Chapter 1

Finsler Metrics

To measure the length of a smooth curve C parametrized by a map $c = c(t)$, $a \le t \le b$, in a manifold M, it suffices to define a nonnegative scalar function $F(x, \cdot)$ on every tangent space $T_x M$. Then the length of C is defined by

$$\mathcal{L}_F(C) = \int_a^b F\Big(c(t), \dot{c}(t)\Big) dt.$$

It is required that $\mathcal{L}_F(C)$ be independent of parametrization. F must be positively homogeneous with degree one,

$$F(x, \lambda y) = \lambda F(x, y), \qquad \lambda > 0.$$

The length structure induces a nonnegative function $d : M \times M \to [0, \infty)$ by

$$d(p, q) := \inf_C L(C),$$

where the infimum is taken over all smooth curves C from p to q. In general, d is irreversible, i.e., $d(p, q) \ne d(q, p)$ for some pairs of points $\{p, q\}$. It is required that F uniquely determine d_F. We impose a convexity condition on F, that is,

$$F(x, y_1 + y_2) \le F(x, y_1) + F(x, y_2), \qquad y_1, y_2 \in T_x M. \tag{1.1}$$

Thus $F_x(\cdot) := F(x, \cdot)$ is a "norm" on $T_x M$ without the reversibility. In order to apply calculus to study the geometric properties of F, we assume that F is differentiable on $TM \setminus \{0\}$. Further, we replace the above convexity condition with a stronger convexity condition, that is, the Hessian,

$[F^2]_{y^iy^j}(x,y)$, is positive definite for any $y \in T_xM \setminus \{0\}$. This strong convexity implies the inequality (1.1). Therefore F_x is still a "norm" on T_xM. Such a "norm" is called a *Minkowski norm.* A scalar function F with the above properties is called a *Finsler metric.* When the norm F_x at every point $x \in M$ is *Euclidean*, the Finsler metric F is called an *Riemannian* metric.

Riemann-Finsler geometry is to study the geometric properties of Finsler metrics on a manifold. Intuitively, the Minkowski norm F_x or its unit ball $\mathcal{U}_x := \{F_x(y) < 1\}$ at each point x is an infinitesimal color pattern and it varies over the manifold. Thus a Finsler manifold is a "colorful" curved space. We study not only the curvature, but also the colors on the space.

1.1 Minkowski Norms

A Finsler metric on a manifold consists of the so-called Minkowski norms on every tangent space. Thus we first study the geometry of Minkowski norms on a vector space.

Let V be a finite dimensional vector space. A function $F = F(y)$ on V is called a *Minkowski norm* if it has the following properties:

(a) $F(y) \geq 0$ for any $y \in \mathrm{V}$, and $F(y) = 0$ if and only if $y = 0$;
(b) $F(\lambda y) = \lambda F(y)$ for any $y \in \mathrm{V}$ and $\lambda > 0$;
(c) F is C^∞ on $\mathrm{V} \setminus \{0\}$ such that for any $y \in \mathrm{V}$, the following bilinear symmetric functional $\mathbf{g}_y$ on V is an inner product,

$$\mathbf{g}_y(u,v) := \frac{1}{2}\frac{\partial^2}{\partial s \partial t}\Big[F^2(y + su + tv)\Big]_{s=t=0}.$$

The inner product $\mathbf{g}_y$ is called the *fundamental form* in the direction y. The pair (V, F) is called a *Minkowski space.* A Minkowski norm F is said to be *reversible* if $F(-y) = F(y)$ for $y \in \mathrm{V}$.

Given a Minkowski space (V, F), let

$$\mathrm{S}_F := \Big\{y \in \mathrm{V} \mid F(y) = 1\Big\}.$$

S_F is a closed hypersurface around the origin, which is diffeomorphic to the standard sphere $\mathrm{S}^{n-1} \subset \mathrm{R}^n$. S_F is called the *indicatrix* of F.

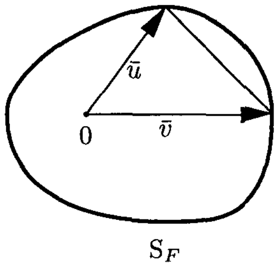

Figure 1.1

Let $u, v \in V \setminus \{0\}$ and $\bar{u} := u/F(u)$, $\bar{v} := v/F(v)$. Assume that $\bar{u} \neq \pm\bar{v}$. Consider

$$\varphi(t) := F^2(t\bar{u} + (1-t)\bar{v}).$$

We have that $\varphi(0) = \varphi(1) = 1$ and

$$\varphi''(t) = 2\mathbf{g}_y(\bar{u} - \bar{v}, \bar{u} - \bar{v}) > 0, \qquad 0 < t < 1,$$

where $y := t\bar{u} + (1-t)\bar{v}$. Thus $\varphi = \varphi(t)$ is strictly convex on $[0, 1]$. By a well-known result in calculus, $\varphi(t) < 1$, $0 < t < 1$, that is,

$$F(t\bar{u} + (1-t)\bar{v}) < 1, \qquad 0 < t < 1.$$

Clearly, when $\bar{u} = -\bar{v}$, the above inequality still holds. Plugging $t = F(u)/(F(u) + F(v))$ into the above inequality yields

$$F(u + v) < F(u) + F(v).$$

In the case when $\bar{u} = \bar{v}$, the following equality holds

$$F(u + v) = F(u) + F(v).$$

Let (V, F) be a Minkowski space. Fix a basis $\{\mathbf{b}_i\}$ for V, and view $F(y) = F(y^i\mathbf{b}_i)$ as a function of $(y^i) \in \mathrm{R}^n$. Then for $y \neq 0$,

$$g_{ij}(y) := \mathbf{g}_y(\mathbf{b}_i, \mathbf{b}_j) = \frac{1}{2}[F^2]_{y^i y^j}(y).$$

