

## STAR-TRIANGLE EQUATIONS AND IDENTITIES IN HYPERGEOMETRIC SERIES

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In this paper, we introduce the cyclic basic hypergeometric series  ${}_{p+1}\Phi_p$  with  $q \rightarrow \omega$  where  $\omega^N = 1$ . This is a terminating series with  $N$  terms, whose summand has period  $N$ . We show how the Fourier transform of the weights of the integrable chiral Potts model are related to the  ${}_2\Phi_1$ , which is summable. We show that  ${}_3\Phi_2$  satisfies certain transformation formulae. We then show that the Saalschützian  ${}_4\Phi_3$  series is summable at argument  $z = \omega$ . This then gives the simplest proof of the star-triangle relation in the chiral Potts model. Finally, we let  $N \rightarrow \infty$ , where the star-triangle equation becomes a two-sided identity for the hypergeometric series.

### 1. Introduction

#### 1.1. Definitions

The generalized hypergeometric series is defined<sup>1,2</sup> as

$${}_{p+1}F_p \left[ \begin{matrix} a_1, \dots, a_{p+1} \\ b_1, \dots, b_p \end{matrix}; z \right] = \sum_{l=0}^{\infty} \frac{(a_1)_l \cdots (a_{p+1})_l}{(b_1)_l \cdots (b_p)_l l!} z^l, \quad (1)$$

where

$$(a)_l = \Gamma(a+l)/\Gamma(a) = a(a+1)\cdots(a+l-1), \quad (2)$$

while the basic hypergeometric series is

$${}_{p+1}\Phi_p \left[ \begin{matrix} \alpha_1, \dots, \alpha_{p+1} \\ \beta_1, \dots, \beta_p \end{matrix}; z \right] = \sum_{l=0}^{\infty} \frac{(\alpha_1; q)_l \cdots (\alpha_{p+1}; q)_l}{(\beta_1; q)_l \cdots (\beta_p; q)_l (q; q)_l} z^l, \quad (3)$$

in which

$$(x; q)_l \equiv \begin{cases} (1-x)(1-xq)\cdots(1-xq^{l-1}), & l \geq 0, \\ 1/[(1-xq^{-1})(1-xq^{-2})\cdots(1-xq^l)], & l < 0. \end{cases} \quad (4)$$

The hypergeometric series  ${}_{p+1}F_p$  can be obtained from  ${}_{p+1}\Phi_p$  by taking the limit

$$q \rightarrow 1 \quad \text{with} \quad \alpha = q^a, \quad (\alpha; q)_l / (q; q)_l \rightarrow (a)_l. \quad (5)$$

**1.2. Known Identities**

A hypergeometric series is summable if the series can be written in terms of ratios of products of Gamma functions, while for the summable basic series it is written in terms of the  $q$ -products defined in (4). The most well-known summation formula is due to Gauss

$${}_2F_1 \left[ \begin{matrix} a, b \\ c \end{matrix} ; 1 \right] = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}, \tag{6}$$

which is a summation formula for  ${}_2F_1$  of unit argument. The other is Saalschütz's theorem

$${}_3F_2 \left[ \begin{matrix} a, b, -n \\ c, d \end{matrix} ; 1 \right] = \frac{(c-a)_n(c-b)_n}{(c-a-b)_n(c)_n}, \quad \text{for } c+d = a+b-n+1, \tag{7}$$

for a terminating Saalschützian  ${}_3F_2$  of unit argument. In general, a series is called Saalschützian if it satisfies the Saalschütz condition

$$1 + a_1 + \dots + a_{p+1} = b_1 + \dots + b_p. \tag{8}$$

Most of the summation formulae for the usual  ${}_p+1F_p$  hypergeometric series have basic series analogues.<sup>1,2</sup> The summability condition on the argument of  $z = 1$  for the hypergeometric series must be replaced by  $z = q$  for the basic series, while the Saalschütz condition is seen from (5) to become

$$q\alpha_1 \dots \alpha_{p+1} = \beta_1 \dots \beta_p, \tag{9}$$

as  $a$  and  $b$  are the exponents of  $\alpha$  and  $\beta$ . As an example, Dougall's theorem summing a terminating  ${}_7F_6$  of unit argument generalizes to Jackson's theorem for terminating  ${}_8\Phi_7$  of argument  $z = q$ .

We shall now show that the basic hypergeometric series at root of unity are intimately related to the integrable chiral Potts model. Indeed, many of the results presented here are implicit in the earlier works.<sup>3-13</sup> Since the notations used in several of these works<sup>6-11</sup> are unconventional, making the connections obscure, we present here the results in more standard notation.

**2. The Cyclic Basic Hypergeometric Series**

**2.1. Definitions**

Since most of the summation formulae are valid only for terminating series, it is straightforward to analytically continue  $q$  to a root of unity without any convergence problems. For  $q \rightarrow \omega \equiv e^{2\pi i/N}$ , we find

$$(x; \omega)_{l+N} = (1 - x^N)(x; \omega)_l, \quad (\omega; \omega)_{l+N} = 0, \tag{10}$$

$$(x; \omega)_{-l} = \omega^{\frac{1}{2}l(l+1)} / [(-x)^l(\omega x^{-1}; \omega)_l], \tag{11}$$

$$(x; \omega)_{l+k} = (x; \omega)_k (\omega^k x; \omega)_l. \tag{12}$$

From (10), we see immediately that the basic series is ill-defined for  $q = \omega$  unless  $\alpha_{p+1} = \omega^{-J}$  for some  $J < N$ . We shall here restrict ourselves to the case with  $\alpha_{p+1} = \omega^{-N+1} = \omega$ , so that there are  $N$  terms in the series.

