the homomorphism.

Every group has a simplest homomorphic mapping realized by letting each element of the group correspond to the identity element.

A less trivial homomorphism is $C_{3v} \to S_2$. The mapping is

$$ e, C_3, C_3^2 \to e, \qquad \sigma_1, \sigma_2, \sigma_3 \to (12)\ , \tag{1-16"} $$

and the kernel is (e, C_3, C_3^2).

An isomorphism of a group with itself, that is a one–to–one correspondence between elements of the group preserving multiplication, is known as an *automorphism*. An automorphism of G can be regarded as a linear transformation in the $|G|$–dimensional vector space. All such transformations form together the group of automorphisms. If the correspondence is brought about by conjugation (see Sec. 1.5),

$$ G \to R_a G R_a^{-1}, $$

then it is called an *inner automorphism*. The inner automorphisms form a subgroup of all automorphisms. The remaining automorphisms are called the *outer automorphisms*.

An example of an automorphism is the map,

$$(e, a, a^2) \xrightarrow{\phi} (e, a^2, a),$$

for which

$$\phi = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

1.5. Conjugate Classes

An element b of a group G is said to be *conjugate* to an element a if we can find an element u in G such that

$$b = uau^{-1}. \tag{1-17}$$

We refer to uau^{-1} as a *conjugate operation* on a with u. From (1-17) we have $a = u^{-1}bu$, that is, if b is conjugate to a, then a is conjugate to b. It is easy to show that if a is conjugate to b, and b is conjugate to c, then a is conjugate to c. Any element is conjugate to itself. These three properties make conjugacy an equivalence relation.

The elements conjugate to one another from a *conjugate class*, or simply a *class*. Each element of G belongs to one of the classes. The number of classes, denoted by N, is an important characteristic of the group. Designating the number of elements in the class i by g_i, we have the result $g = \sum_{i=1}^{N} g_i$.

How do we find the classes?

Method 1: Multiply the b–th column of the group table, such as Table 1.2, from the left with the element R_b^{-1}. This gives a new table where the elements in each row belong to the same class. The new table is referred to as the *class structure table.*

Method 2: Two elements which lie symmetrically with respect to the diagonal line in the group table belong to the same class.

When a class contains the inverse of all elements in the class, the class is said to be *ambivalent.*

Remarks :

1. In every group, the identity e forms a class by itself.
2. In an Abelian group, each element forms a class by itself:

$$a = uau^{-1} = uu^{-1}a = a .$$

Example 1: Group R_3. Suppose that there are two rotations through the same angle φ but about different axes, say z and n axes (see Fig. 1.5). These two rotations are related by

$$R_n(\varphi) = R(z \to \mathbf{n}) R_z(\varphi) R^{-1}(z \to \mathbf{n}) , \tag{1-18}$$

where $R(z \to \mathbf{n})$ is a rotation which takes the z–axis into the $\mathbf{n}$–axis. Therefore, all rotations through the same angle φ belong to the same class.

The geometric meaning of (1-18) is that a rotation through angle φ about the axis $\mathbf{n}$ can be thought of as the net result of three rotations: first the rotation $R^{-1}(z \to \mathbf{n})$ takes the axis $\mathbf{n}$ to z, then $R_z(\varphi)$ performs a rotation through angle φ about the z–axis, and finally $R(z \to \mathbf{n})$ takes z back to $\mathbf{n}$.

All the classes of R_3 are ambivalent on account of (1-2b). The number of classes of R_3 is infinite.

Example 2: The group C_{3v} has three ambivalent classes, namely,

$$e, \quad (C_3, C_3^2), \quad (\sigma_1, \sigma_2, \sigma_3).$$

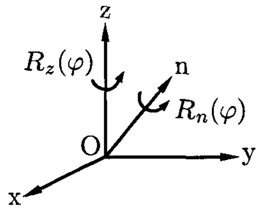

Fig. 1.5. Rotations through the same angle, but about different axes.