Here $[F^2]_{y^i y^j}(y)$ denote the partial derivative of F^2 with respect to y^i and y^j. We have

$$\mathbf{g}_y(u, v) = g_{ij}(y)u^i v^j, \qquad u = u^i\mathbf{b}_i, \ v = v^j\mathbf{b}_j$$

and

$$F(y) = \sqrt{g_{ij}(y)y^i y^j}, \qquad y = y^i \mathbf{b}_i.$$

Let us take a look at some special Minkowski norms. Let $\langle\ ,\ \rangle$ be an inner product on a vector space V with a basis $\{\mathbf{b}_i\}$, and let

$$\alpha := \sqrt{\langle y, y\rangle} = \sqrt{a_{ij}y^i y^j}, \qquad y = y^i \mathbf{b}_i,$$

where $a_{ij} = \langle \mathbf{b}_i, \mathbf{b}_j\rangle$. Clearly, α is a Minkowski norm with $\mathbf{g}_y(u,v) = \langle u, v\rangle$ independent of $y \in \mathrm{V} \setminus \{0\}$. α is called an *Euclidean norm*, and the pair (V, α) is called an *Euclidean space.* In each dimension, all Euclidean spaces are linearly isometric to each other. The standard Euclidean norm $|\cdot|$ on R^n is defined by

$$|y| := \sqrt{\sum_{i=1}^{n} (y^i)^2}, \qquad y = (y^i) \in \mathrm{R}^n.$$

There are many interesting non-Euclidean norms on a vector space. Let $\alpha = \sqrt{a_{ij}y^i y^j}$ be an Euclidean norm on a vector space V and $\beta = b_i y^i \in \mathrm{V}^*$ be a linear functional on V. Let

$$F := \alpha(y) + \beta(y). \tag{1.2}$$

By a direct computation, one obtains

$$g_{ij} = \frac{F}{\alpha}\Big\{a_{ij} - \frac{y_i}{\alpha}\frac{y_j}{\alpha} + \frac{\alpha}{F}\Big(b_i + \frac{y_i}{\alpha}\Big)\Big(b_j + \frac{y_j}{\alpha}\Big)\Big\}, \tag{1.3}$$

where $y_i := a_{ij}y^j$. From (1.3), one can see that (g_{ij}) is positive definite if and only if the length of β is less than 1, i.e.,

$$\|\beta\|_\alpha := \sqrt{a^{ij}b_i b_j} < 1,$$

where $(a^{ij}) := (a_{ij})^{-1}$. A Minkowski norm in the form (1.2) is called the *Randers norm* ([79]).

For further computation, we need the following lemma from linear algebra.

Lemma 1.1.1 *Let $G = (g_{ij})$ and $H = (h_{ij})$ be symmetric $n \times n$ matrices and $C = (c_i)$ be an n-vector. Assume that H is invertible with $H^{-1} = (h^{ij})$,* *and*

$$g_{ij} = h_{ij} + \delta c_i c_j.$$

Then

$$\det(g_{ij}) = (1 + \delta c^2)\det(h_{ij}),$$

where $c := \sqrt{h^{ij}c_i c_j}$. *If* $1 + \delta c^2 \neq 0$, *then* G *is invertible. The inverse matrix* $G^{-1} = (g^{ij})$ *is given by*

$$g^{ij} = h^{ij} - \frac{\delta c^i c^j}{1 + \delta c^2},$$

where $c^i := h^{ij}c_j$.

Applying Lemma 1.1.1 to (1.3), we obtain

$$\det(g_{ij}) = \Big(\frac{\alpha + \beta}{\alpha}\Big)^{n+1}\det(a_{ij}). \tag{1.4}$$

By (1.4), we can also show that (g_{ij}) is positive definite if and only if $\alpha(y) + \beta(y) > 0$ for any $y \in \mathrm{V} \setminus \{0\}$, if and only if $\|\beta\|_\alpha < 1$.

Using an Euclidean norm $\alpha = \sqrt{a_{ij}y^i y^j}$ and a 1-form $\beta = b_i y^i$ on a vector space V, one can define more general Minkowski norms—the (α, β)-*norms*:

$$F = \alpha\phi\Big(\frac{\beta}{\alpha}\Big). \tag{1.5}$$

Here the function $\phi = \phi(s)$ is a C^∞ positive function on some symmetric open interval $I = (-b_o, b_o)$. It is easy to see that $F = \alpha\phi(\beta/\alpha)$ is positively homogeneous of degree one. Let us find the condition for the positivity of $g_{ij} := \frac{1}{2}[F^2]_{y^i y^j}$. Assume that $b := \|\beta\|_\alpha < b_o$. Using a Maple program, one can easily compute the matrix (g_{ij}):

$$g_{ij} = \rho a_{ij} + \rho_0 b_i b_j + \rho_1(b_i\alpha_j + b_j\alpha_i) - s\rho_1\alpha_i\alpha_j,$$

where $\alpha_i = \alpha_{y^i}$ and

$$\rho = \phi^2 - s\phi\phi', \qquad \rho_0 = \phi\phi'' + \phi'\phi',$$

$$\rho_1 = -s(\phi\phi'' + \phi'\phi') + \phi\phi',$$

where the functions are evaluated on $s := \beta/\alpha$ with $|s| \leq b < r$. By Lemma 1.1.1, we find a formula for $\det(g_{ij})$.

$$\det\left(g_{ij}\right) = \phi^{n+1}\left(\phi - s\phi'\right)^{n-2}\Big[(\phi - s\phi') + (b^2 - s^2)\phi''\Big]\det\left(a_{ij}\right).$$

Lemma 1.1.2 *$F = \alpha\phi(\beta/\alpha)$ is a Minkowski norm for any Riemannian metric α and 1-form β with $\|\beta\|_\alpha < b_o$ if and only if $\phi = \phi(s)$ satisfies the following conditions:*

$$\phi(s) > 0, \qquad (\phi(s) - s\phi'(s)) + (b^2 - s^2)\phi''(s) > 0, \tag{1.6}$$

where s and b are arbitrary numbers with $|s| \leq b < b_o$.

