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Introduction

1.1 Classical mechanics is challenging again

In the latter half of the 19th century, physics was widely considered to be complete. The basic pillar of the physics was Newtonian Mechanics, and this was augmented by the theories of Lagrange, Hamilton and Jacobi. So at this time, increased focus was directed at the epistemology of physics, particularly mechanics. Ernst MACH (1838-1916), to mention just one prominent thinker, thoroughly expounded epistemological aspects of mechanics, in his book, "The Science of Mechanics – a Critical Account of its Development".

At the beginning of the 20th century, of all the major epistemological issues waiting to be either clarified or settled, two proved to be very fruitful for the future of physics. The first of these was the question of the existence of the ether. Despite its strange properties, the ether had to have in order to be compatible with several physical processes, the ether had until then been considered indispensable in electrodynamics, particularly for the propagation of the electromagnetic waves, discovered not so long ago by Heinrich Hertz. This ether happened to be a kind of manifestation of Newton's absolute space. The second issue was the controversy between advocates of an atomistic world view – who believed, for instance, in the kinetic theory of gases where typically statistical methods are used – and the supporters of a continuum picture of nature – as in the theories of elasticity and of electromagnetism, which both are typically formulated in terms of partial differential equations[1]. The question of the existence of the ether was settled by Albert Einstein in his special and general theories

[1] L. Boltzmann, Vorlesungen über die Prinzipe der Mechanik (Lectures on the principles of mechanics, see Bibliography).

of relativity (1905 and 1916), who was partly motivated by Mach's criticism of Newton's absolute space. The controversy between atomistic and continuum views was settled by quantum mechanics, the new mechanics originating in 1900 with M. Planck's explanation of the spectrum of the blackbody radiation.

But there was also a third issue at the turn of the century, one that was to have a major impact on physics only much later. The question of the stability of planetary motion was studied in particular by Henri POINCARÉ (1854-1912). In 1889, he submitted to the Swedish Academy a prize-winning paper on the stability of the motion of three gravitationally interacting bodies. In later work, he concluded that the stability question is of a fundamental nature. In his book, "Science and Method", he anticipated an essential characteristic of chaotic behavior: "small differences in the initial conditions" may "produce very great ones in the phenomena". A number of scientists continued investigating the idea without obtaining crucial results.

Half a century passed before the conception of classical mechanics changed fundamentally. This occurred with the appearance of chaotic behavior in deterministic equations such as those of Newton. It all started with E. Lorenz's observation in 1963 that a simplified atmospheric model – derived from hydrodynamic equations and consisting of three coupled nonlinear first-order ordinary differential equations – turned out to show quite different numerical solutions for extremely tiny changes in the initial conditions. This discovery precipitated a tremendous amount of research into dynamical systems worldwide. We mention only two research fields directly related to classical mechanics: chaotic behavior in a particular nonlinear two-dimensional Hamiltonian system (M. Hénon and C. Heiles (1964), F.G. Gustavson (1966)); and chaos in the restricted 3 body-systems with gravitational interactions (M. Hénon (1966)). We discuss both these systems in detail later. At the same time, the celebrated Kolmogorov-Arnold-Moser (KAM) theorem (1954-63), solved Poincaré's convergence problem in the power series treatment of dynamical systems that are not exactly solvable.

As a result of this discovery, the generally accepted presumption of **predictability of mechanical systems** was overturned. At least since Pierre Simon LAPLACE (1749-1827) expressed that predictability in 1814, it had been universally believed that given all the initial conditions, and given sufficiently powerful calculational tools, one could predict the future state of any classical system. This is still *true in principle*, but the fact remains that due to extreme sensitivity to changes in the initial conditions, it is *practically not possible* to predict the future for many systems, since the initial values are always only known to a certain accuracy.

The new picture also has implications for other fields of physics. For example, in statistical physics, it sheds new light on ergodic theory, and allows a new understanding of the arrow of time – that is, the apparent irreversible direction of time in the macroscopic world, despite the microscopic time reversibility of the fundamental laws.