Example 3: Classes of the permutation group. Let $a = (ij)$ and $p = \begin{pmatrix} i \\ p_i \end{pmatrix}$ so that $p^{-1} = \begin{pmatrix} p_i \\ i \end{pmatrix}$. It is clear that

$$b = pap^{-1} = p(ij)p^{-1} = (p_i p_j),$$

since pap^{-1} represents the permutation $p_i \to i \to j \to p_j$ and $p_j \to j \to i \to p_i$, which is equivalent to $p_i \leftrightarrow p_j$. Similarly we have

$$p(ij)(klm\ldots)p^{-1} = p(ij)p^{-1}p(klm\ldots)p^{-1} = (p_i p_j)(p_k p_l p_m \ldots)\ . \tag{1-19}$$

For example, consider letting $a = (12)(345)$ and $p = (135)$. Permuting the indices in (12)(345) according to what is specified by p, that is $1 \to 3 \to 5 \to 1$, we get

$$pap^{-1} = (32)(541) = (23)(154).$$

As mentioned before, in expressing a permutation by a product of independent cycles, the order of the cycle factors is irrelevant. We can always write the cycles in the order of decreasing length of the cycles. Suppose that the longest one is an i_1-cycle, the next one is an i_2–cycle $(i_1 > i_2 > \ldots)$. We can use $(i_1 i_2 \ldots)$ to represent the *cycle structure* of a permutation. The components of the cycle structure satisfy $n = i_1 + i_2 + \ldots + i_n$. For example, the cycle structure of the permutation (237)(14)(58)(5) is $(3221) = (32^21)$ and $3 + 2^2 + 1 = 8$.

From (1-19) we know that conjugate elements of a permutation group have the same cycle structure, and vice versa. Thus elements belonging to the same class have the same permutation parity. The cycle structure can be used to write down the conjugate classes of a permutation group. For instance

Group S_3 has N=3 classes

$$\begin{array}{lll} 1.\ (111) = (1^3), & g_1 = 1: & e\ . \\ 2.\ (21), & g_2 = 3: & (12), (23), (13)\ . \\ 3.\ (3), & g_3 = 2: & (123), (132)\ . \end{array} \tag{1-20}$$

Group S_4 has $N = 5$ classes

$$\begin{array}{lll} 1.\ (1^4), & g_1 = 1: & e\ . \\ 2.\ (21^2), & g_2 = 6: & (12), (13), (14), (23), (24), (34)\ . \\ 3.\ (31), & g_3 = 8: & (123), (132), (124), (142), \\ & & (134), (143), (234), (243)\ . \\ 4.\ (4), & g_4 = 6: & (1234), (1243), (1324), \\ & & (1342), (1423), (1432)\ . \\ 5.\ (2^2), & g_5 = 3: & (12)(34), (13)(24), (14)(23)\ . \end{array} \tag{1-21}$$

From Eq. (1-10) one sees that the permutations a and a^{-1} have the same cycle structure, and thus belong to the same class. In other words, all classes of S_n are ambivalent classes.

Note: Two elements belonging to the same class of group G, may belong to different classes of a subgroup of G. For example, (12)(34), (13)(24) and (14)(23) belong to the same class of S_4, but each forms a class by itself for the four-group (1-13), which is a subgroup of S_4, since in the four-group we cannot find an element u such that $(12)(34) = u(13)(24)u^{-1}$.

Let us put the cycle structure in the following way

$$\underbrace{(\cdot)\cdots(\cdot)}_{f_1}\quad\underbrace{(\cdot\cdot)\cdots(\cdot\cdot)}_{f_2}\quad\underbrace{(\cdot\cdot\cdot)\cdots(\cdot\cdot\cdot)}_{f_3}\cdots,$$

that is, there are f_1 1–cycles, f_2 2–cycles,..., and f_n n–cycles, $n = \sum_{k=1}^{n} kf_k$. It can be shown (Hamermesh, 1962) that the number of the permutations of S_n which have this cycle structure is

$$g_{(f_1 f_2\dots)} = \frac{n!}{f_1!f_2!\dots f_n!2^{f_2}3^{f_3}\cdots n^{f_n}}\,. \tag{1-22}$$

Letting $f_1 = n-k, f_k = 1$, and all other $f_i = 0$, we get the number of elements belonging to the k-cycle class

$$g_{(k)} = \frac{n!}{(n-k)!k} = \binom{n}{k}(k-1)! \tag{1-23}$$

where $\binom{n}{k}$ is the binary combination coefficient.

Ex. 1.9. From group Table 1.2 construct the class structure table of S_3.

Ex. 1.10. Calculate the number of elements contained in each class of S_4 and S_5.

Ex. 1.11. Find the classes of the group C_{4v}.

