Chapter 1

Spinor geometry

It has been argued that the most natural way of discussing *spinors* is by means of the *theory of representations of groups* (see, for example, Cartan, 1966; Carmeli and Malin, 2000). Indeed, most of the literature currently available on spinors follows this precedent and develops the theory exceedingly well. However, we are interested in spinors applied to general relativity theory (a geometrical theory) and, in this sense, a geometrical approach to studying spinors will be the most appropriate one.

1.1 Minkowski space

We define *Minkowski space* $\mathbb{M}$ to be a four-dimensional vector space over the real number field $\mathbb{R}$ with a flat *Lorentzian metric* $\eta_{\mathbf{ab}}$ given by

$$\eta_{\mathbf{ab}} = diag(1, -1, -1, -1). \tag{1.1}$$

(Note that boldface indices will always represent numerical values: lower-case Latin indices will range over $0, 1, 2, 3$, and upper-case Latin indices will range over $0, 1$.) At each point on $\mathbb{R}^4$ there exists a set of *basis* vectors $\mathbf{e}_0, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \in \mathbb{M}$ called a *tetrad*, which define uniquely any $\mathbf{U} \in \mathbb{M}$ by

$$\mathbf{U} = U^0\mathbf{e}_0 + U^1\mathbf{e}_1 + U^2\mathbf{e}_2 + U^3\mathbf{e}_3 \tag{1.2}$$

for $U^0, U^1, U^2, U^3 \in \mathbb{R}$ not all zero unless $\mathbf{U} = \mathbf{0}$. We can write (1.2) more concisely as

$$\mathbf{U} = U^{\mathbf{a}}\mathbf{e}_{\mathbf{a}} \tag{1.3}$$

where it is understood that from now, and henceforth, the *Einstein summation convention* will be adopted whenever indices are repeated as in (1.3).

An *inner product (scalar product)* on $\mathbb{M}$ is a mapping $\mathbb{M} \times \mathbb{M} \to \mathbb{R}$. Thus for any $\mathbf{U}$, $\mathbf{V}$, $\mathbf{W}$, $\in \mathbb{M}$ and $a, b \in \mathbb{R}$ we have

$$\mathbf{U} \cdot \mathbf{V} = \mathbf{V} \cdot \mathbf{U} \tag{1.4}$$

$$\mathbf{U} \cdot (a\mathbf{V} + b\mathbf{W}) = a\mathbf{U} \cdot \mathbf{V} + b\mathbf{U} \cdot \mathbf{W} \tag{1.5}$$

$$\mathbf{U} \cdot \mathbf{V} = 0 \quad \forall \mathbf{V} \in \mathbb{M} \Leftrightarrow \mathbf{U} = 0. \tag{1.6}$$

Hence the inner product is *symmetric* by (1.4) and *bilinear* from (1.4) and (1.5). Equation (1.6) implies that the inner product is also *non-degenerate*, and we say that $\mathbb{M}$ is *non-singular*. The *orthonormalisation* conditions on the tetrad $\mathbf{e_a}$ are

$$\mathbf{e_a} \cdot \mathbf{e_b} = \begin{cases} 1 & \text{if } \mathbf{a} = \mathbf{b} = 0 \\ -1 & \text{if } \mathbf{a} = \mathbf{b},\ \mathbf{a} = 1, 2, 3 \\ 0 & \text{if } \mathbf{a} \neq \mathbf{b} \end{cases} \tag{1.7}$$

which can be written compactly as

$$\mathbf{e_a} \cdot \mathbf{e_b} = \eta_{\mathbf{ab}} \tag{1.8}$$

where $\eta_{\mathbf{ab}} = \eta^{\mathbf{ab}}$ is given by (1.1).

We shall adopt the convention of referring to a tetrad $\mathbf{e_a}$ which satisfies (1.8) as a *Minkowski tetrad*, and coordinates $U^{\mathbf{a}} \in \mathbb{R}$ representing any $\mathbf{U} \in \mathbb{M}$ will be called *Minkowski coordinates*.