Chaotic behavior is ubiquitous even in rather simple mechanical systems. Most textbooks on classical mechanics only consider the small minority of so-called integrable systems. In the light of the 'chaos revolution', this cannot be considered adequate in a modern approach. Therefore, we introduce concepts and tools necessary to understand integrability and chaotic behavior quite early in the treatment, along with examples of chaotic systems. Presenting this modern view of classical mechanics is our chief goal.

However, even today, what the exact implications of chaos are is not completely settled – the situation remains controversial. Even though, as just said, chaotic behavior is ubiquitous in classical model systems, apart from the unreliability of weather forecasts, it is hard to point out consequences in everyday experience. This has several causes. One of these is touched on at the end of this book. Regarding the others, we mention only that in real systems there exist much more and even different degrees of freedom than in the simplifying examples of this book. The opportunity for energy flow into such 'additional' degrees of freedom may change the behavior of that part of the system that when isolated, manifests chaotic behavior.

1.2 On the scientific method

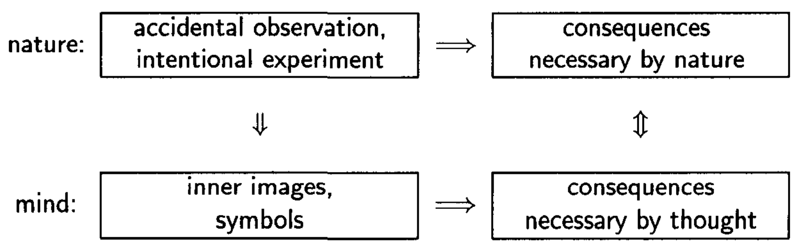
The laws of physics inevitably have a mathematical form, since physics aims to be quantitative, and often even precise. But behind the mathematical form stand concepts and reasoning. A (theoretical) physicist's everyday routine is mainly concerned with mathematics. But particularly when new theories are formed or old ones are scrutinized, it is the concepts and their understanding that is important. What is the relation between physical reality and its mathematical image?

Everyone who interacts with their environment actively – i.e. who is observing, conceptualizing or even changing the nature surrounding them – develops an imagination that goes beyond simple cognition of their environment. They construct pictures of nature – for example, suppositions are made about connections between processes; conclusions are drawn about influences between processes. A picture of the world emerges that often also includes metaphysical currents, perhaps even religious ones. Physics is restricted to the rational part of the world view. We quote below views held by outstanding physicists on the relation between exterior world and the individual's knowledge of the world. Presumably, most physicists share these views. The selection is far from complete: it can only touch on the epistemology of mechanics (and physics). Our selection attempts to illustrate the philosophy basic to this book.

In the following, we refer to introductions of two important books on the principles of mechanics – written by Heinrich HERTZ (1857-1894) and Ludwig BOLTZMANN (1844-1906) at around the turn of the 20th century (see Bibliography). Both these physicists are certainly among the most prominent physicists of that era.

In his book on mechanics, Hertz starts with a model of the epistemological process in physics[2]:

> *"As a basis ... we make use of ... preceding experiences, obtained by an accidental observation or by an intentional experiment. The method, however, that we constantly use to deduce the future from the past, thereby obtaining the desired foresight, is this: we attribute inner images or symbols to the exterior objects; and this we do in such a manner that the consequences of the images necessary by thought are just the images of the consequences necessary by nature of the objects represented. In order that this demand can be fulfilled, certain reconciliations (accords) between nature and our mind have to exist. Experience teaches us that this demand can be fulfilled and that hence such accords exist."*



Hertz's scheme of the process of gaining knowledge

In his book on the principles of mechanics, Boltzmann refers to Hertz's view, and presents his own.

[2]In the following, all citations that do not refer to an English edition were translated by the present author.