Proof. Assume that (1.6) is satisfied. Then by taking $b = s$ in (1.6), we see that the following inequality holds for any s with $|s| < b_o$,

$$\phi(s) - s\phi'(s) > 0. \tag{1.7}$$

Consider the following family of functions,

$$\phi_t(s) := 1 - t + t\phi(s).$$

Let $F_t := \alpha\phi_t(\beta/\alpha)$ and $g^t_{ij} := \frac{1}{2}[F_t^2]_{y^iy^j}(y)$. Note that for any $0 \leq t \leq 1$ and any s, b with $|s| \leq b < b_o$,

$$\phi_t - s\phi_t' = 1 - t + t\Big[\phi - s\phi'\Big] > 0,$$

$$(\phi_t - s\phi_t') + (b^2 - s^2)\phi_t'' = 1 - t + t\Big[(\phi - s\phi') + (b^2 - s^2)\phi''\Big] > 0.$$

Thus $\det(g^t_{ij}) > 0$ for all $0 \leq t \leq 1$. Since (g^0_{ij}) is positive definite, we conclude that (g^t_{ij}) is positive definite for any $t \in [0, 1]$. Therefore, F_t is a Minkowski norm for all $t \in [0, 1]$.

Conversely, assume that $F = \alpha\phi(\beta/\alpha)$ is a Minkowski norm for any Riemannian metric α and 1-form β with $b := \|\beta\|_\alpha < b_o$. Then $\phi(s) > 0$ for any s with $|s| < b_o$. If $n = even$, then $\det(g_{ij}) > 0$ implies that (1.6) holds for any s with $|s| \leq b$. If $n = odd > 1$, then $\det(g_{ij}) > 0$ implies that the following inequality holds for any s with $|s| \leq b$,

$$\phi(s) - s\phi'(s) \neq 0.$$

Since $\phi(0) > 0$, the above inequality implies that the inequality (1.7) holds for any s with $|s| \leq b$. Since the number b can be arbitrary with $0 \leq b < b_o$,

we conclude that (1.7) holds for any s with $|s| < b_o$. Finally, we can see that $\det(g_{ij}) > 0$ implies that (1.6) holds for any s and b with $|s| \le b < b_o$.
Q.E.D.

Sabau–Shimada have studied certain (α, β)-norms and they have also computed the Hessian g_{ij} for these metrics [83].

Let us take a look at some special (α, β)-norms. Let $\alpha = \sqrt{a_{ij}y^i y^j}$ and $\beta = b_i y^i$ be a Euclidean norm and a 1-form on a vector space V, respectively. Let $y_i := a_{ij}y^j$ and $b := \|\beta\|_\alpha$. Then $|\beta(y)| \le b\alpha(y)$ for any $y \in$ V. Let

$$\phi := 1 + \varepsilon s + ks^2,$$

where ε and k are constants. The (α, β)-norm defined by ϕ is given by

$$F = \alpha + \varepsilon\beta + k\frac{\beta^2}{\alpha}.$$

By the above formulas, one obtains

$$\begin{aligned} g_{ij} = & \frac{(\alpha^2 - k\beta^2)F}{\alpha^3}a_{ij} + \frac{6kF + (\varepsilon^2 - 4k)\alpha}{\alpha}b_i b_j \\ & + \frac{\varepsilon\alpha^3 - 3\varepsilon k\alpha\beta^2 - 4k^2\beta^3}{\alpha^4}\Big\{(b_i y_j + b_j y_i) - \frac{\beta}{\alpha^2}y_i y_j\Big\}, \end{aligned}$$

and

$$\det(g_{ij}) = \Big(\frac{\alpha^2 - k\beta^2}{\alpha^3}\Big)^n F^{n+1}\frac{[(1 + 2kb^2)\alpha^2 - 3k\beta^2]\alpha}{(\alpha^2 - k\beta^2)^2}\det(a_{ij}).$$

Observe that

$$\phi(s) - s\phi'(s) + (b^2 - s^2)\phi''(s) = 1 + 2kb^2 - 3ks^2.$$

By Lemma 1.1.2, F is a Minkowski norm for any α and β with $\|\beta\|_\alpha < b_o$ if and only if

$$1 + \varepsilon s + ks^2 > 0, \qquad 1 + 2kb^2 - 3ks^2 > 0,$$

for any numbers s and b with $|s| \le b < b_o$. Where $\varepsilon = 2$ and $k = 1$, F can be expressed as

$$F = \frac{(\alpha + \beta)^2}{\alpha}.$$

Thus this function is a Minkowski norm if $\|\beta\|_\alpha < 1$.

Now we are going to construct Minkowski norms by shifting a Minkowski norm. Let (V, Φ) be a Minkowski space and let $v \in \mathrm{V}$ with $\Phi(-v) < 1$. Then the shifted set, $\mathrm{S}_\Phi + \{v\}$, contains the origin of V.

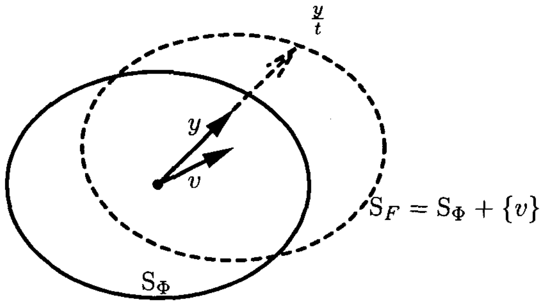

Figure 1.2

We can define a function $F : \mathrm{V} \to [0, \infty)$ as follows: for any $y \in \mathrm{V} \setminus \{0\}$, $F(y)$ is the unique positive number $t > 0$ such that

$$\frac{y}{t} \in \mathrm{S}_\Phi + \{v\}.$$

It is easy to see that F has the following properties:

(a) $F(y) > 0$ for any $y \in \mathrm{V} \setminus \{0\}$,
(b) $F(\lambda y) = \lambda F(y)$ for any $\lambda > 0$,
(c) $\mathrm{S}_F = \mathrm{S}_\Phi + \{v\}$.

For any $y \in \mathrm{V} \setminus \{0\}$, $F(y)$ can be determined by the following equation,

$$F(y) = \Phi\Big(y - F(y)\, v\Big). \tag{1.8}$$

Moreover, F is a Minkowski norm, i.e., the Hessian $g_{ij} := \frac{1}{2}[F^2]_{y^i y^j}$ is positive definite. The proof is left for the reader. F is called the *Minkowski norm generated by* (Φ, v). One can easily show that if $F = F(y)$ is generated by (Φ, v), then $\Phi = \Phi(y)$ is generated by $(F, -v)$.