A vector $\mathbf{U} \in \mathbb{M}$ is called

$$\left.\begin{array}{ll} \text{timelike} & \text{if } \mathbf{U} \cdot \mathbf{U} > 0 \\ \text{null} & \text{if } \mathbf{U} \cdot \mathbf{U} = 0 \\ \text{spacelike} & \text{if } \mathbf{U} \cdot \mathbf{U} < 0. \end{array}\right\} \tag{1.9}$$

In Minkowski coordinates $\mathbf{U}$ is timelike or null if

$$\mathbf{U} \cdot \mathbf{U} = (U^0)^2 - (U^1)^2 - (U^2)^2 - (U^3)^2 \geq 0. \tag{1.10}$$

It can be easily shown that a timelike vector cannot be orthogonal to a null vector (see Exercise 1.1) or, indeed, to another timelike vector. Because of this, timelike vectors and null vectors can be separated into two classes i.e. *future-pointing* and *past-pointing*, then $\mathbb{M}$ is said to be *time-orientated*. We call the Minkowski tetrad $\mathbf{e_a}$ *orthochronous* if $\mathbf{e}_0$ is a future-pointing timelike vector, or more succinctly, a *future-timelike vector*.

Consider two Minkowski tetrads $\mathbf{e_a}$ and $\mathbf{e_{a'}}$. Each can be related to the other by a general linear combination of vectors written compactly as

$$\mathbf{e_a} = \Lambda_{\mathbf{a}}{}^{\mathbf{a}'}\mathbf{e_{a'}} \tag{1.11}$$

where $(\Lambda_{\mathbf{a}}{}^{\mathbf{a}'})$ is a transformation matrix that is real and non-singular. Consequently, the determinant of this matrix is non-zero. If det $(\Lambda_{\mathbf{a}}{}^{\mathbf{a}'}) > 0$ then $\mathbf{e_a}$ and $\mathbf{e_{a'}}$ are defined to be *equally orientated*, and thus the tetrads are *proper*. If det $(\Lambda_{\mathbf{a}}{}^{\mathbf{a}'}) < 0$ then $\mathbf{e_a}$ and $\mathbf{e_{a'}}$ are defined to be *unequally orientated*, and the tetrads are *improper*. $\mathbb{M}$ is said to be *orientated* depending on whether $\mathbf{e_a}$ is proper or improper.

It will be assumed throughout that our orthonormal basis will be *right-handed*, that is, $\mathbf{e_a}$ is proper in addition to being orthochronous — otherwise $\mathbf{e_a}$ would be deemed to be *left-handed*. These two requirements, taken together, also form what is called a *restricted Minkowski tetrad*. (Note that if $\mathbf{e_a}$ was neither proper nor orthochronous then our basis would still be right-handed, but it would no longer be restricted.)

We define a *Lorentz transformation* to be *restricted* if an *active Lorentz transformation* is both orientated and time-orientated on $\mathbb{M}$. The adjective 'active' refers to a linear transformation of $\mathbb{M}$ such that the first equality of (1.10) is presumed. A *passive Lorentz transformation* is a map

$$P : U^{\mathbf{a}} \mapsto U^{\mathbf{a}'} \tag{1.12}$$

where $U^{\mathbf{a}} \in \mathbb{R}$ are Minkowski coordinates defined by some Minkowski tetrad $\mathbf{e_a} \in \mathbb{M}$. If the tetrads that form (1.12) are themselves restricted then (1.12) is *restricted*. If the two tetrads in question are given by (1.11) we can write (1.12) as

$$U^{\mathbf{a}'} = U^{\mathbf{a}}\Lambda_{\mathbf{a}}{}^{\mathbf{a}'}. \tag{1.13}$$

Although active Lorentz transformations are coordinate independent it is still permissible to discuss them in terms of coordinates. To do this, we define an active Lorentz transformation as a map

$$A : U^{\mathbf{a}} \mapsto V^{\mathbf{a}'}. \tag{1.14}$$

If also $\mathbf{e_a} \mapsto \mathbf{e_{a'}} \in \mathbb{M}$ then we can write $U^{\mathbf{a}} = V^{\mathbf{a}'}$. Thus the passive transformation generated by $\mathbf{e_{a'}} \mapsto \mathbf{e_a}$ sends $U^{\mathbf{a}}$ to $V^{\mathbf{a}}$. With the aid of (1.13) we arrive at

$$U^{\mathbf{a}'} = V^{\mathbf{b}'}\Lambda_{\mathbf{b}}{}^{\mathbf{a}'} \tag{1.15}$$

where in this particular case, summation is over $\mathbf{b}'$ and $\mathbf{b}$. Thus the active Lorentz transformation is given by

$$V^{\mathbf{b}'} = U^{\mathbf{a}'} A_{\mathbf{a}'}{}^{\mathbf{b}} \tag{1.16}$$

where the matrix

$$(A_{\mathbf{a}'}{}^{\mathbf{b}}) = (\Lambda_{\mathbf{b}}{}^{\mathbf{a}'})^{-1}. \tag{1.17}$$

1.2 The null cone and Riemann sphere

Our aim is to establish a coordinate representation of what will come to be known as a *spin-vector*; that is, the object which is considered to be the simplest of the generic class of spinors.