> *"It has hardly ever been doubted ..., that our thoughts are mere images of the objects (rather symbols for these) having at most a certain kinship to them, but can never coincide with them and can be related only to them like the letters to the sound or the note to the tone. Because of the limitedness of our intellect, they can only reflect a small part of the objects.*
>
> *We may now proceed in two ways: 1. We may leave the images more general. Then there is less risk that they later turn out to be false, since they are more adaptable to the newly discovered facts; yet because of their generality the images become indeterminate and pale and their further development will be connected with a certain uncertainty and ambiguity. 2. We particularize the images and finish them in detail to a certain degree. Then we have to add much more arbitrariness (hypothetical), thus perhaps not fitting to new experiences; however, we have the advantage, that the images are as clear and distinct as possible and we can deduce from them all consequences with complete certainty and uniqueness."*

Furthermore, Boltzmann points out that the requirement *"to register only the directly given phenomena and not to add something arbitrary"*, to omit all hypothetical, does not lead beyond *"simply marking every phenomenon"*.

So, when constructing images, symbols, and models, a certain freedom in the choice of hypotheses exists. If the results of distinct hypotheses do not differ, then criteria from outside physical reasoning are used for the selection. Something like aesthetics of explanations, models, and theories also exists. A very important criterion, according to Mach, is the "economy of thought" in theories, in particular, that the number of suppositions should be minimal. Even Isaac NEWTON (1642-1727), in his book "Philosophiae naturalis principia mathematica" ("Principia" for short), advocated simplicity of explanation. The first two of Newton's rules are[3] (The Principia, Book III, Rules of Reasoning in Philosophy):

Rule I:

> *We are to admit no more causes of natural things than such as are both true and sufficient to explain their appearances.*

[3] In the present book, all the citations from Newtons "Principia" are taken from the 1848 translation from Latin, "The Principia", by A. Motte (cf. Bibliography). Occasionally we use more modern words here.

Rule II:

> *Therefore to the same natural effects we must, as far as possible, assign the same causes.*

The hypotheses and postulates resulting from creation of the images of nature are refined and made more precise in a dialectic way. Progress in physics results from an interplay between gaining experience with the process considered (increasingly detailed experience in the sense of Hertz) and improving the model (image) of the process accordingly by comparing the consequences with the increasing experience of physical reality. If the model's predictions (the consequences by thought) do not fit well enough, then one has to check the experience and the modelling. Thus, although originally imprecise, ordinary concepts are refined, their meaning becomes more exact. (For example, the concepts momentum, force and energy of a body are easily distinguished by a physicist – but in common usage, these three concepts are almost always confounded).

With regards to observations and experiments, it was Galileo GALILEI (1564-1642) who initiated the scientific method through his demand to *measure what is measurable and to render measurable what is not measurable.* In Galileo's time, systematic observation was already established in astronomy, as was the use of mathematics to represent and summarize the results; but observation of the motion of bodies under controlled and controllable conditions, and the mathematical representation of physical process such as those in Galileo's experiments (cf. Section 3.3), was completely new.

In the development of physical models and theories, one may distinguish the following stages. For a specific process, the first stage is to find out which properties and facts are relevant and which are irrelevant. For example, for a freely falling apple, its color does not affect the time the apple takes to hit the ground. Details found (or assumed) to be irrelevant, are, if necessary, eliminated or their importance diminished in the experiment as well as in the conceptual image of the process, thus arriving at a model process. In this way, the set of relevant physical quantities is worked out. At this point – the second stage – it is fundamental for progress in physics that mathematical images be attached to these quantities, that means, a quantity can have certain values represented by one or several real numbers. Thus the quantities – and consequently the repetitions of the process – become comparable, and therefore they become measurable. The central aim of modelling of a process is – in the third stage – to find the mathematical laws that connect the mathematical images selected. If such laws have been found, one has to compare the solutions of the equations (their 'consequences necessary by thought') with the results of the process in nature (the 'consequences necessary by nature'). This comparison becomes the input for a new, refined study of the process. In the course of refining the conceptualization of the process (e.g. in terms of a model), the stages just mentioned are repeated in the light of the newly emerged picture. For instance, regarding again the free fall experiment, the refined investigation may proceed under the assumption that the air resistance is not important to the 'basic' laws. Hence, the air resistance will be reduced, or even removed, by using an appropriate experimental setup. As knowledge about a given process increases, connections may be spotted with other processes. In this way, one arrives at a theory encompassing a variety of processes. For example, Newton's falling apple and the motion of the moon around the earth both lead to the theory of gravity.