Example 1.1.3 For a Euclidean norm $\Phi = |y|$ on V and a vector $v \in \mathrm{V}$ with $|v| < 1$, the solution of (1.8) is a Randers norm,

$$F = \frac{\sqrt{(1 - |v|^2)|y|^2 + \langle y, v\rangle^2} - \langle y, v\rangle}{1 - |v|^2}.$$

1.2 Finsler Metrics

We are now ready to introduce Finsler metrics on a manifold. Throughout this book, we always assume that manifolds are C^∞ (smooth), connected and finite dimensional.

Let M be a manifold. For a point $x \in M$, denote by T_xM the *tangent space* of M at x. The *tangent bundle* TM of M is the union of tangent spaces with a natural differential structure,

$$TM := \bigcup_{x \in M} T_xM.$$

Denote the elements in TM by (x, y) where $y \in T_xM$.

Roughly speaking, a Finsler metric on a manifold M is a C^∞ function on the slit tangent bundle $TM_o := TM \setminus \{0\}$, whose restriction to each tangent space T_xM is a Minkowski norm.

Definition 1.2.1 Let M be a manifold. A function $F = F(x, y)$ on TM is called a *Finsler metric* on M if it has the following properties:

(a) $F(x, y)$ is C^∞ on TM_o;
(b) $F_x(y) := F(x, y)$ is a Minkowski norm on T_xM for any $x \in M$.

The pair (M, F) is called a *Finsler manifold.*

A Finsler metric $F = F(x, y)$ on a manifold M is said to be *reversible* if $F(x, -y) = F(x, y)$ for all $y \in T_xM$. We usually do not impose the reversibility condition on Finsler metrics. A Finsler metric F on M is said to be *Riemannian*, if the restriction of F , $F_x(y) := F(x, y)$, is a Euclidean norm on T_xM for any $x \in M$, that is,

$$F_x(y) = \sqrt{\langle y, y \rangle_x}, \qquad y \in T_xM,$$

where $\langle \ , \ \rangle_x$ is an inner product on T_xM. We usually denote a Riemannian metric by a family of inner products $g_x = \langle y, y \rangle_x$ on tangent spaces T_xM. Clearly, Riemannian metrics are reversible Finsler metrics.

Riemannian metrics are among the most important Finsler metrics. Let us take a look at some special Riemannian metrics.

Example 1.2.2 Let $| \cdot |$ be the standard Euclidean norm on R^n,

$$|y| := \sqrt{\sum_{i=1}^{n} (y^i)^2}.$$

Define $F = F(x, y)$ by

$$F := |y|, \qquad y \in T_x\mathrm{R}^n \cong \mathrm{R}^n.$$

F is a Finsler metric on R^n, which is called the *standard Euclidean metric.*

Example 1.2.3 Let $\mathrm{B}^n \subset (\mathrm{R}^n, |\cdot|)$ be the standard unit ball and let

$$\alpha_{-1} := \frac{\sqrt{|y|^2 - (|x|^2|y|^2 - \langle x, y\rangle^2)}}{1 - |x|^2}, \qquad y \in T_x\mathrm{B}^n \cong \mathrm{R}^n. \tag{1.9}$$

α_{-1} is a Riemannian metric on B^n, which is called the *Klein metric.* The pair $(\mathrm{B}^n, \alpha_{-1})$ is called the *Klein model.*

Example 1.2.4 Let $\mathrm{S}^n \subset (\mathrm{R}^{n+1}, |\cdot|)$ be the standard unit sphere. For $x \in \mathrm{S}^n$, we identify $T_x\mathrm{S}^n$ with a hypersurface in R^{n+1} in a natural way. Let

$$\alpha_{+1} := |y|_o, \qquad y \in T_x\mathrm{S}^n \subset \mathrm{R}^{n+1}. \tag{1.10}$$

Here $|\cdot|_o$ denotes the Euclidean norm on R^{n+1}. Let S^n_+ and S^n_- denote the upper and lower hemispheres, respectively, and let $\psi_\pm : \mathrm{R}^n \to \mathrm{S}^n_\pm$ be the projection map defined by

$$\psi_\pm(x) := \Big(\frac{x}{\sqrt{1+|x|^2}}, \frac{\pm 1}{\sqrt{1+|x|^2}}\Big).$$

$\psi_\pm$ sends straight lines in R^n to great circles on $\mathrm{S}^n_\pm$.

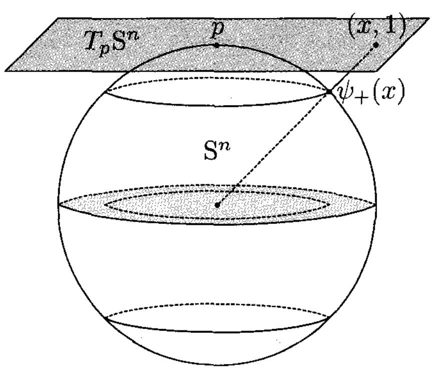

Figure 1.3

The pull-back metric on R^n from S^n_+ by ψ_+ is given by

$$\alpha_{+1} = \frac{\sqrt{|y|^2 + (|x|^2|y|^2 - \langle x, y\rangle^2)}}{1 + |x|^2}, \qquad y \in T_x\mathrm{R}^n \cong \mathrm{R}^n. \tag{1.11}$$

The pair $(\mathrm{R}^n, \alpha_{+1})$ is called the *projective spherical model.*

The Riemannian metrics in Examples 1.2.2, 1.2.3 and 1.2.4 can be expressed in one single formula.

$$\alpha_\mu := \frac{\sqrt{|y|^2 + \mu(|x|^2|y|^2 - \langle x, y\rangle^2)}}{1 + \mu|x|^2}, \qquad y \in T_x\mathrm{B}^n(r_\mu) \cong \mathrm{R}^n, \tag{1.12}$$

where $r_\mu := 1/\sqrt{-\mu}$ if $\mu < 0$ and $r_\mu := +\infty$ if $\mu \geq 0$. The metric α_μ can be expressed as $\alpha_\mu = \sqrt{a_{ij}y^iy^j}$, where

$$a_{ij} = \frac{1}{1+\mu|x|^2}\Big\{\delta_{ij} - \frac{\mu x^i x^j}{1+\mu|x|^2}\Big\}.$$