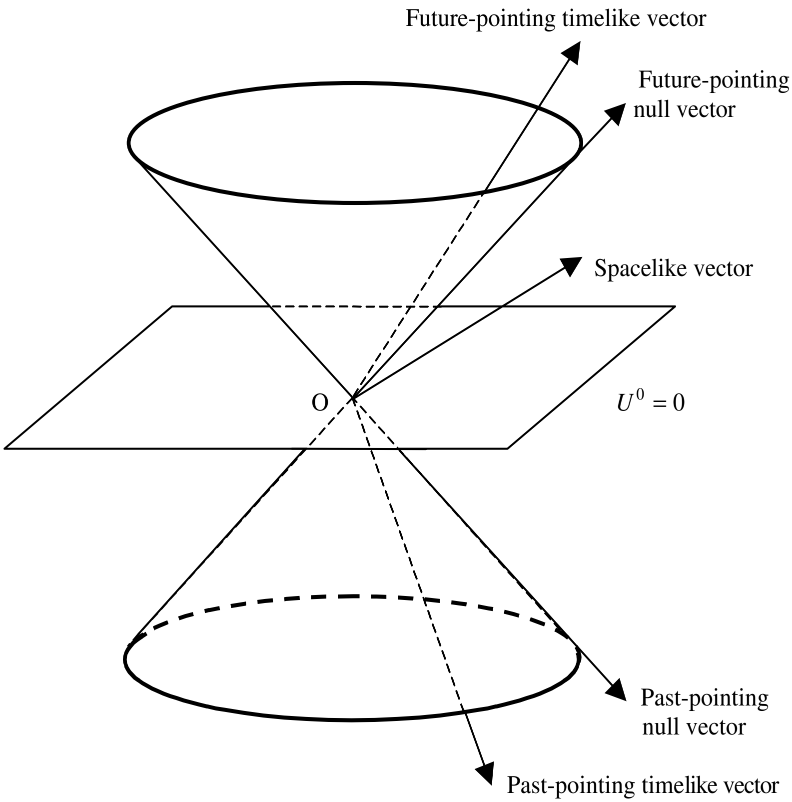

Fig. 1.1 The null cone

The set of all null vectors forms a null cone (see Fig. 1.1) with coordinates represented by

$$(U^0)^2 - (U^1)^2 - (U^2)^2 - (U^3)^2 = 0. \tag{1.18}$$

For any timelike or null vector $\mathbf{U} \in \mathbb{M}$ given by (1.2) we associate two distinct and opposite *directions* relative to the origin O; namely, *future-timelike/null directions* and *past-timelike/null directions.* This will be elucidated upon presently.