**Definition:** A cyclic basic hypergeometric series is a terminating series of  $N$  terms

$${}_{p+1}\Phi_p \left[ \begin{matrix} \omega, \alpha_1, \dots, \alpha_p \\ \beta_1, \dots, \beta_p \end{matrix}; z \right] = \sum_{l=0}^{N-1} \frac{(\alpha_1; \omega)_l \cdots (\alpha_p; \omega)_l}{(\beta_1; \omega)_l \cdots (\beta_p; \omega)_l} z^l, \tag{13}$$

whose summand is periodic in  $N$ .

Using (10), we find that the requirement for periodic summand is satisfied if

$$z^N = \prod_{j=1}^p \frac{1 - \beta_j^N}{1 - \alpha_j^N}. \tag{14}$$

Unlike the ordinary basic hypergeometric series in (3), where the dependences on the parameters  $\alpha_l$  and  $\beta_l$  are elementary, the periodicity requirement makes the dependences on these parameters very complicated, with an extremely complex  $N$ -sheeted Riemann sheet structure.

Because of this periodicity, we may change the indices of the summation  $l \rightarrow -l$  in (13) and then let  $l \rightarrow l + k$  while using (11), to find

$${}_{p+1}\Phi_p \left[ \begin{matrix} \omega, \alpha_1, \dots, \alpha_p \\ \beta_1, \dots, \beta_p \end{matrix}; z \right] = {}_{p+1}\Phi_p \left[ \begin{matrix} \omega, \omega\beta_1^{-1}, \dots, \omega\beta_p^{-1} \\ \omega\alpha_1^{-1}, \dots, \omega\alpha_p^{-1} \end{matrix}; \frac{\beta_1 \cdots \beta_p}{z\alpha_1 \cdots \alpha_p} \right] \tag{15}$$

$$= {}_{p+1}\Phi_p \left[ \begin{matrix} \omega, \omega^k \alpha_1, \dots, \omega^k \alpha_p \\ \omega^k \beta_1, \dots, \omega^k \beta_p \end{matrix}; z \right] \frac{(\alpha_1; \omega)_k \cdots (\alpha_p; \omega)_k}{(\beta_1; \omega)_k \cdots (\beta_p; \omega)_k} z^k. \tag{16}$$

Since  $\alpha_{p+1} = \omega$ , the series in (13) is called a Saalschützian if

$$\omega^2 \alpha_1 \alpha_2 \cdots \alpha_p = \beta_1 \beta_2 \cdots \beta_p, \quad z = \omega. \tag{17}$$

Clearly, if the left-hand side of (15) is a Saalschützian, so is the right-hand side, and vice versa.

### 2.2. Cyclic basic series ${}_2\Phi_1$

It has been found<sup>3, 15</sup> that the Boltzmann weights of the integrable chiral Potts model can be written in product form, i.e.

$$W(n) = \gamma^n \frac{(\alpha; \omega)_n}{(\beta; \omega)_n} = \left( \frac{\beta}{\gamma\alpha} \right)^{-n} \frac{(\omega/\beta; \omega)_{-n}}{(\omega/\alpha; \omega)_{-n}} = W^*(-n), \quad \gamma^N = \frac{1 - \beta^N}{1 - \alpha^N}. \tag{18}$$

The Boltzmann weight of an edge connecting spin  $a$  and spin  $b$  is chiral, namely,  $W(a - b) \neq W(b - a)$ , and arrows are introduced to indicate the direction from spin  $a$  to  $b$ , as shown in Fig. 1. Here we have introduced  $W^*(a - b) = W(b - a)$  to indicate the operation of arrow-reversing.

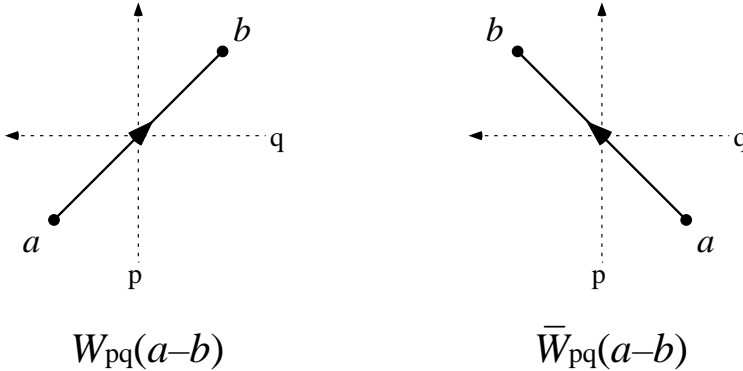


Fig. 1. Boltzmann Weights

Since the weights are periodic with period  $N$ , their Fourier transforms may be written as

$$W^{(f)}(k) = \sum_{n=0}^{N-1} \omega^{nk} W(n) = {}_2\Phi_1 \left[ \begin{matrix} \omega, \alpha \\ \beta \end{matrix}; \gamma \omega^k \right]. \tag{19}$$

2.2.1. Recursion Formula

It has been found originally in Canberra<sup>3</sup> that

$$\frac{W^{(f)}(k)}{W^{(f)}(0)} = {}_2\Phi_1 \left[ \begin{matrix} \omega, \alpha \\ \beta \end{matrix}; \gamma \omega^k \right] {}_2\Phi_1 \left[ \begin{matrix} \omega, \alpha \\ \beta \end{matrix}; \gamma \right]^{-1} = \frac{(\omega/\beta)^k (\gamma; \omega)_k}{(\omega \alpha \gamma / \beta; \omega)_k} = \frac{(\beta/\gamma \alpha; \omega)_{-k}}{\alpha^k (\omega/\gamma; \omega)_{-k}}. \tag{20}$$

