Chapter 1

The Poincaré Group

1.1 Notation

Throughout this book we use the following notation. A four vector is specified by its four components as follows

$$A \equiv A^{\mu} = (A^0, \vec{A}) \ . \tag{1.1.1}$$

The metric tensor is given by

$$g^{\mu\nu} = g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \ . \tag{1.1.2}$$

This metric tensor is used to raise and lower indeces according to

$$A_{\mu} = A^{\nu} g_{\nu\mu} \quad A^{\mu} = A_{\nu} g^{\nu\mu} \tag{1.1.3}$$

Here, as throughout the rest of the book we have summed over repeated upper and lower indeces.

Using this summation convention, we write the inner product as

$$A.B = A^{\mu} g_{\mu\nu} B^{\nu} = A^{\mu} B_{\mu} = A^0 B^0 - \vec{A} \cdot \vec{B} \ . \tag{1.1.4}$$

the "length" of a vector is given by

$$A^2 = A.A = (A^0)^2 - \vec{A} \cdot \vec{A} \tag{1.1.5}$$

A Lorentz transformation Λ is a linear transformation mapping Minkowski spacetime onto itself and therefore preserves the inner product. In other words we always have for any two four vectors x and y

$$(\Lambda x) \cdot (\Lambda y) = x \cdot y \quad . \tag{1.1.6}$$

If we write out the Lorentz transformation explicitly so that

$$(\Lambda x)^\mu = \Lambda^\mu_{\cdot\nu} x^\nu \tag{1.1.7}$$

then we find that the real matrix $\Lambda^\mu_{\cdot\nu}$ must satisfy

$$\Lambda^\mu_{\cdot\nu} \Lambda_{\mu\lambda} = g_{\nu\lambda} \tag{1.1.8}$$

1.2 The Poincaré Group

The Poincaré group is the set of Lorentz transformations and space-time translations (Λ, a) such that

$$x^\mu \rightarrow x'^\mu = \Lambda^\mu_{\cdot\nu} x^\nu + a^\mu \ . \tag{1.2.9}$$

The group multiplication law is given by

$$(\Lambda_2, a_2)(\Lambda_1, a_1) = (\Lambda_2 \Lambda_1,\ \Lambda_2 a_1 + a_2) \ . \tag{1.2.10}$$

The generators of these transformations are called P^λ, $M^{\mu\nu}$ and are such that the corresponding unitary operators that represent these transformations may be written

$$\begin{aligned} U(a,1) &= \exp(iP \cdot a) \\ U(0,\Lambda) &= \exp(iM^{\mu\nu}\Lambda_{\mu\nu}) \ . \end{aligned} \tag{1.2.11}$$

They satisfy the following commutation relations. (See problem 1.1)

$$\begin{aligned} [M_{\mu\nu}, M_{\rho\sigma}] &= -i\,(g_{\mu\rho} M_{\nu\sigma} + g_{\nu\sigma} M_{\mu\rho} - g_{\nu\rho} M_{\mu\sigma} - g_{\mu\sigma} M_{\nu\rho}) \\ [M_{\mu\nu}, P_\sigma] &= i\,(g_{\nu\sigma} P_\mu - g_{\mu\sigma} P_\nu) \\ [P_\mu, P_\nu] &= 0 \ . \end{aligned} \tag{1.2.12}$$

In 1939 Wigner [1.1] presented a complete classification of all the unitary irreducible representations of the Poincaré group. These are the possible elementary states in any theory which respects this symmetry. We now give his classification. However, we do not follow his method. Instead we find a set of operators whose eigenvalues label these irreducible representations. These so-called *Casimir operators* are constructed from the generators and commute with all the generators.

A familiar example, from ordinary quantum mechanics, is the Casimir operator J^2 for the rotation group. It commutes with the three generators $\vec{J}$ of the rotation group and its eigenvalues $j(j+1)\hbar^2$ label the "simplest" states of particles with angular momentum.