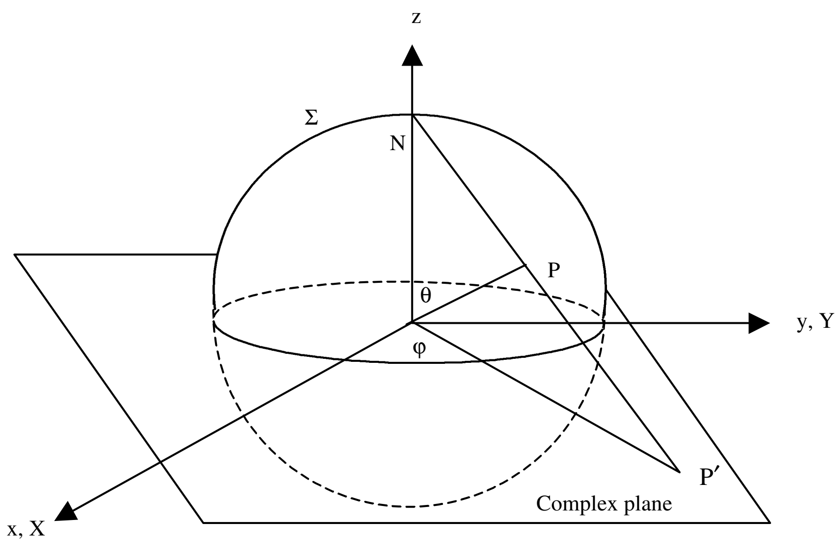

Fig. 1.2 Stereographic projection of the Riemann sphere

Consider the null cone as consisting of two distinct, but connected, halves separated by the hyperplane $U^0 = 0$, the upper part being the *future null cone* and the lower half being the *past null cone*. Furthermore, we can associate with each half of the null cone an abstract space with components that are either future null directions or past null directions. If we now form an intersection with, say, the past null cone and some hyperplane $T =$ constant then topologically, this would represent a sphere in $\mathbb{R}^3$. Indeed, in the space-time of special relativity, the observer at the origin would consider this sphere as his entire field of vision, i.e. his *celestial sphere*. Hence null rays would constitute points on the celestial sphere. Points on the hyperplane T = constant define the direction of any vector given by (1.2), provided $T \neq 0$. Timelike directions are given by points inside the sphere and spacelike directions are given by points outside.

Consider now the *anti-celestial sphere*, i.e. the abstract space representing a sphere in $\mathbb{R}^3$ when the future null cone is intersected by the hyperplane $T = 1$, for example. Mathematically, we might think of this as the *Riemann sphere*, Σ, given by the equation

$$x^2 + y^2 + z^2 = 1. \tag{1.19}$$

By means of *stereographic projection* a one-to-one correspondence can be set up between the *extended complex plane* $\tilde{\mathbb{C}}$ and Σ (see Fig. 1.2). The coordinates x, y, z on Σ can then be replaced with a single complex parameter on the complex plane $\mathbb{C}$. Clearly $\tilde{\mathbb{C}}$ is formed by adding to $\mathbb{C}$ an extra 'point' ∞ — which represents the 'north pole' of Σ.

The geometrical construction is as follows. $\mathbb{C}$ intersects Σ at the 'equator' $z = 0$ where we think of $\mathbb{C}$ as being embedded in Euclidean space $\mathbb{R}^3$, $T = 1$. Each point on Σ is stereographically projected, or mapped, from the north pole, N, with coordinates $(1, 0, 0, 1)$, to $\mathbb{C}$. Thus the mapping carries the point P, with coordinates $(1, x, y, z)$, to a point P' with coordinates $(1, X, Y, 0)$. Each point on $\mathbb{C}$ can then be identified by a single complex number, sometimes called a *stereographic coordinate*

$$\xi = X + iY. \tag{1.20}$$

Using simple trigonometry we can see that

$$\frac{NP}{NP'} = \frac{x}{X} = \frac{y}{Y} = 1 - z. \tag{1.21}$$

Comparing (1.21) with (1.20) implies that

$$\xi = \frac{x + iy}{1 - z}. \tag{1.22}$$

In spherical polar coordinates (θ, ϕ) equation (1.22) can be written as

$$\xi = e^{i\phi} \cos \frac{\theta}{2}, \tag{1.23}$$

which is achieved by applying the standard equations

$$x = \sin\theta \cos\phi, \quad y = \sin\theta \sin\phi, \quad z = \cos\theta \tag{1.24}$$

to (1.22). The inverse relations with respect to (1.22) are directly obtainable with the aid of (1.19). Thence,

$$x = \frac{\xi + \overline{\xi}}{\xi\overline{\xi} + 1}, \quad y = \frac{i(\overline{\xi} - \xi)}{\xi\overline{\xi} + 1}, \quad z = \frac{\xi\overline{\xi} - 1}{\xi\overline{\xi} + 1}. \tag{1.25}$$

1.3 Spin transformations and spin matrices

It will be preferable to represent the single complex parameter, ξ, by a pair of complex components

$$\xi = \frac{\zeta}{\eta}. \tag{1.26}$$

The reason for doing this is so as to circumvent the problem of using an infinite coordinate to represent the north pole on the Riemann sphere. (It will be shown later that the pair (ζ, η) can be treated as components of a spin-vector.) With the complex components (1.26), (1.25) can be rewritten as

$$x = \frac{\zeta\overline{\eta} + \overline{\zeta}\eta}{\zeta\overline{\zeta} + \eta\overline{\eta}}, \quad y = \frac{i(\overline{\zeta}\eta - \zeta\overline{\eta})}{\zeta\overline{\zeta} + \eta\overline{\eta}}, \quad z = \frac{\zeta\overline{\zeta} - \eta\overline{\eta}}{\zeta\overline{\zeta} + \eta\overline{\eta}}. \tag{1.27}$$