1.6. Cosets and Lagrange's Theorem

1.6.1 Left and right cosets

Let $H = (h_1 = e, h_2, \dots h_{|H|})$ be a *proper* subgroup of G so that $|H| < |G|$. Suppose that a is an element of G which is not contained in H. Form the products $ah_1, ah_2, \dots, ah_{|H|}$, and denote the set of these h elements aH. The products ah_i are all different, for if $ah_i = ah_j$, we would have $h_i = h_j$. Also none of them is contained in H, for if $ah_i = h_j$, then $a = h_j h_i^{-1}$, and a would belong to H, contrary to our assumption. If H and aH have not exhausted the group G, then we proceed as before; pick some element b of G which belongs neither to H nor to aH, and form the set bH. The set bH will again yield $|H|$ new elements of G. Continuing this process, the group G is decomposed into several disjoint sets, each containing $|H|$ elements, denoted symbolically as

$$\begin{aligned} G &= H + aH + bH + \dots + dH \\ &= (e + a + b + \dots + d)H \end{aligned} \tag{1-24}$$

where the plus sign "+" should be understood as the union "∪" for sets. The sets $\{aH, bH, \dots\}$ are called the *left cosets* of H in G, and the $\{a, b, \dots\}$ are called the *representatives* of the cosets.

Suppose that $ah_i = a_i$ for $i = 1, 2, \dots |H|$, that is

$$aH = (a_1, a_2, \dots, a_{|H|})\ , \quad a_1 = a\ .$$

One has that the cosets aH and $a_k H$ coincide, that is

$$aH = a_k H, \quad \text{for any } a_k \in aH,$$

since

$$a_k h_i = a h_k h_i = a h_{ki}.$$

Therefore, any element in the coset aH can be chosen as the coset representative, and for given H, the left coset decomposition of G is unique.

Analogously, we have the right coset decomposition

$$G = H + Ha' + Hb' + \ldots + Hd'. \tag{1-25a}$$

This leads to

1.6.2 Lagrange's Theorem

Lagrange's Theorem: If H is a subgroup of G, then $|H|$ divides $|G|$ so

$$|G|/|H| = m, \tag{1-25b}$$

where m is an integer and is called the *index* of the subgroup H.

From Eq. (1-25b) we immediately know that the order $|R|$ of any element R of G divides $|G|$ so that $|G|/|R|$ =integer. If $|G|$ is a prime number, then $|R|$ is necessarily either equal to one (for $R = e$) or $|G|$ (for $R \neq e$). Hence we have

Theorem 1.1: Let G be a finite group of order p where p is a prime number, then G is a cyclic group.

The left coset decomposition of S_3 is

$$\begin{aligned} S_3 &= (e + (13) + (23))[e, (12)] = (e + (123) + (132))[e, (12)] \\ &= [e, (12)] + [(13), (123)] + [(23), (132)]. \end{aligned}$$

Its right coset decomposition is

$$\begin{aligned} S_3 &= [e, (12)](e + (13) + (23)) = [e, (12)](e + (132) + (123)) \\ &= [e, (12)] + [(13), (132)] + [(23), (123)]. \end{aligned}$$

As is seen in the example above, the left cosets $a_i H$ do not in general coincide with the right cosets Ha_i.

1.6.3 Double cosets

In a manner similar to (1-24) we can partition a group G into *double cosets* with respect to the subgroups H and K,

$$G = H(e + s_2 + s_3 + \cdots + s_q)K \ , \tag{1-26a}$$

where s_i are double coset representatives. As with left and right cosets, two double cosets either coincide or have no elements in common since if $Hs_iK = Hs_\ell K$, we would have $s_i = s_\ell$. Also similar to the single coset case, any element in a double coset can be chosen as its representative. Suppose

$$HaK = \{a_{ij} : \ i = 1, \cdots, |H|; j = 1, \cdots, |K|\} \ , \ a_{ij} = h_i a k_j, \ a_{11} = a \ .$$

Pick up another element, say $b = a_{ij}$, in HaK and form another coset

$$HbK = \{b_{\ell m} : \ \ell = 1, \cdots, |H|; m = 1, \cdots, |K|\} \ .$$

where

$$b_{\ell m} = h_\ell b k_m = h_\ell a_{ij} k_m = h_\ell h_i a k_j k_m = h_{\ell m} a h_{jm} \ .$$

Therefore $HaK = HbK$. In other words, all the elements in the same coset are on an equal footing.