The first to formulate the symbols and images of classical mechanics was Isaac Newton. In this book, we will start from his laws and dwell mainly on the elaboration of the (mathematical) consequences that, according to the philosophical sketch just presented, should fit the experimental observation - the consequences in nature.

1.3 Time, space and motion

The meaning of the basic concepts time, space and motion is hard to outline exactly. Philosophers, as well as physicists, have devoted studies to these concepts[4]. In physics, these concepts have a more precise usage – and are also more restricted in content (see Boltzmann's view above) – than in everyday life. So the physical concepts of *time* and *space* have different and fewer characteristics. There are no 'good' or 'bad' times. And physical space is not attributed with 'wideness', as in common speech[5]. The central properties of time and space in physics are to be measurable and mappable onto mathematical entities. Although these quantities are prerequisites to describe physical processes, their conceptualization may change as physics progresses. At the end of the 19th century, concepts of both time and space seemed to be well established – and were revised by Einstein's theories of relativity.

Below, we present Newton's concepts – essentially, the underlying concepts of classical mechanics – expounded in his "Principia", and confront them with Mach's critique in "The Science of Mechanics" (see Bibliography).

[4]An extensive treatment of the history of space is given by M. Jammer's "Concepts of Space" (cf. Bibliography).

[5]Unfortunately, physicists, too, keep talking about 'empty' space.

Newton on **time** (The Principia, Book I, Definitions, Scholium):

> *Absolute, true, and mathematical time, of itself and from its own nature, flows equably without relation to anything external, and by another name is called "duration"; relative, apparent, and common time is some sensible and external (whether accurate or unequable) measure of duration by the means of motion, which is commonly used instead of true time, such as an hour, a day, a month, a year.*

These sentences may make sense intuitively; however, they seem to be inconsequential if one takes into account Newton's intention *"to investigate only actual facts"*. Presumably, the statements reflect Newton's contemporary philosophical and ideological background. Mach[6] argues, that Newton's concept of time is an abstract (useless) concept:

> *"This absolute time can be measured by comparison with no motion; it has therefore neither a practical nor a scientific value; and no one is justified in saying that he knows aught about it. It is an idle metaphysical conception."*

Newton on **space** (ibid.):

> *Absolute space in its own nature, without relation to anything external, remains always similar and immovable. Relative space is some movable dimension or measure of the absolute spaces; which our senses determine by its position to bodies; and which is vulgarly taken for immovable space; ...*

And on **motion**, which connects space and time, Newton wrote (ibid.):

> *Absolute motion is the translation of a body from one absolute place into another; and relative motion, the translation from one relative place into another.*

The objections raised by Mach to absolute space:

> *No one can say anything about absolute space and absolute motion, they are just objects in thoughts that cannot be demonstrated in practice. As has been pointed out in detail, all our principles of mechanics are experiences about relative positions and motions of bodies.*

[6]Mach took the view that science should not ask "why?" but "how?" (R. Carnap, An Introduction to the Philosophy of Science, Dover Publications, New York 1995).