The proof of this recursion relation has been given in our Taniguchi lectures.<sup>4</sup> This was later extended to a more general case by Kashaev et al.<sup>7</sup>

$${}_2\Phi_1 \left[ \begin{matrix} \omega, \omega^m \alpha \\ \omega^n \beta \end{matrix}; \gamma \omega^k \right] / {}_2\Phi_1 \left[ \begin{matrix} \omega, \alpha \\ \beta \end{matrix}; \gamma \right] = \frac{(\omega/\beta)^k (\beta; \omega)_n (\gamma; \omega)_k (\omega \alpha / \beta; \omega)_{m-n}}{(\gamma \omega^k)^n (\alpha; \omega)_m (\omega \alpha \gamma / \beta; \omega)_{m-n+k}}. \tag{21}$$

2.2.2. Baxter Formula

Consider the determinant whose elements are the weights in (18), i.e.

$$D = \det_{1 \leq l, k \leq N} W(l - k), \tag{22}$$

Baxter gave the following formula<sup>5</sup> without proof:

$$D = \Phi_0^N N^{\frac{1}{2}N} \prod_{j=1}^{N-1} \left[ \frac{(\alpha - \omega^{-1-j}\beta)}{(1 - \omega^{-1-j}\beta)(1 - \omega^j\alpha)} \right]^j, \tag{23}$$

where

$$\Phi_0 \equiv e^{i\pi(N-1)(N-2)/12N}. \tag{24}$$

A detailed proof was subsequently given by us in Ref. 14. Since this is a cyclic determinant, we find

$$D = \prod_{j=0}^{N-1} W^{(f)}(j) = [W(0)^{(f)}]^N \prod_{j=1}^{N-1} \left[ \frac{W^{(f)}(j)}{W^{(f)}(0)} \right]. \tag{25}$$

Baxter’s formula (23) and the recursion formula (20) may now be used to prove the following theorem.

**Theorem 1:** Every cyclic basic hypergeometric series  ${}_2\Phi_1$  is summable, and is given by

$${}_2\Phi_1 \left[ \begin{matrix} \omega, \alpha \\ \beta \end{matrix} ; \gamma \right] = \omega^\ell N^{\frac{1}{2}} \Phi_0 \left( \frac{\omega}{\beta} \right)^{\frac{1}{2}(N-1)} \frac{p(\omega\alpha/\beta)p(\gamma)}{p(\alpha)p(\omega/\beta)p(\omega\alpha\gamma/\beta)}, \tag{26}$$

where  $\ell$  takes  $N$  different integer values for the  $N$  different Riemann sheets, and

$$p(\alpha) = \prod_{j=1}^{N-1} (1 - \omega^j \alpha)^{j/N}. \tag{27}$$

Here summable mean that the series is expressible as products. It is worthwhile to emphasize that the basic hypergeometric function  ${}_2\Phi_1$  is an  $N$ -valued function of  $\alpha$  and  $\beta$  with a complicated Riemann surface. The function  $p(\alpha)$  has  $N - 1$  branch points at  $\alpha = \omega^j$  for  $j = 1, \dots, N - 1$ . Due to the appearance of the composite functions—particularly,  $p(\gamma)$  with  $\gamma$  found from (18) to be an  $N$ -valued function of  $\alpha$  and  $\beta$ —we can see that it is non-trivial to describe the Riemann surface. It is rather amazing even to us that Baxter and others (see Refs. 17, 18 and citations quoted there) have somehow found a way out without the detailed knowledge of the Riemann surfaces.

**2.3. Transformation formula for  ${}_3\Phi_2$**

We shall now derive a transformation formula for the cyclic basic series  ${}_3\Phi_2$ . Using the convolution theorem, we may write

$$\begin{aligned} {}_3\Phi_2 \left[ \begin{matrix} \omega, \alpha_1, \alpha_2 \\ \beta_1, \beta_2 \end{matrix} ; \gamma \right] &= \sum_{l=0}^{N-1} \left[ \frac{(\alpha_1; \omega)_l}{(\beta_2; q)_l} u^l \right] \left[ \frac{(\alpha_2; q)_l}{(\beta_1; q)_l} \left( \frac{z}{u} \right)^l \right] \\ &= N^{-1} \sum_{k=0}^{N-1} {}_2\Phi_1 \left[ \begin{matrix} \omega, \alpha_1 \\ \beta_2 \end{matrix} ; \omega^{-k} u \right] {}_2\Phi_1 \left[ \begin{matrix} \omega, \alpha_2 \\ \beta_1 \end{matrix} ; \frac{\omega^k \gamma}{u} \right], \end{aligned} \tag{28}$$

where

$$u^N = \frac{1 - \beta_2^N}{1 - \alpha_1^N}, \quad \frac{z^N}{u^N} = \frac{1 - \beta_1^N}{1 - \alpha_2^N}. \tag{29}$$

Now we can use the recursion formula (20) to obtain

$$\begin{aligned}
 {}_3\Phi_2 \left[ \begin{matrix} \omega, \alpha_1, \alpha_2 \\ \beta_1, \beta_2 \end{matrix}; z \right] &= N^{-1} {}_2\Phi_1 \left[ \begin{matrix} \omega, \alpha_1 \\ \beta_2 \end{matrix}; u \right] {}_2\Phi_1 \left[ \begin{matrix} \omega, \alpha_2 \\ \beta_1 \end{matrix}; \frac{z}{u} \right] \\
 &\times {}_3\Phi_2 \left[ \begin{matrix} \omega, z/u, \beta_2/\alpha_1 u \\ \omega/u, \omega\alpha_2 z/\beta_1 u \end{matrix}; \frac{\omega\alpha_1}{\beta_1} \right]. \tag{30}
 \end{aligned}$$

Clearly, we may change  $\beta_2$  to  $\beta_1$  with  $u \rightarrow \gamma_1$ , and obtain a different transformation formula. If we let  $z = \omega$  in (30), then it is easily seen that the  ${}_3\Phi_2$  on the right-hand side of the equation becomes a  ${}_2\Phi_1$ . As the three  ${}_2\Phi_1$  are summable, the cyclic hypergeometric series  ${}_3\Phi_2$  is also summable for argument  $z = \omega$ . We conclude:

**Theorem 2:** Every cyclic basic hypergeometric series  ${}_3\Phi_2$  has a transformation formula given by (30), and is summable for  $z = \omega$ .