There is however one essential difference between the left (or right) cosets and the double cosets. In any left (or right) coset, an element of G occurs only once, while in some double cosets an element of G may occurs more than once. We use $d(a)$ to denote the number of times, or the "frequency", that the element a occurs in the double coset HaK. It is clear that all elements in the same double coset have the same frequency, since they are on an equal footing. It can be shown (Zhang & Li, 1986) that $d(a)$ is equal to the number of elements in the intersection of the right coset Ha and the left coset aK,

$$d(a) = |a^{-1}Ha \cap K| = |Ha \cap aK| \ . \tag{1-26b}$$

A special but very useful case is when $H = K$,

$$G = H(e + s_2 + s_3 + \cdots + s_q)H \ . \tag{1-26c}$$

In applications (see Sec. 8.7), the most interesting cases are those when all $d(s_i)$ are either equal to 1 or $|H|$. According to (1-26b), $d(s_i) = 1$ when the left and right cosets s_iH and Hs_i have nothing in common except the coset generator s_i, that is when all elements in the double coset Hs_iH are distinct. On the other hand, when the left and right cosets are the same, $s_iH = Hs_i$, $d(s_i) = |H|$. For example for the identity e, $d(e) \equiv |H|$. In the case where $d(s_i) = |H|$, the double coset Hs_iH reduces to the left coset s_iH or right coset Hs_i.

Suppose that in the q double cosets, there are p cosets which are equal to single cosets. For this case the double coset decomposition (1-26c) can be re–written as

$$G = H(e + s_2 + \cdots + s_p) + H(s_{p+1} + \cdots + s_q)H \ . \tag{1-26d}$$

The decomposition (1-26d) is more convenient, since each element of G occur only once in the right cosets and double cosets. For examples and applications of the double–coset decomposition see Sec. 8.7.

1.7. Invariant Subgroups

If the left coset and the right coset of H for any element a of G are the same,

$$aH = Ha, \quad a \in G \ , \tag{1-27}$$

then we say H is an *invariant* (or *normal*) *subgroup* of G. The invariant subgroup can also be defined in the following way. If $H = (h_1, h_2, \dots h_{|H|})$ and

$$ah_ia^{-1} \in H \ , \quad i = 1, 2, \dots, h \ , \quad a \in G \ , \tag{1-28}$$

that is, if H is invariant under the group of inner automorphisms, then H is an invariant subgroup of G.

Equation (1-28) tells us that an invariant subgroup H contains either all or none of the elements in a class of G. The converse is also true. If a subgroup H of G consists of entire classes of G, then H is an invariant subgroup of G. Therefore, any subgroup of an Abelian group is an invariant subgroup. Thus we have the definitions

Simple group G :	if $G \not\supset$ an invariant subgroup.
Semi–simple group G :	if $G \not\supset$ an invariant Abelian subgroup.

where $\not\supset$ means "does not contain."

Example:

1. The group R_3 is simple, but R_2 is not.
2. The group S_3 is not semi–simple, since it contains the Abelian invariant subgroup $\mathcal{A}_3 = (e, (123), (132))$.

The *center* of a group G is the set of elements which commute with every element of G. The center of a group G is an invariant subgroup of G.

1.8. Factor Groups*

Let H be an invariant subgroup of G. We use G/H to denote the set of cosets of H in G:

$$G/H = \{H, a_2H, \ldots, a_mH\}. \tag{1-29}$$

Define a product on the set G/H according to the rule

$$aH \times bH = abH. \tag{1-30}$$

Then the set G/H is a group under the multiplication rule (1-30), called the *factor* or *quotient group* of G relative to the invariant subgroup H. The identity of the factor group G/H is the subgroup H.

Theorem 1.2: If $G \to G'$ is a homomorphic mapping with kernel H, then H is an invariant subgroup of G and the factor group G/H is isomorphic to G'.

Proof: If $a \to e', b \to e'$, then $ab \to e'$. Also $aa^{-1} = e \to e'$ so that if $a \to e'$, then $a^{-1} \to e'$. Therefore, H is a subgroup. Since $H \to e', aH \to a'e' = a', Ha \to e'a' = a'$. Thus aH coincides with Ha, namely H is an invariant subgroup of G. Obviously, G/H is isomorphic to G'. **QED**

Example: $S_3/\mathcal{A}_3 = \{\mathcal{A}_3, \mathcal{B}\}$ is a factor group of order 2, where

$$\mathcal{A}_3 = ((e, (123), (132)), \quad \mathcal{B} = ((12), (13), (23)).$$

The multiplication relations are

$$\mathcal{A}_3\mathcal{A}_3 = \mathcal{A}_3, \quad \mathcal{B}\mathcal{B} = \mathcal{A}_3, \quad \mathcal{B}\mathcal{A}_3 = \mathcal{A}_3\mathcal{B} = \mathcal{B}\ . \tag{1-31}$$

Therefore $S_3/\mathcal{A}_3 \approx S_2$,

$$\mathcal{A}_3 \to e, \quad \mathcal{B} \to (12). \tag{1-32}$$

In the homomorphic mapping (1-32) of $S_3 \to S_2$, the kernel is $\mathcal{A}_3$, which is an invariant subgroup of S_3.