1.3 The Casimir Operators

For the Poincaré group we have two Casimir operators. They are

$$P^2 = P_\mu P^\mu \tag{1.3.13}$$

and

$$w^2 = w_\mu w^\mu \ . \tag{1.3.14}$$

where the *Pauli-Lubanski vector* w_μ is defined by

$$w_\sigma = \frac{1}{2}\epsilon_{\sigma\mu\nu\lambda}M^{\mu\nu}P^\lambda \ . \tag{1.3.15}$$

Writing this out we find

$$w^0 = \vec{P}\cdot\vec{J} \quad , \quad \vec{w} = P_0\vec{J} - \vec{P}\times\vec{N} \tag{1.3.16}$$

where

$$\vec{J} = (M_{32},\, M_{13},\, M_{21}) \tag{1.3.17}$$

are the three components of angular momentum and (with units such that $\hbar = 1$ satisfy

$$[J_1, J_2] = iJ_3 \ \text{ and cyclic permutations} \ . \tag{1.3.18}$$

and

$$\vec{N} = (M_{01},\, M_{02},\, M_{03}) \tag{1.3.19}$$

are the *boosts* in the three Cartesian directions.

1.4 The Irreducible Representations

The irreducible unitary representations of the Poincaré group are classified according to the eigenvalues of P^2 and w^2 . They fall into several classes according to the eigenvalues of P^2 and P_0. These are (here we have units such that $\hbar = c = 1$)

1. a) $P^2 = m^2 > 0$ and $P_0 > 0$

1. b) $P^2 = m^2 > 0$ and $P_0 < 0$

2. a) $P^2 = 0$ and $P_0 > 0$

2. b) $P^2 = 0$ and $P_0 < 0$

3. $P^2 = 0$ and $P_0 = 0$
   In this case $P_\mu = (0, 0, 0, 0)$

4. $P^2 = m^2 < 0$
   This case corresponds to faster-than-light particles or "tachyons".

Case 3. corresponds to particles with continuous spin and is also unphysical. Thus, only cases 1. and 2. are of interest. For case 1. we have $m \neq 0$ and thus we can transform to a Lorentz frame in which the three-momentum $\vec{p} = 0$. In this *rest frame* the eigenvalues of P^μ are

$$p^\mu = (m, 0, 0, 0) \tag{1.4.20}$$

so that

$$p^2 = p_\mu p^\mu = m^2 \tag{1.4.21}$$

and

$$-w^2 = -w_0^2 + \vec{w}^2 = \vec{w}^2 = p_0^2 J^2 = m^2 s(s+1) \ . \tag{1.4.22}$$

Now, the eigenvalues of J^2 in the rest frame of the particle are just the values of the total intrinsic angular momentum or spin, namely $s(s+1)$. This means that massive particles may be classified according to their mass and spin. We next consider the simplest cases, namely spin 0 and spin 1/2.

1.5 Problems

1.1 Show that the generators of the Poincarè group satisfy the commutation relations given by (1.5.23)

$$\begin{aligned}
[M_{\mu\nu}, M_{\rho\sigma}] &= -i\,(g_{\mu\rho} M_{\nu\sigma} + g_{\nu\sigma} M_{\mu\rho} - g_{\nu\rho} M_{\mu\sigma} - g_{\mu\sigma} M_{\nu\rho}) \\
[M_{\mu\nu}, P_\sigma] &= i\,(g_{\nu\sigma} P_\mu - g_{\mu\sigma} P_\nu) \\
[P_\mu, P_\nu] &= 0 \ . \qquad (1.5.23)
\end{aligned}$$

1.2 Verify that P^2 and w^2 are indeed Casimir operators, i.e. that they commute with P_μ and $M_{\mu\nu}$.

Bibliography

[1.1] E.P. Wigner, Ann. Math., **40**, 149, (1939).