Let $\alpha = \sqrt{a_{ij}(x)y^iy^j}$ be a Riemannian metric and $\beta = b_i(x)y^i$ be a 1-form on an n-dimensional manifold M. Let

$$\|\beta_x\|_\alpha := \sup_{y\in T_xM} \frac{\beta(x,y)}{\alpha(x,y)} = \sqrt{a^{ij}(x)b_i(x)b_j(x)}.$$

Consider the following function

$$F := \alpha\phi(s), \qquad s = \frac{\beta}{\alpha}, \tag{1.13}$$

where $\phi = \phi(s)$ is a positive C^∞ function on $(-b_o, b_o)$ satisfying

$$\phi(s) - s\phi'(s) + (b^2 - s^2)\phi''(s) > 0, \qquad |s| \leq b < b_o.$$

Then by Lemma 1.1.2, F is a Finsler metric if $\|\beta_x\|_\alpha < b_o$ for any $x \in M$. A Finsler metric in the form (1.13) is called an *(α, β)-metric.*

Let $\phi = 1 + s$. Then $F = \alpha\phi(s)$, where $s = \beta/\alpha$, becomes

$$F = \alpha + \beta.$$

Note that ϕ is positive on $(-1, 1)$ and $\phi(s) - s\phi'(s) + (b^2 - s^2)\phi''(s) = 1$. Thus $F = \alpha + \beta$ is a Finsler metric if and only if $\|\beta_x\|_\alpha < 1$ for any $x \in M$.

One can prove it directly using the formula (1.3) for g_{ij}. The Finsler metric $F = \alpha + \beta$ with $\sup_{x \in M} \|\beta_x\|_\alpha < 1$ is called a *Randers metric* on M.

A typical example of Randers metrics is defined on the ball $\mathrm{B}^n(r_\mu) \subset \mathrm{R}^n$:

$$F := \frac{\sqrt{|y|^2 + \mu(|x|^2|y|^2 - \langle x, y\rangle^2)} + \sqrt{-\mu}\langle x, y\rangle}{1 + \mu|x|^2}, \tag{1.14}$$

where $\mu < 0$ and $r_\mu := 1/\sqrt{-\mu}$. The metric when $\mu = -1$ is of particular interest.

$$F := \frac{\sqrt{|y|^2 - (|x|^2|y|^2 - \langle x, y\rangle^2)} + \langle x, y\rangle}{1 - |x|^2}. \tag{1.15}$$

The metric F in (1.15) is called the *Funk metric* on $\mathrm{B}^n(1)$. It has many special geometric properties.

Let $\phi := (1+s)^2$. $F = \alpha\phi(s)$, where $s = \beta/\alpha$, becomes

$$F = \frac{(\alpha + \beta)^2}{\alpha}.$$

Note that ϕ is positive on $(-1, 1)$ and for any s, b with $|s| \le b < 1$,

$$\begin{aligned} \phi(s) - s\phi'(s) + (b^2 - s^2)\phi''(s) &= 1 - 3s^2 + 2b^2 \\ &> 1 - s^2 + 2(b^2 - s^2) > 0. \end{aligned}$$

Thus F is a Finsler metric if and only if $\|\beta_x\|_\alpha < 1$ for any $x \in M$.

A typical example of metrics in the above form is defined on the ball $\mathrm{B}^n(r_\mu) \subset \mathrm{R}^n$:

$$F = \frac{(\sqrt{|y|^2 + \mu(|x|^2|y|^2 - \langle x, y\rangle^2)} + \sqrt{-\mu}\langle x, y\rangle)^2}{(1 + \mu|x|^2)^2\sqrt{|y|^2 + \mu(|x|^2|y|^2 - \langle x, y\rangle^2)}}, \tag{1.16}$$

where $\mu < 0$ and $r_\mu := 1/\sqrt{-\mu}$. The reader should try to find α and β so that $F = (\alpha + \beta)^2/\alpha$. The metric when $\mu = -1$ is of particular interest.

$$F := \frac{(\sqrt{|y|^2 + -(|x|^2|y|^2 - \langle x, y\rangle^2)} + \langle x, y\rangle)^2}{(1 - |x|^2)^2\sqrt{|y|^2 - (|x|^2|y|^2 - \langle x, y\rangle^2)}}. \tag{1.17}$$

The metric in (1.17) was constructed by L. Berwald [17]. It has many special geometric properties. We will discuss it later in the book.

One may construct a product Finsler metrics from a pair of Finsler manifolds. Let (M_1, F_1) and (M_2, F_2) be Finsler manifolds. A Finsler metric F on $M := M_1 \times M_2$ is called a *product Finsler metric* of F_1 and F_2 if at any point $x = (x_1, x_2) \in M$,

$$F(x,y) = \begin{cases} F_1(x_1, y_1) & \text{if } y = y_1 \oplus 0 \in T_xM \\ F_2(x_2, y_2) & \text{if } y = 0 \oplus y_2 \in T_xM \end{cases}$$

where $T_xM \cong T_{x_1}M_1 \oplus T_{x_2}M_2$. In this case, (M, F) is called a *product Finsler manifold* of (M_1, F_1) and (M_2, F_2).

For a pair of Finsler manifolds, there is no canonical way to define a product Finsler metrics on the product manifold. When the Finsler metrics are Riemannian, we can define the product Finsler metrics in the following way.

Example 1.2.5 Let α_1 and α_2 be Euclidean norms on vector spaces V_1 and V_2 respectively. Let $f : [0, \infty) \times [0, \infty) \to [0, \infty)$ be a C^∞ function satisfying

$$f(\lambda s, \lambda t) = \lambda f(s,t), \quad \forall \lambda > 0, \quad \text{and} \quad f(s,t) > 0, \quad \forall (s,t) \neq (0,0). \tag{1.18}$$

Define a function $F : \mathrm{V} := \mathrm{V}_1 \oplus \mathrm{V}_2 \to [0, \infty)$ by

$$F(y) := \sqrt{f\Big([\alpha_1(y_1)]^2, \ [\alpha_2(y_2)]^2\Big)},$$

where $y = y_1 \oplus y_2 \in \mathrm{V} = \mathrm{V}_1 \oplus \mathrm{V}_2$. $F = F(y)$ has the following properties

(a) $F(y) \geq 0$ for any $y \in V$, and $F(y) = 0$ if and only if $y = 0$;
(b) $F(\lambda y) = \lambda F(y)$ for any $y \in \mathrm{V}$ and $\lambda > 0$;
(c) $F(y)$ is C^∞ on $\mathrm{V} \setminus \{0\}$.