If (1.27) is multiplied by $\zeta\overline{\zeta} + \eta\overline{\eta}$ — the denominator of (1.27) — then any point on the null cone with coordinates (T, X, Y, Z) can be represented in terms of the complex pair (ζ, η) by

$$\begin{aligned} T &= \frac{1}{\sqrt{2}}(\zeta\overline{\zeta} + \eta\overline{\eta}), \quad X = \frac{1}{\sqrt{2}}(\zeta\overline{\eta} + \eta\overline{\zeta}) \\ Y &= \frac{i}{\sqrt{2}}(\overline{\zeta}\eta - \eta\overline{\zeta}), \quad Z = \frac{1}{\sqrt{2}}(\zeta\overline{\zeta} - \eta\overline{\eta}). \end{aligned} \tag{1.28}$$

Notice that the factor $\frac{1}{\sqrt{2}}$ has been introduced so as to be consistent with analogous expressions to be given in the next chapter.

Let

$$\begin{aligned} \zeta &\mapsto \hat{\zeta} = a\zeta + b\eta \\ \eta &\mapsto \hat{\eta} = c\zeta + d\eta \end{aligned} \tag{1.29}$$

be a general complex linear transformation of components ζ and η, where $a, b, c, d \in \mathbb{C}$ and $ad - bc \neq 0$. Hence a conformal transformation of the Riemann sphere must be a globally defined holomorphic transformation

$$\xi \mapsto \hat{\xi} = \frac{a\xi + b}{c\xi + d}. \tag{1.30}$$

Without loss of generality we may take $ad - bc = 1$, then this condition together with (1.30) (or independently (1.29)), are referred to as *spin transformations.*

The non-singular matrix $\mathbf{S}$ defined by

$$\mathbf{S} = \begin{pmatrix} a\,b \\ c\,d \end{pmatrix}, \quad ad - bc = 1 \tag{1.31}$$

is called the *spin matrix.* Thus (1.29) can be rewritten as

$$\begin{pmatrix} \hat{\zeta} \\ \hat{\eta} \end{pmatrix} = \mathbf{S} \begin{pmatrix} \zeta \\ \eta \end{pmatrix}. \tag{1.32}$$

Clearly, a double application of (1.32) yields yet another spin transformation. $\mathbf{S}$ being non-singular ensures that $\mathbf{S}^{-1}$ exists. And a group is indicated; this group is called the *special linear group*, commonly referred to as $SL(2, \mathbb{C})$.

1.4 Flagpoles and flag planes

Let us suppose that (1.28) is the coordinate representation of a future-pointing null vector $\mathbf{A}$ defined by the directed line segment $\overrightarrow{OQ}$ on the future null cone (see Fig. 1.3). In this context $\mathbf{A}$ is referred to as a 'flagpole'. On the Riemann sphere itself $\mathbf{A}$ has coordinates (ζ, η). Clearly, if we refer to (1.28), $\mathbf{A}$ remains invariant under a *transformation of phase* $\zeta \mapsto e^{i\theta}\zeta$, $\eta \mapsto e^{i\theta}\eta$, and consequently provided θ is real the same null vector is defined.

If we now introduce a real spacelike vector $\mathbf{B}$, which is a tangent vector to the Riemann sphere and orthogonal to $\mathbf{A}$, then each positive multiple of $\mathbf{B}$ lies in a two-plane $s\mathbf{A} + t\mathbf{B}, (s, t \in \mathbb{R})$. This plane is tangent to the future null cone because $\mathbf{A} \cdot \mathbf{B} = 0$.

In actuality, $s\mathbf{A} + t\mathbf{B}$ represents a *half-plane* for $t > 0$. This is required if we wish to discuss the 'orientation' of $\mathbf{B}$ (not to be confused with our previous use of orientation). This half-plane is our 'flag plane'. Combining the flagpole and flag plane forms a structure known as a *null flag.*

Consider the spin transformation (1.32) with $\mathbf{S} = diag(e^{i\theta}, e^{i\theta})$:

$$\begin{pmatrix} \hat{\zeta} \\ \hat{\eta} \end{pmatrix} = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{i\theta} \end{pmatrix} \begin{pmatrix} \zeta \\ \eta \end{pmatrix}. \tag{1.33}$$

For $0 \leq \theta \leq \pi$ we have

$$\begin{pmatrix} \hat{\zeta} \\ \hat{\eta} \end{pmatrix} = -\begin{pmatrix} \zeta \\ \eta \end{pmatrix}, \tag{1.34}$$

however, the flag plane has performed a 2π rotation! That is: a phase change of π leaves the flag plane invariant and takes $\mathbf{S}$ to $-\mathbf{I}$ (where $\mathbf{I}$ is the identity matrix).