### 2.4. The Saalschützian ${}_4\Phi_3$ and the Star-Triangle Relation

#### 2.4.1. Summation Formula

Consider a Saalschützian  ${}_4\Phi_3$  for argument  $z = \omega$ , and use the convolution theorem to express it as

$$\begin{aligned}
 {}_4\Phi_3 \left[ \begin{matrix} \omega, \omega^a \alpha_1, \omega^b \alpha_2, \omega^c \alpha_3 \\ \omega^a \beta_1, \omega^b \beta_2, \omega^c \beta_3 \end{matrix}; \omega \right] &= \\
 N^{-1} \sum_{k=0}^{N-1} {}_3\Phi_2 \left[ \begin{matrix} \omega, \omega^a \alpha_1, \omega^b \alpha_2 \\ \omega^a \beta_1, \omega^b \beta_2 \end{matrix}; \omega^k \gamma \right] {}_2\Phi_1 \left[ \begin{matrix} \omega, \omega^c \alpha_3 \\ \omega^c \beta_3 \end{matrix}; \frac{\omega^{1-k}}{\gamma} \right], \tag{31}
 \end{aligned}$$

where

$$\gamma^N = \frac{(1 - \alpha_3^N)}{(1 - \beta_3^N)} = \frac{(1 - \beta_1^N)(1 - \beta_2^N)}{(1 - \alpha_1^N)(1 - \beta_2^N)} = \gamma_1^N \gamma_2^N. \tag{32}$$

The first part of this equation and the Saalschütz condition  $\omega^2 \alpha_1 \alpha_2 \alpha_3 = \beta_1 \beta_2 \beta_3$ , may be used to solve  $\alpha_3$  and also  $\beta_3$ . This gives

$$\alpha_3^N = \frac{1 - \gamma^N}{1 - (\alpha_1 \alpha_2 \gamma / \beta_1 \beta_2)^N}, \quad \beta_3 = \frac{\omega^2 \alpha_1 \alpha_2 \alpha_3}{\beta_1 \beta_2}. \tag{33}$$

Now we use the transformation formula (30) for  ${}_3\Phi_2$  thrice, i.e.

$$\begin{aligned}
 {}_3\Phi_2 \left[ \begin{matrix} \omega, \omega^a \alpha_1, \omega^b \alpha_2 \\ \omega^a \beta_1, \omega^b \beta_2 \end{matrix}; \gamma \omega^k \right] &= \hat{A} {}_3\Phi_2 \left[ \begin{matrix} \omega, \omega^k \gamma / u, \omega^{b-a} \bar{\beta}_3^* / u \\ \omega / u, \omega^{+k+b-a} \bar{\alpha}_3^* \gamma / u \end{matrix}; \frac{\omega \alpha_1}{\beta_1} \right] \\
 &= \hat{A} \hat{B} {}_3\Phi_2 \left[ \begin{matrix} \omega, \bar{\gamma}_1^*, \omega^k \bar{\gamma}_2 \\ \omega / \bar{\gamma}_1, \omega^{k+1} / \bar{\gamma}_2 \end{matrix}; \frac{\omega^b \beta_2}{\omega^a \alpha_1} \right] \\
 &= \hat{A} \hat{B} \hat{C} {}_3\Phi_2 \left[ \begin{matrix} \omega, \omega^{b-a} \bar{\alpha}_2^*, \bar{\alpha}_1^* \\ \omega^{b-a} \bar{\beta}_2^*, \bar{\beta}_1^* \end{matrix}; \frac{\omega^{1-k}}{\gamma} \right], \tag{34}
 \end{aligned}$$

where the first line is identical to (30) in which

$$\left\{ \begin{matrix} \bar{\alpha}_1 = \omega \alpha_2 / \beta_3, \\ \bar{\beta}_1 = \beta_2 / \alpha_3, \end{matrix} \right\} \left\{ \begin{matrix} \bar{\alpha}_2 = \omega \alpha_1 / \beta_3, \\ \bar{\beta}_2 = \beta_1 / \alpha_3, \end{matrix} \right\} \left\{ \begin{matrix} \bar{\alpha}_3 = \omega / \bar{\beta}_3^* = \omega \alpha_1 / \beta_2, \\ \bar{\beta}_3 = \omega / \bar{\alpha}_3^* = \beta_1 / \alpha_2, \end{matrix} \right\} \tag{35}$$

with superscript \* denoting the arrow-reversing operation described in (18), and

$$\hat{A} = N^{-1} {}_2\Phi_1 \left[ \begin{matrix} \omega, \omega^a \alpha_1 \\ \omega^b \beta_2 \end{matrix}; u \right] {}_2\Phi_1 \left[ \begin{matrix} \omega, \omega^b \alpha_2 \\ \omega^a \beta_1 \end{matrix}; \frac{\omega^k \gamma}{u} \right]. \tag{36}$$

If we denote

$$\gamma_j^N = \frac{1 - \beta_j^N}{1 - \alpha_j^N}, \quad \bar{\gamma}_j^N = \frac{\bar{\beta}_j^N}{(\bar{\gamma}_j^* \bar{\alpha}_j)^N} = \frac{1 - \bar{\beta}_j^N}{1 - \bar{\alpha}_j^N}, \tag{37}$$

then by using (35) and (33), it is easy to verify that

$$\bar{\gamma}_1 = \gamma_2 / \bar{\gamma}_3^*, \quad \bar{\gamma}_2 = \gamma_1 / \bar{\gamma}_3, \tag{38}$$

which in turn can be used to show

$$\bar{\gamma}_1 \bar{\gamma}_2 = \gamma / \bar{\gamma}_3 \bar{\gamma}_3^* = \gamma \beta_3 / \omega \alpha_3 \text{ leading to } \gamma_3 = \omega / \gamma = \bar{\gamma}_1^* \bar{\gamma}_2^*. \tag{39}$$