Let $n_1 = \dim \mathrm{V}_1$, $n_2 = \dim \mathrm{V}_2$ and $n = n_1 + n_2 = \dim \mathrm{V}$. We shall assume the following ranges of indices:

$$1 \leq a, b, c \leq n_1, \qquad n_1 + 1 \leq \alpha, \beta, \gamma \leq n, \qquad 1 \leq i, j, k \leq n.$$

Let $\{\mathbf{b}_a\}$ and $\{\mathbf{b}_\alpha\}$ be bases for V_1 and V_2 respectively. Then $\{\mathbf{b}_i\}$ is a basis for V. Express

$$\alpha_1(y_1) = \sqrt{\bar{g}_{ab} y^a y^b}, \qquad \alpha_2(y_2) = \sqrt{\bar{g}_{\alpha\beta} y^\alpha y^\beta},$$

where $y_1 = y^a \mathbf{b}_a$ and $y_2 = y^\alpha \mathbf{b}_\alpha$. Then $g_{ij} := \frac{1}{2}[F^2]_{y^i y^j}$ are given by

$$(g_{ij}) = \begin{pmatrix} 2f_{ss}\bar{y}_a\bar{y}_b + f_s\bar{g}_{ab} & 2f_{st}\bar{y}_a\bar{y}_\beta \\ 2f_{st}\bar{y}_b\bar{y}_\alpha & 2f_{tt}\bar{y}_\alpha\bar{y}_\beta + f_t\bar{g}_{\alpha\beta} \end{pmatrix}, \tag{1.19}$$

where $\bar{y}_a := \bar{g}_{ab}y^b$ and $\bar{y}_\alpha := \bar{g}_{\alpha\beta}y^\beta$. By an elementary argument, one can see that (g_{ij}) is positive definite if and only if $f(s,t)$ satisfies the following conditions:

$$f_s > 0, \qquad f_t > 0, \qquad f_s + 2sf_{ss} > 0, \qquad f_t + 2tf_{tt} > 0, \tag{1.20}$$

and

$$f_s f_t - 2ff_{st} > 0. \tag{1.21}$$

In this case,

$$\det(g_{ij}) = h\Big([\alpha_1]^2,\ [\alpha_2]^2\Big)\det(\bar{g}_{ab})\det(\bar{g}_{\alpha\beta}), \tag{1.22}$$

where

$$h := (f_s)^{n_1-1}(f_t)^{n_2-1}\Big\{f_s f_t - 2ff_{st}\Big\}.$$

There are lots of functions f satisfying (1.18), (1.20) and (1.21). For example, one can take

$$f(s,t) := \frac{1}{1+\varepsilon}\Big\{s + t + \varepsilon\Big(s^k + t^k\Big)^{\frac{1}{k}}\Big\}, \tag{1.23}$$

where ε is a nonnegative number and k is a positive integer.

By the above construction, one can construct infinitely many product Finsler metrics on $M = M_1 \times M_2$ for any given pair of Riemannian manifolds (M_1, α_1) and (M_2, α_2). Just take a function f as in (1.23) and define

$$F(x,y) := \sqrt{f\Big([\alpha_1(x_1,y_1)]^2,\ [\alpha_2(x_2,y_2)]^2\Big)},$$

where $x = (x_1, x_2) \in M$ and $y = y_1 \oplus y_2 \in T_xM$. Then F is a reversible product Finsler metric on M.

1.3 Length Structure and Volume Form

Every Finsler metric on a manifold defines a *length structure* on piecewise smooth curves. Let M be a C^∞ manifold. A map $c : I = [a,b] \to M$ is called a *piecewise C^∞ curve* if it is continuous and there is a partition $a = t_0 < t_1 < \cdots < t_{n-1} < t_n = b$ such that c restricted to each $[t_{i-1}, t_i]$ is C^∞. For $t \in [t_{i-1}, t_i]$, let $\dot{c}(t) := \frac{dc}{dt}(t) \in T_{c(t)}M$ denote the tangent vector of c. c is said to be *regular* if $\dot{c}(t) \neq 0$, $\forall t \in [t_{i-1}, t_i]$.

Two regular maps $c : I \to M$ and $\bar{c} : \bar{I} \to M$ are said to be *equivalent* if there is a one-to-one and onto piecewise C^∞ map $\varphi : I \to \bar{I}$ such that $\varphi'(t) > 0$ and $\bar{c}(\varphi(t)) = c(t)$, $\forall t \in [t_{i-1}, t_i]$. A (piecewise) C^∞ curve C in a manifold M is an equivalence class of regular (piecewise) C^∞ maps from an interval I into M. For the sake of simplicity, we do not distinguish a regular map $c = c(t)$ and the curve C represented by c.

For a C^∞ curve (represented by) $c : I = [a,b] \to M$, its reverse $c_- : I \to M$ is defined by $c_-(t) := c(b + a - t)$. The class represented by c_- is different from that represented by c. All C^∞ curves in this book are *oriented*.

Consider a piecewise C^∞ curve C from p to q in (M, F). Let C be a piecewise C^∞ curve represented by $c = c(t)$ with $c(a) = p$ and $c(b) = q$. The length of C is defined by

$$\mathcal{L}_F(C) := \int_a^b F\Big(c(t), \dot{c}(t)\Big) dt.$$

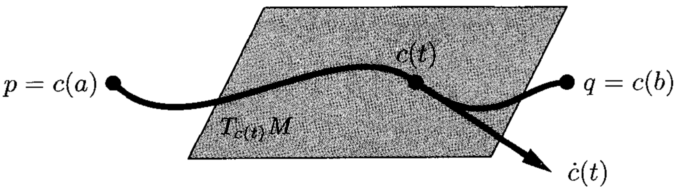

Figure 1.4

If C is represented by another map $\bar{c} = \bar{c}(\bar{t})$ with $\bar{c}(\bar{a}) = p$ and $\bar{c}(\bar{b}) = q$, then there is a positive function $\bar{t} = \varphi(t)$ such that $\bar{c}(\bar{t}) = c(t)$ with $\bar{a} = \varphi(a)$ and $\bar{b} = \varphi(b)$. Then

$$d\bar{t} = \varphi'(t)dt, \qquad \dot{c}(t) = \dot{\bar{c}}(\bar{t})\varphi'(t).$$