And as to uniform motion:

> *Motion can be uniform with respect to other motion. The question of whether a motion as such is uniform has no meaning at all...*

Also, Mach did not consider *Newton's pail experiment* as compelling proof of absolute space: a bucket filled with water is spun, then suddenly stopped; before being spun, the water surface is flat, but as being spun, the surface forms a concave shape that remains for some time after stopping. According to Mach, the curvature of the surface of the rotating water only indicates motion relative to the fixed stars, and not relative to a fictitious absolute space[7]. The a priori existence of space and time is a feature of Newtonian mechanics; in the general theory of relativity, space and time are 'produced' by the masses. Although an absolute conception of space and time is certainly the background for Newton's laws, these laws can be applied to physical situations without referring to that background[8].

In classical physics, one describes the behavior of (the images of the) objects in a 'space', and the change of their positions in that space with 'time': **Space** and **time** or, to be more precise, their images are the very basic concepts of physics. **Motion** connects space and time. The epistemological background of the concepts of space and time is certainly interesting; nevertheless, it is the mapping of space and time onto mathematical entities that is essential for theoretical mechanics. In our experience, we know, for example, that if we agree on some point of reference $\mathbf{O}$ in space, and an initial instant of time t_0, then every point $\mathbf{r}$ in space is uniquely labelled by three real numbers, while one real number suffices to label time. Furthermore, our experience is reflected in the points in space forming a vector space $\mathbb{R}^3 = \{\mathbf{r} = (x, y, z) \mid x, y, z \text{ real}\}$, where usually one sets $\mathbf{O} = (0, 0, 0)$; and instances in time are simply represented by the real axis ($\mathbb{R}^1 = \{t \mid t \text{ real}\}$) with $t_0 = 0$. Two more characteristics of physical space that are to be reflected in the mathematical picture are: the distance between two points, and the angle between two straight lines[9]. A mathematical object suitable for representing physical space in classical mechanics is the

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[7] This controversy was resolved in favor of Mach by the general theory of relativity.

[8] The situation is somewhat similar to quantum mechanics, where the interpretation seem to be not important in practice.

[9] We do not want to discuss the mathematical meaning of 'straight'. In physical reality, in the context of classical mechanics, light rays between the objects may serve to define straight lines in space ('in vacuo').

Euclidean vector space in 3 dimensions $\mathbb{E}^3$.

The mathematical properties of $\mathbb{E}^3$ are summarized in Appendix A. A point $\mathbf{r} \in \mathbb{R}^3$ is represented in the Euclidean space $\mathbb{E}^3$ by the *radius vector* $\mathbf{r}$, whose components are the *Cartesian coordinates* of that point. This Euclidean space is the mathematical image of the physical *configuration space.* If the motion of a body is restricted, a one or two dimensional space (motion along a line or on a surface) may suffice. On the other hand, it will turn out to be advantageous to extend the set of coordinates (to obtain, say, a 'phase space'), and hence to use a higher dimensional space.

For a complete representation of the motion of a body in classical mechanics, one needs to know its position for any time. To measure the time dependence, one compares the motion at various stages with a time standard, which is usually – by definition – a *periodic motion* (e.g. the motion of a pendulum or oscillations in an atomic clock). The mathematical picture of particle or body motion is the function $\mathbf{r}(t)$ for object's position. For an extended object, in many cases it is sufficient to consider a representative (mathematical) point of the body. Of particular importance in classical mechanics are the first and second time derivatives of the radius vector: the rate of change of the radius vector $\mathbf{r}(t)$ with time[10],

$$\text{the } \textit{velocity } \mathbf{v} = \frac{d\mathbf{r}}{dt} = \dot{\mathbf{r}},$$

and the rate of change of the velocity vector with time,

$$\text{the } \textit{acceleration } \dot{\mathbf{v}} = \frac{d^2\mathbf{r}}{dt^2} = \ddot{\mathbf{r}}.$$

The fundamental laws of classical mechanics are formulated in terms of the variables $\mathbf{r}(t)$, $\dot{\mathbf{r}}(t)$, and $\ddot{\mathbf{r}}(t)$.

The stage is now set with time and space – so raise the curtain, and let physical systems play their part!

[10]Denoting the time derivative by a dot is due to Newton.