Using (29) in which  $z = \gamma$ , it is easy to verify that

$$\frac{1 - (\bar{\alpha}_3^* \gamma / u)^N}{1 - (\bar{\beta}_3^* / u)^N} = \left( \frac{\gamma_2}{\bar{\gamma}_3^*} \right)^N = \bar{\gamma}_1^N, \quad \frac{1 - (1/u)^N}{1 - (\gamma/u)^N} = \left( \frac{\bar{\gamma}_3^* \alpha_1}{\gamma_2 \beta_1} \right)^N = (\bar{\gamma}_1^*)^N. \tag{40}$$

From (18) and (35), it follows that  $\bar{\gamma}_1 \bar{\gamma}_1^* = \omega \alpha_1 / \beta_1$ . As a consequence, we may use the transformation formula (30) again to obtain the second equality in (34) with

$$\hat{B} = N^{-1} {}_2\Phi_1 \left[ \begin{matrix} \omega, \omega^k \gamma / u \\ \omega / u \end{matrix}; \bar{\gamma}_1^* \right] {}_2\Phi_1 \left[ \begin{matrix} \omega, \omega^{b-a} \bar{\beta}_3^* / u \\ \omega^{k+b-a} \bar{\alpha}_3^* \gamma / u \end{matrix}; \bar{\gamma}_1 \right]. \tag{41}$$

Furthermore, from (37), we find that

$$\frac{1 - (1/\bar{\gamma}_j)^N}{1 - (\bar{\gamma}_j^*)^N} = \bar{\alpha}_j^N, \quad \frac{1 - (1/\bar{\gamma}_j^*)^N}{1 - (\bar{\gamma}_j)^N} = (\bar{\alpha}_j^*)^N, \tag{42}$$

while from (35) and (18) we obtain  $\bar{\alpha}_1 \bar{\alpha}_2^* = \beta_2 / \alpha_1$ . Using the transformation formula (30) for the third time, we arrive at (34) with

$$\hat{C} = N^{-1} {}_2\Phi_1 \left[ \begin{matrix} \omega, \bar{\gamma}_1^* \\ \omega / \bar{\gamma}_1 \end{matrix}; \bar{\alpha}_1 \right] {}_2\Phi_1 \left[ \begin{matrix} \omega, \omega^k \bar{\gamma}_2 \\ \omega^{k+1} / \bar{\gamma}_2^* \end{matrix}; \omega^{b-a} \bar{\alpha}_2^* \right]. \tag{43}$$

Denoting

$$W_i(n) = \gamma_i^n \frac{(\alpha_i; \omega)_n}{(\beta_i; \omega)_n}, \quad \bar{W}_i(n) = \bar{W}_i^*(-n) = \bar{\gamma}_i^n \frac{(\bar{\alpha}_i; \omega)_n}{(\bar{\beta}_i; \omega)_n} \tag{44}$$

and using recursion formula (20) for  ${}_2\Phi_1$  in these constants  $\hat{A}$ ,  $\hat{B}$  and  $\hat{C}$ , we find

$$\begin{aligned} {}_3\Phi_2 \left[ \begin{matrix} \omega, \omega^a \alpha_1, \omega^b \alpha_2 \\ \omega^a \beta_1, \omega^b \beta_2 \end{matrix}; \omega^k \gamma \right] &= \frac{\bar{W}_3^*(b-a) \bar{W}_2^*(b-a)}{W_1(a) W_2(b)} \\ &\times \frac{\omega^{-kb} D \alpha_3^{-k}(\gamma; \omega)_k}{(\gamma \beta_3 / \omega \alpha_3; \omega)_k} {}_3\Phi_2 \left[ \begin{matrix} \omega, \omega^{b-a} \bar{\alpha}_2^*, \bar{\alpha}_1^* \\ \omega^{b-a} \bar{\beta}_2^*, \bar{\beta}_1^* \end{matrix}; \frac{\omega^{1-k}}{\gamma} \right], \end{aligned} \tag{45}$$

where  $D = [\hat{A}\hat{B}\hat{C}]_0$  with  $a = b = c = k = 0$  in (36), (41) and (43). Next, the recursion formula (20) is used to write

$${}_2\Phi_1 \left[ \begin{matrix} \omega, \omega^c \alpha_3 \\ \omega^c \beta_3 \end{matrix}; \frac{\omega^{1-k}}{\gamma} \right] = \frac{R \omega^{ck} (\gamma \beta_3 / \omega \alpha_3; \omega)_k}{D W_3(c) \alpha_3^{-k}(\gamma; \omega)_k}, \quad \frac{R}{D} = {}_2\Phi_1 \left[ \begin{matrix} \omega, \alpha_3 \\ \beta_3 \end{matrix}; \cdot \right] \frac{\omega}{\gamma}. \tag{46}$$

Substituting these two equations into the convolution formula (31) and using

$$\frac{W(n+a)}{W(a)} = \gamma^n \frac{(\omega^a \alpha; \omega)_n}{(\omega^a \beta; \omega)_n}, \tag{47}$$

we find

$$W_1(a)W_2(b)W_1(c) {}_4\Phi_3 \left[ \begin{matrix} \omega, \omega^a \alpha_1, \omega^b \alpha_2, \omega^c \alpha_3 \\ \omega^a \beta_1, \omega^b \beta_2, \omega^c \beta_3 \end{matrix}; \omega \right] = N^{-1} R \overline{W}_3(a-b) \sum_{k=0}^{N-1} \sum_{l=0}^{N-1} \omega^{k(c-b-l)} \overline{W}_2(a-b-l) \overline{W}_1(-l), \tag{48}$$

where the summation over  $k$  can be carried out resulting in the delta function  $N\delta_{l,c-b}$ . This then proves the following theorem:

**Theorem 3:** Every cyclic Saalschützian basic hypergeometric series  ${}_4\Phi_3$  is summable for  $z = \omega$ .

2.4.2. Star-Triangle Relation

Furthermore, (48) is also the star-triangle equation

$$\sum_{d=0}^{N-1} W_1(a-d)W_2(b-d)W_3(c-d) = R \overline{W}_3(a-b)\overline{W}_2(a-c)\overline{W}_1(b-c), \tag{49}$$

shown in Fig. 2. The weights  $W$  and  $\overline{W}$  are defined in (44), in which the parameters  $\alpha_i, \beta_i$  and  $\bar{\alpha}_i, \bar{\beta}_i$  are related by (35), while  $\gamma_i$  and  $\bar{\gamma}_i$  are related by (38) and (39). These relations are very symmetric.