We have

$$\int_a^b F\Big(c(t),\dot c(t)\Big)dt = \int_a^b F\Big(\bar c(\bar t),\dot{\bar c}(\bar t)\varphi'(t)\Big)dt$$
$$= \int_a^b F\Big(\bar c(\bar t),\dot{\bar c}(\bar t)\Big)\varphi'(t)dt = \int_{\bar a}^{\bar b} F\Big(\bar c(\bar t),\dot{\bar c}(\bar t)\Big)d\bar t.$$

Thus, $\mathcal{L}_F(C)$ is well-defined.

For a pair of points $p, q \in M$, define

$$d_F(p,q) := \inf_C \mathcal{L}_F(C), \tag{1.24}$$

where the infimum is taken over all piecewise C^∞ curves C issuing from p to q. d_F is a function on $M \times M$ with the following properties:

(a) $d_F(p,q) \geq 0$;
(b) $d_F(p,q) = 0$ if and only if $p = q$;
(c) $d_F(p,q) \leq d_F(p,r) + d_F(r,q)$.

The proofs of (a) and (c) are elementary. To prove (b), it suffices to prove the following fact [55]. At every point $x_o \in M$, there is a local coordinate system $\varphi : \mathcal{U} \subset M \to U \subset \mathrm{R}^n$ such that

$$\lambda^{-1}|(y^i)| \leq F(x,y) \leq \lambda|(y^i)|, \qquad y = y^i\frac{\partial}{\partial x^i} \in T_x\mathcal{U},$$

where $\lambda > 1$ is a constant. Choosing a smaller neighborhood $\mathcal{U}$ if necessarily, one can easily show that there is a constant $C > 1$ such that

$$C^{-1}|\varphi(x_2) - \varphi(x_1)| \leq d_F(x_1,x_2) \leq C|\varphi(x_2) - \varphi(x_1)|, \tag{1.25}$$

where $x_1, x_2 \in \mathcal{U}$. This implies (b).

The function d_F is called the *distance function* of F. Let $\Delta \subset M \times M$ denote the diagonal. It can be shown that d_F is C^∞ on $\mathcal{U} \setminus \Delta$ for some neighborhood $\mathcal{U}$ of Δ in $M \times M$. The proof is very technical, so it is omitted here. Conversely, d_F determines the Finsler metric F by

$$F(x,y) = \lim_{\varepsilon\to 0^+}\frac{d_F(x,c(\varepsilon))}{\varepsilon}, \qquad y \in T_xM, \tag{1.26}$$

where $c(t)$ is an arbitrary C^∞ curve with $c(0) = x$ and $\dot c(0) = y$.

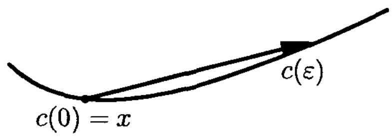

Figure 1.5

Clearly, if F is reversible, then the induced distance function d_F is reversible, i.e., $d_F(p,q) = d_F(q,p)$, for all pairs of points $p, q \in M$ and vice versa.

Every Finsler metric $F = F(x,y)$ on an n-dimensional manifold M defines a volume form. At a point $x \in M$, let $\{\mathbf{b}_i\}$ be a basis for T_xM and $\{\theta^i\}$ be the basis for T_x^*M dual to $\{\mathbf{b}_i\}$. Then the following n-form at $x \in M$ is, up to an orientation, well-defined,

$$dV_F := \sigma_F(x)\theta^1 \wedge \cdots \wedge \theta^n,$$

where

$$\sigma_F(x) := \frac{\mathrm{Vol}(\mathrm{B}^n(1))}{\mathrm{Vol}\Big((y^i) \in \mathrm{R}^n \mid F(x, y^i\mathbf{b}_i) < 1\Big)}. \tag{1.27}$$

Here $\mathrm{Vol}(\cdot)$ denotes the Euclidean volume function on subsets in R^n so that for the unit cubic $\mathcal{U} = [0,1]^n$, $\mathrm{Vol}(\mathcal{U}) = 1$. The n-form dV_F is called the *Finsler volume form.* For an open subset $\mathcal{U} \subset M$, the volume of $\mathcal{U}$ is defined by

$$\mathrm{Vol}_F(\mathcal{U}) := \int_{\mathcal{U}} dV_F.$$

When F is reversible, the induced distance function d_F is reversible and the Hausdorff measure of d_F can be defined in the usual way. An important fact is that the *Hausdorff measure* of an open subset $\mathcal{U}$ with respect to d_F is equal to $\mathrm{Vol}_F(\mathcal{U})$. See [22].

In general, $\sigma_F(x)$ can't be expressed in terms of elementary functions though $F = F(x,y)$ sometimes is. Nevertheless, $\sigma_F(x)$ is computable for Randers metrics including Riemannian metrics.

First, let us consider a Riemannian metric α on an n-dimensional manifold M. Let

$$\alpha = \sqrt{a_{ij}(x)y^i y^j}, \qquad y = y^i \frac{\partial}{\partial x^i}|_x \in T_xM.$$

Let A be a matrix such that $A^T A = (a_{ij})$. Then the linear transformation $x = Ay : \mathrm{R}^n \to \mathrm{R}^n$ sends the convex domain $\mathcal{U}_x := \{(y^i) \in \mathrm{R}^n \mid \sqrt{a_{ij}(x)y^i y^j} < 1\}$ onto the unit ball $\mathrm{B}^n(1)$. We may assume that the Jacobian matrix has positive determinant $\det(A) > 0$. We have

$$\det(A) = \sqrt{\det(a_{ij}(x))}.$$

Observe that

$$\begin{aligned}\mathrm{Vol}(\mathrm{B}^n(1)) &= \int_{\mathrm{B}^n(1)} dx^1 \cdots dx^n \\ &= \int_{\mathcal{U}_x} \det(A) dy^1 \cdots dy^n = \sqrt{\det(a_{ij}(x))}\mathrm{Vol}(\mathcal{U}_x).\end{aligned}$$