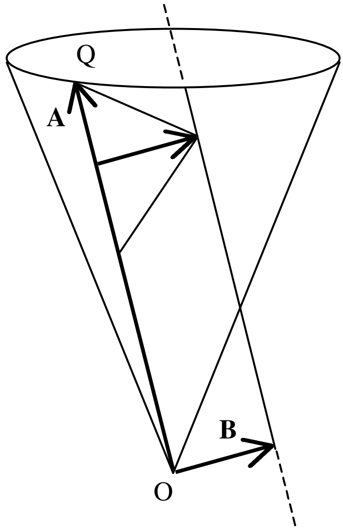

Fig. 1.3 The spin-vector as a null flag

For $\pi \leq \theta \leq 2\pi$ (1.33) becomes

$$\begin{pmatrix} \hat{\zeta} \\ \hat{\eta} \end{pmatrix} = \begin{pmatrix} \zeta \\ \eta \end{pmatrix}. \tag{1.35}$$

Thus a further 2π rotation must necessarily be performed by the flag plane in order that the null flag is returned to its original state.

A glance at (1.34) and (1.35) shows clearly that a sign ambiguity exists in the local geometry of $\mathbb{M}$; an ambiguity which cannot be resolved utilising geometrical reasoning. Indeed, the scope of local geometry must be extended to allow for the existence of *spinorial objects*. We will not enter into a detailed discussion here regarding these objects, and the reader is referred to the appropriate literature in the bibliography. However, roughly speaking, a null flag can be regarded as comprising of two spin-vectors — the spin-vector itself being a spinorial object. For example, consider the two arbitrary vectors $\boldsymbol{\alpha}$ and -$\boldsymbol{\alpha}$ defined by some null flag. We can send $\boldsymbol{\alpha}$ into -$\boldsymbol{\alpha}$ by performing a continuous 2π rotation. A further 2π rotation results in the original spin-vector. (It is of some interest to mention that an analogous description of flags has been given by Payne, 1952. In that paper flags are axes, the flag plane is the blade of the axe and the flagpole is the axe handle.)

1.5 Spin-space

Spin-vectors $\boldsymbol{\zeta}$, $\boldsymbol{\eta}$, etc. are elements belonging to *spin-space* S. We define S to be a two-dimensional vector space over the complex field $\mathbb{C}$. Furthermore, there exists on S a two-form $[\,,\,]$ with the following properties. Namely,

(1) a *skew-symmetric* (*alternating*) inner product, which is

(2) *bilinear*, and

(3) *non-degenerate.*

Let us consider these properties in more detail. Property (1) can be represented in a straightforward manner by

$$[\boldsymbol{\zeta}, \boldsymbol{\eta}] = -[\boldsymbol{\eta}, \boldsymbol{\zeta}] \tag{1.36}$$

for all $\boldsymbol{\zeta}$, $\boldsymbol{\eta} \in S$ and $[\boldsymbol{\zeta}, \boldsymbol{\eta}] \in \mathbb{C}$. Property (2) is directly represented by the relations:

$$\lambda[\boldsymbol{\zeta}, \boldsymbol{\eta}] = [\lambda\boldsymbol{\eta}, \boldsymbol{\zeta}] \tag{1.37}$$

$$[\boldsymbol{\zeta} + \boldsymbol{\eta}, \boldsymbol{\phi}] = [\boldsymbol{\zeta}, \boldsymbol{\phi}] + [\boldsymbol{\eta}, \boldsymbol{\phi}] \tag{1.38}$$

for $\lambda \in \mathbb{C}$. In combination, relations (1.37) and (1.38) imply that $[\,,\,]$ is linear in the second argument, and (1.36) ensures linearity in the first argument. Hence bilinearity ensues.