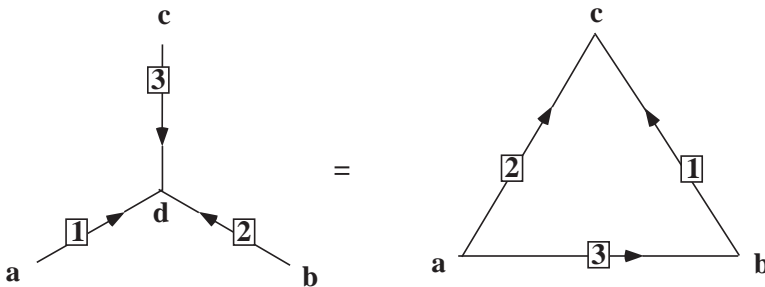


Fig. 2. Star-Triangle Relation, with  $\gamma_1 = \bar{\gamma}_2 \bar{\gamma}_3$ ,  $\gamma_2 = \bar{\gamma}_1 \bar{\gamma}_3^*$  and  $\gamma_3 = \bar{\gamma}_1^* \bar{\gamma}_2^*$ .

2.4.3. The Constant R

The constant  $R$  was originally given in Ref. 3. A proof was published in Ref. 16 and their proof can also be used here. By defining the matrices

$$(\mathcal{A}_2)_{b,d} = W_2(b-d), \quad (\mathcal{A}_3)_{c,d} = W_3(c-d), \quad (\mathcal{A}_1^a)_{d,d'} = \delta_{d,d'} W_1(a-d),$$

$$(\bar{\mathcal{A}}_1)_{b,c} = \bar{W}_1(b-c), \quad (\bar{\mathcal{A}}_2^a)_{c,c'} = \delta_{c,c'} \bar{W}_2(a-c), \quad (\bar{\mathcal{A}}_3^a)_{b,b'} = \delta_{b,b'} \bar{W}_3(a-b), \quad (50)$$

the star-triangle equation (49) may be expressed as

$$(\mathcal{A}_2 \mathcal{A}_1^a \mathcal{A}_3^T)_{b,c} = R (\bar{\mathcal{A}}_3^a \bar{\mathcal{A}}_1 \bar{\mathcal{A}}_2^a)_{b,c} \quad \text{or} \quad \mathcal{A}_2 \mathcal{A}_1^a \mathcal{A}_3^T = R \bar{\mathcal{A}}_3^a \bar{\mathcal{A}}_1 \bar{\mathcal{A}}_2^a. \quad (51)$$

The determinants are also equal, i.e.

$$R^N = \prod_{l=0}^{N-1} \frac{W_1(l)}{\bar{W}_2(l) \bar{W}_3(l)} \frac{\det \mathcal{A}_2 \det \mathcal{A}_3}{\det \bar{\mathcal{A}}_1} \quad (52)$$

This gives the constant  $R$  in terms of determinants of matrices  $\mathcal{A}$  defined in (50), which can be evaluated by Baxter’s formula (23). Alternatively,  $R$  is seen from (46) to be a product of seven  ${}_2\Phi_1$ . It can also be evaluated using (26), which is much more tedious, and after many cancellations, this yields the same result. We have thus avoided the complexity in the Riemann surface by relegating it to the multiplicative constant  $R$  in the star-triangle equation.

#### 2.4.4. Rapidity Lines

To form commuting transfer matrices, it is necessary to assign rapidity lines to the weights. There are two possible weights shown in Fig. 1. There are two essentially different choices for the directions of the arrows.

##### Original Choice

By assigning the rapidity lines as we originally did in Ref. 3, also shown in Fig. 3, then it is easily seen from Figs. 1 and 3 that

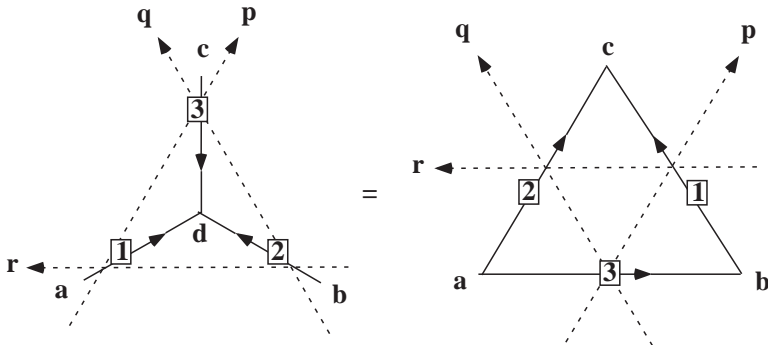


Fig. 3. Original choice of the rapidities.

$$\begin{cases} W_1(n) = W_{pr}(n), \\ \bar{W}_1(n) = \bar{W}_{pr}(n), \end{cases} \quad \begin{cases} W_2(n) = \bar{W}_{qr}(n), \\ \bar{W}_2(n) = W_{qr}(n), \end{cases} \quad \begin{cases} W_3(n) = \bar{W}_{pq}^*(n), \\ \bar{W}_3(n) = W_{pq}(n). \end{cases} \quad (53)$$