Ex. 1.12. Show that the factor group S_4/F is isomorphic to S_3.

1.9. Direct Product and Semi–Direct Product Groups

Let there be two independent (thus commuting with one another) groups $G = \{a\}$ and $G' = \{a'\}$ with different multiplication rules. Form the $|G||G'|$ pairs (a, a') (or aa') and define the product of pairs by

$$(a, a')(b, b') = (ab, a'b')\ .$$

Those $|G||G'|$ pairs form a group, called the *direct product* of G and G' and denoted $G \times G'$. From the definition, $G \times G'$ is isomorphic to $G' \times G$.

As a special case of the above, the groups G and G' can be subgroups of a larger group. For example, let G_1 and G_2 be two subgroups of G that commute with one another and let $H_i^{(1)} \in G_1, H_j^{(2)} \in G_2$

$$\left[H_i^{(1)}, H_j^{(2)}\right] = 0, \quad i = 1, \ldots, |G_1|, \quad j = 1, \ldots, |G_2|\ . \tag{1-33}$$

The $|G_1||G_2|$ products $\{H_i^{(1)}H_j^{(2)}\}$ form the direct product group $G_1 \times G_2$. $G_1 \times G_2$ is a subgroup of G. We denote this relation by $G \supset G_1 \times G_2$. Since $[G_1, G_2] = 0$, both G_1 and G_2 are invariant subgroups of $G_1 \times G_2$.

Let G be a group with subgroups G_1 and G_2 such that

1. The coset $H_i^{(2)}G_1 = G_1H_i^{(2)}$, for any $H_i^{(2)} \in G_2$,
2. Any element of G can be expressed as $R = H_i^{(1)}H_j^{(2)}$,
3. The intersection of G_1 and G_2 is the identity. The group G is called the *semi–direct product* group of G_1 and G_2, denoted by

$$G = G_1 \wedge G_2.$$

Notice that G_1 is an invariant subgroup of G, but G_2 is not necessarily invariant. In the expression $G_1 \wedge G_2$ we always write the invariant subgroup first.

Example 1: The cyclic $\mathcal{C}_6$ of order 6.

$$\mathcal{C}_6 = G_1 \times G_2, \quad G_1 = (e, a^2, a^4), \quad G_2 = (e, a^3).$$

Example 2: The group $S_n(n = n_1 + n_2)$ has two commuting subgroups $S_{n_1}(1, 2, \dots n_1)$ and $S_{n_2}(n_1 + 1, \dots n)$. We have $S_n \supset S_{n_1} \times S_{n_2}$.
Example 3: The rotation group R_3 commutes with the space inversion group G_I. The direct product of R_3 and G_I is called the *orthogonal group* in three–dimensional space. It is written as

$$O_3 = R_3 \times G_I.$$

Example 4: $\mathcal{A}_3 = (e, (123), (132))$ is an invariant subgroup of S_3. The permutation group S_3 is a semi–direct product of $\mathcal{A}_3$ and S_2,

$$S_3 = \mathcal{A}_3 \wedge S_2. \tag{1-34}$$

Example 5: $S_4 = F \wedge S_3, F$ being the four–group.
Most point groups are semi–direct product groups, see (8-27).

The notion of direct or semi–direct products seems to be like an inverse to the notion of factor group. The exact relationship is follows.

1. If $G = G_1 \times G_2$, then $G/G_1 = G_2$, and $G/G_2 = G_1$.
2. If $G = G_1 \wedge G_2$, then $G/G_1 = G_2$.
3. If $G/G_1 = G_2$, then

$$G = G_1 \wedge G_2, \text{ if } G_2 \text{ is a subgroup of } G,$$
$$G = G_1 \times G_2, \text{ if } G_2 \text{ is an invariant subgroup of } G.$$

4. If $G/G_1 = G_2, G/G_2 = G_1$, then $G = G_1 \times G_2$.

For example, from $S_3 = \mathcal{A}_3 \wedge S_2$ and $S_4 = F \wedge S_3$, we have $S_3/\mathcal{A}_3 = S_2$, and $S_4/F = S_3$ respectively.

Ex. 1.13. Show that $O_3/R_3 = Z_2$.