It is evident that for two linearly dependent spin-vectors $\boldsymbol{\kappa} = \lambda\boldsymbol{\zeta}$, say, it immediately follows from (1.36) and (1.37) that

$$[\boldsymbol{\zeta}, \boldsymbol{\kappa}] = 0. \tag{1.39}$$

Therefore a necessary and sufficient condition for two spin-vectors to be linearly dependent is

$$[\boldsymbol{\zeta}, \boldsymbol{\eta}] = 0 \Rightarrow \boldsymbol{\zeta} = \mathbf{0} \tag{1.40}$$

for all $\boldsymbol{\eta} \in S$, or

$$[\boldsymbol{\zeta}, \boldsymbol{\eta}] \neq 0 \tag{1.41}$$

for all $\boldsymbol{\zeta}$, $\boldsymbol{\eta} \in S$. This then is the explicit form of the non-degeneracy property (3). Notice the set $\{\boldsymbol{\zeta}, \boldsymbol{\eta}\}$ of spin-vectors (not proportional to each other) forms a spanning set for S.

Let $(\boldsymbol{o}, \boldsymbol{\iota})$ constitute a *spin basis* for S where $\boldsymbol{o}$ and $\boldsymbol{\iota}$ are two arbitrary spin-vectors such that

$$[\boldsymbol{o}, \boldsymbol{\iota}] = 1 = -[\boldsymbol{\iota}, \boldsymbol{o}] \tag{1.42}$$

by (1.36). Then we define the components of some spin-vector $\boldsymbol{\zeta} \in S$ by

$$\boldsymbol{\zeta} = \zeta^0 \boldsymbol{o} + \zeta^1 \boldsymbol{\iota} \tag{1.43}$$

where

$$\zeta^0 = [\boldsymbol{\zeta}, \boldsymbol{\iota}], \qquad \zeta^1 = -[\boldsymbol{\zeta}, \boldsymbol{o}] \tag{1.44}$$

and

$$\boldsymbol{o} = (1, 0), \qquad \boldsymbol{\iota} = (0, 1). \tag{1.45}$$

Remark: in this section, as in the whole of this book, we have adopted the convention that both spin-vector and its corresponding components are represented by the same Greek kernel letter. No ambiguity will result with regard to previous sections. All that has taken place is a relabelling of components i.e. in the case of (1.43) $\zeta \to \zeta^0$ and $\eta \to \zeta^1$.

By direct calculation one can easily show that the component form of the inner product (1.36) is

$$[\boldsymbol{\zeta}, \boldsymbol{\eta}] = \zeta^0 \eta^1 - \zeta^1 \eta^0 \tag{1.46}$$

where (η^0, η^1) are the components of $\boldsymbol{\eta}$.

This result will be useful in our study of the *Levi–Civita spinor* — which will be defined later. Before we continue with our explanation of spin bases, it will be advantageous to discuss what has come to be known as 'abstract index notation' (Penrose, 1968; Penrose and Rindler, 1986). This deals with the inherent ambiguities involving indices.

1.6 Exercises

1.1 Let $\mathbf{T}$ be a timelike vector with components $(T, 0, 0, 0)$ and $\mathbf{V}$ be a vector orthogonal to $\mathbf{T}$. Relative to an orthonormal basis in Minkowski space show that a timelike vector cannot be orthogonal to a null vector.

1.2 Obtain equation (1.13).

1.3 Obtain equation (1.23) by applying trigonometry to Fig. 1.2.

1.4 Using the *antipodal transformation* $(x, y, z) \mapsto (-x, -y, -z)$ and the equivalent transformation for polar coordinates $(\theta, \phi) \mapsto (\pi - \theta, \pi + \phi)$ obtain the antipodal counterparts of equations (1.22), (1.23), and (1.25).

[Hint: you should find in all cases that the effect of these transformations is $\xi \mapsto -\overline{\xi}^{-1}$.]

1.5 What geometrical inference can be made regarding the above antipodal transformations?

1.6 A child's swing is fixed to the branch of a tree by looping two pieces of rope over the branch and connecting the free ends to each of the four corners of the swing's seat. The seat is then rotated 720° about an axis through the centre of the seat causing the rope to become twisted. How can the rope be disentangled without rotating the seat again? Note that chopping down the tree is not an option!

[Remark: the above exercise is a variation of Paul Dirac's scissors problem. Other variations can also be found (see Penrose and Rindler, 1984; Misner, Thorne and Wheeler, 1973; Huggett and Tod, 1985).]

1.7 Verify (1.46).