This means

$$\begin{cases} \alpha_1 = \alpha_{pr}, \\ \beta_1 = \beta_{pr}, \\ \gamma_1 = \gamma_{pr}, \end{cases} \quad \begin{cases} \alpha_2 = \bar{\alpha}_{qr}, \\ \beta_2 = \bar{\beta}_{qr}, \\ \gamma_2 = \bar{\gamma}_{qr}, \end{cases} \quad \begin{cases} \alpha_3 = \omega/\bar{\beta}_{pq}, \\ \beta_3 = \omega/\bar{\alpha}_{pq}, \\ \gamma_3 = \gamma_{pq}^*, \end{cases} \\ \begin{cases} \bar{\alpha}_1 = \bar{\alpha}_{pr}, \\ \bar{\beta}_1 = \bar{\beta}_{pr}, \\ \bar{\gamma}_1 = \bar{\gamma}_{pr}, \end{cases} \quad \begin{cases} \bar{\alpha}_2 = \alpha_{qr}, \\ \bar{\beta}_2 = \beta_{qr}, \\ \bar{\gamma}_2 = \gamma_{qr}, \end{cases} \quad \begin{cases} \bar{\alpha}_3 = \alpha_{pq}, \\ \bar{\beta}_3 = \beta_{pq}, \\ \bar{\gamma}_3 = \gamma_{pq}. \end{cases} \end{cases} \tag{54}$$

Consequently, we find from (35)

$$\begin{cases} \bar{\alpha}_{pr} = \bar{\alpha}_{qr}\bar{\alpha}_{pq}, \\ \bar{\beta}_{pr} = \bar{\beta}_{qr}\bar{\beta}_{pq}/\omega, \end{cases} \quad \begin{cases} \alpha_{qr} = \alpha_{pr}\bar{\alpha}_{pq}, \\ \beta_{qr} = \beta_{pr}\bar{\beta}_{pq}/\omega, \end{cases} \quad \begin{cases} \alpha_{pq} = \omega\alpha_{pr}/\bar{\beta}_{qr}, \\ \beta_{pq} = \beta_{pr}/\bar{\alpha}_{qr}. \end{cases} \tag{55}$$

From the relations for  $\bar{\alpha}_{pr}$  and  $\bar{\beta}_{pr}$ , we see that we would like to have the products  $\bar{\alpha}_{qr}\bar{\alpha}_{pq}$  and  $\bar{\beta}_{qr}\bar{\beta}_{pq}$  independent of  $q$ . For this to happen, we must have  $\bar{\alpha}_{qr}$  and  $\bar{\alpha}_{pq}$  containing the same  $q$ -dependent factor, say  $x_q$ , one in the denominator, the other in the numerator, such that the dependence on  $q$  cancels out upon multiplication. A similar reasoning holds for  $\bar{\beta}_{pq}$ . In fact, we find the only choices are

$$\bar{\alpha}_{pq} = x_q/x_p, \quad \bar{\beta}_{pq} = \omega y_p/y_q. \tag{56}$$

Using this in the second and third brackets of (55), we find

$$\alpha_{pq} = \omega x_p/y_q, \quad \beta_{pq} = \omega x_q/y_p. \tag{57}$$

It is easily verified that these choices satisfy the Saalschütz condition in (17). The periodicity requirement on the argument at  $z = \omega$

$$\omega = \gamma_1\gamma_2\gamma_3 = \prod_{j=1}^3 [(1 - \beta_j^N)/(1 - \alpha_j^N)]^{1/N} \tag{58}$$

can be satisfied, if

$$x_s^N + y_s^N = k(1 + x_s^N y_s^N), \quad s = p, q, r. \tag{59}$$

Solving this equation for  $x_s$  and substituting the solution into (37), we find

$$\gamma_{pq} = \mu_p y_q / \mu_q y_p \quad \bar{\gamma}_{pq} = \omega \mu_p x_p \mu_q / y_q, \quad \mu_s = (1 - k x_s^N) / k'. \tag{60}$$

This reproduces exactly the integrable solution found earlier.<sup>3</sup>

*Other Distinct Choice*

Only by flipping the directions of the middle rapidity line  $q$ , do we find a distinct arrangement of weights. This results in the equation

$$\sum_{d=0}^{N-1} W_{pr}(a-d)W_{rq}(b-d)W_{qp}(c-d) = R \bar{W}_{qp}(a-b)\bar{W}_{rq}(c-a)\bar{W}_{pr}(b-c). \tag{61}$$

Flipping other lines merely gives permutations of these rapidity lines in the two star-triangle equations, as can be seen from Fig. 1 and Fig. 4. To have the relation

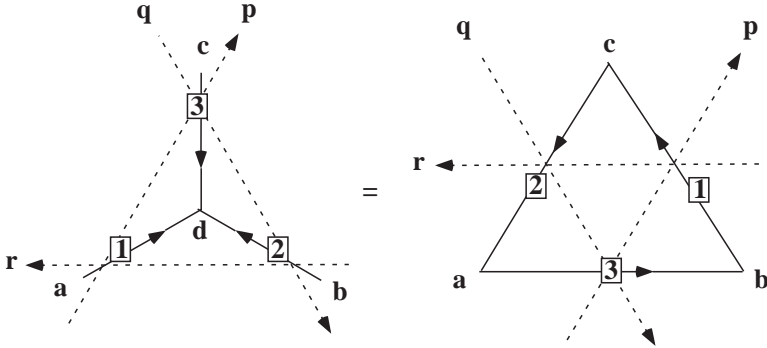


Fig. 4. Flipping the arrow of  $q$ .

(35) to hold, we must have

$$\bar{\alpha}_{pr} = \omega \alpha_{rq} / \beta_{qp}, \quad \bar{\beta}_{pr} = \beta_{rq} / \alpha_{qp}, \tag{62}$$

for which to be satisfied, we must choose

$$\alpha_{pr} = \rho f_p g_r, \quad \beta_{pr} = g_p f_r. \tag{63}$$

The Saalschütz condition in (17) yields  $\omega^2 \rho^3 = 1$ . It is then easy to verify that it is not possible to find  $\rho$ ,  $f_s$  and  $g_s$  satisfying condition (58). Since the relation (61) is more symmetric than (49) when comparing Fig. 3 with Fig. 4, the extra symmetry requirement on the weights makes a solution impossible in the present case.

### 3. The $N \rightarrow \infty$ Limits

#### 3.1. Star-Triangle Equation as a Double-Sided Hypergeometric Identity

In the limit  $N \rightarrow \infty$ , with  $\alpha_i = \omega^{a_i}$  and  $\beta_i = \omega^{b_i}$ , and allowing the spin  $n$  in (44) and thus the summation index  $d$  in (49) to run through all integers, we find<sup>12,13</sup> that if the Saalschützian condition

$$a_1 + a_2 + a_3 + 2 = b_1 + b_2 + b_3. \tag{64}$$

and condition resulting from (58)

$$\sin \pi a_1 \sin \pi a_2 \sin \pi a_3 = \sin \pi b_1 \sin \pi b_2 \sin \pi b_3, \tag{65}$$

are satisfied, the star-triangle equation (49) becomes

$$\sum_{n=-\infty}^{\infty} \frac{(a_1)_{m_1+n} (a_2)_{m_2+n} (a_3)_{m_3+n}}{(b_1)_{m_1+n} (b_2)_{m_2+n} (b_3)_{m_3+n}} = R_{\infty} \frac{(\bar{a}_1)_{m_1-m_2} (\bar{a}_2)_{m_2-m_3} (\bar{a}_3)_{m_1-m_3}}{(\bar{b}_1)_{m_1-m_2} (\bar{b}_2)_{m_2-m_3} (\bar{b}_3)_{m_1-m_3}}, \tag{66}$$

where

$$\begin{cases} \bar{a}_1 = 1 + a_2 - b_3, \\ \bar{b}_1 = b_2 - a_3, \end{cases} \quad \begin{cases} \bar{a}_2 = 1 + a_1 - b_3, \\ \bar{b}_2 = b_1 - a_3, \end{cases} \quad \begin{cases} \bar{a}_3 = 1 + a_1 - b_2, \\ \bar{b}_3 = b_1 - a_2. \end{cases} \tag{67}$$

### 3.1.1. Double-Sided Hypergeometric Identity

The equation (66) may be rewritten as the double-sided summation identity<sup>13</sup>

$$\sum_{n=-\infty}^{\infty} \prod_{i=1}^3 \frac{\Gamma(a_i + n)}{\Gamma(b_i + n)} = \frac{G(a_1, a_2, a_3 | b_1, b_2, b_3)}{\prod_{i=1}^3 \prod_{j=1}^3 \Gamma(b_i - a_j)}, \tag{68}$$

where

$$\begin{aligned} G(a_1, a_2, a_3 | b_1, b_2, b_3) &= \frac{\pi^5}{\sin \pi a_2 \sin \pi a_3 \prod_{i=1}^3 \sin \pi(b_i - a_1)} \\ &= \prod_{j=2}^3 \Gamma(a_j) \Gamma(1 - a_j) \prod_{i=1}^3 \Gamma(b_i - a_1) \Gamma(1 - b_i + a_1), \end{aligned} \tag{69}$$

provided the two conditions (64) and (65) are satisfied. If we let  $a_i \rightarrow a_i + m$  and  $b_j \rightarrow a_j + m$ , then these two conditions are still satisfied. This shows that the above two-sided identity holds for infinitely many different values of  $a_i$  and  $b_j$  and is rather unusual.

### 3.2. Its Dual (Fourier Transform)

If, instead of demanding that the spin values  $n$  in (44) and  $d$  in (49) remain integers, we let  $d, n, N \rightarrow \infty$ , while keeping the ratios  $y = 2\pi n/N$  and  $x = 2\pi d/N$  finite, we find that the summation over  $N$  values in (49) becomes an integral over the interval  $[0, 2\pi]$ . More specifically, we find the weight (18) to become<sup>13</sup>

$$W(a, b, x) = \left( \frac{\sin \pi b}{\sin \pi a} \right)^{\left( \frac{x}{2\pi} - \lfloor \frac{x}{2\pi} \rfloor \right)} \left| \sin \frac{1}{2} x \right|^{a-b}. \tag{70}$$

in this limit. For  $a_i$  and  $b_i$  satisfying the two conditions (64) and (65), we let

$$W_i(x) = W(a_i, b_i, x), \quad \overline{W}_i(x) = W(\bar{a}_i, \bar{b}_i, x), \tag{71}$$

and the star-triangle relation (49) becomes<sup>13</sup>

$$\begin{aligned} \frac{1}{2\pi} \int_0^{2\pi} dw W_1(x-w) W_2(y-w) W_3(z-w) \\ = R_\infty \overline{W}_3(x-y) \overline{W}_1(y-z) \overline{W}_2(x-z). \end{aligned} \tag{72}$$

Since the weights are chiral, namely,  $W(-x) \neq W(x)$ , it is not possible to have both the weights and their Fourier transforms real. Thus the Fourier transform of (72) is an identity similar to (66), but not identical, and vice versa.

### 3.3. Open Problems

Finally, the weights in Sections 3.1 and 3.2 define integrable models, which are limiting cases of the original chiral Potts model, and are chiral extensions of the models in the works of Fateev and Zamolodchikov.<sup>19,20</sup> Since there are sets of  $N$  functional relations for the chiral Potts models, we expect there may then be infinitely many such functional relations for these models, and perhaps some more physical quantities in these  $\infty$ -state models can be evaluated.

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