That yields,

$$\mathrm{Vol}(\mathcal{U}_x) = \frac{\mathrm{Vol}(\mathrm{B}^n(1))}{\sqrt{\det\left(a_{ij}(x)\right)}}.$$

Then $dV_\alpha = \sigma_\alpha(x) dx^1 \cdots dx^n$ is given by

$$\sigma_\alpha(x) = \sqrt{\det\left(a_{ij}(x)\right)}.$$

Consider a Randers metric $F = \alpha + \beta$, where $\alpha = \sqrt{a_{ij}(x)y^i y^j}$ is a Riemannian metric and $\beta = b_i(x)y^i$ is a 1-form on an n-dimensional manifold M. Let $\Omega_x := \{(y^i) \in \mathrm{R}^n \mid F(x, y^i \frac{\partial}{\partial x^i}|_x) < 1\}$. By an elementary argument using linear algebra, we obtain

$$\mathrm{Vol}(\Omega_x) = \frac{\mathrm{Vol}(\mathrm{B}^n(1))}{\left(1 - \|\beta_x\|_\alpha^2\right)^{(n+1)/2} \sqrt{\det\left(a_{ij}(x)\right)}},$$

where $\|\beta_x\|_\alpha$ denotes the norm of β at x with respect to α_x. Plugging the above formula into (1.27) yields

$$\sigma_F(x) = \left(1 - \|\beta_x\|_\alpha^2\right)^{(n+1)/2} \sigma_\alpha(x).$$

Thus

$$dV_F = \left(1 - \|\beta_x\|^2_\alpha\right)^{(n+1)/2} dV_\alpha. \tag{1.28}$$

Note that for any open subset $\Omega \subset M$,

$$\int_\Omega dV_F \leq \int_\Omega dV_\alpha.$$

Equality holds if and only if $F = \alpha$.

Example 1.3.1 Consider the Randers metric $F = \alpha + \beta$ on the unit ball $\mathrm{B}^n(1) \subset \mathrm{R}^n$, where $\alpha = \sqrt{a_{ij}(x)y^iy^j}$ and $\beta = b_i(x)y^i$ are given by

$$\alpha := \frac{\sqrt{|y|^2 - (|x|^2|y|^2 - \langle x, y\rangle^2)}}{1 - |x|^2},$$

$$\beta := \frac{\langle x, y\rangle}{1 - |x|^2} + \frac{\langle a, y\rangle}{1 + \langle a, x\rangle},$$

where $y \in T_x\mathrm{R}^n \cong \mathrm{R}^n$, $a \in \mathrm{B}^n(1)$, $|\cdot|$ and $\langle\ ,\ \rangle$ denote the standard Euclidean norm and inner product in R^n, respectively. When $a = 0$, F is the Funk metric defined in (1.15).

Applying Lemma 1.1.1 to the following matrix

$$a_{ij} = \frac{1}{1 - |x|^2}\left\{\delta_{ij} + \frac{x^ix^j}{1 - |x|^2}\right\},$$

we obtain

$$\det(a_{ij}) = \frac{1}{(1 - |x|^2)^{n+1}}.$$

Then

$$dV_\alpha = \left(1 - |x|^2\right)^{-\frac{n+1}{2}} dx^1 \cdots dx^n.$$

By Lemma 1.1.1,

$$a^{ij} = (1 - |x|^2)\left\{\delta^{ij} - x^ix^j\right\}.$$

Then the norm of β is given by

$$\|\beta_x\|_\alpha = \sqrt{a^{ij}(x)b_i(x)b_j(x)} = 1 - \frac{(1 - |x|^2)(1 - |a|^2)}{(1 + \langle a, x\rangle)^2}. \tag{1.29}$$

Plugging the above formulas into (1.28) yields

$$dV_F = \left[\frac{1-|a|^2}{(1+\langle a,x\rangle)^2}\right]^{\frac{n+1}{2}} dx^1\cdots dx^n.$$

1.4 Navigation Problem

In this section, we will discuss Randers metrics from a navigation point of view. We shall see that non-Riemannian metrics are not avoidable even though we live in a Riemannian world.

Consider an object moving in a metric space, such as Euclidean space, pushed by an interval force and an external force field. The shortest time problem is to determine a curve from one point to another in the space, along which it takes the least time for the object to travel. This problem in some special cases was studied by E. Zermelo [102], hence called the *Zermelo navigation problem*. Here we shall discuss the navigation problem in the most general case. Suppose that an object on a Finsler manifold (M, Φ) is pushed by an internal force U with constant length, $\Phi(x, U_x) = c$, and while it is pushed by an external force field V with $\Phi(x, -V_x) < c$. The combined force at x is $T_x := U_x + V_x$. The condition, $\Phi(x, -V_x) < c$, guarantees that the object can move forward in any direction.

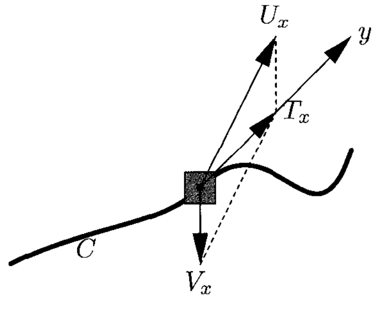

Figure 1.6

Due to the friction, the object moves on M at a speed proportional to the combined force T. For the sake of simplicity, one may assume that $c = 1$ and the velocity vector at any point $x \in M$ is equal to T_x. Given a pair of points $p, q \in M$, let C be an arbitrary piecewise C^∞ curve in M. Since $\Phi(x, U_x) = 1$, we have

$$\Phi\Big(x,\ T_x - V_x\Big) = \Phi(x, U_x) = 1. \tag{1.30}$$

On the other hand, for any vector $y \in T_xM \setminus \{0\}$, there is a unique solution $F = F(x, y) > 0$ to the following equation

$$\Phi\Big(x,\ \frac{y}{F} - V_x\Big) = 1. \tag{1.31}$$

Observe that for any $\lambda > 0$,

$$1 = \Phi\Big(x,\ \frac{\lambda y}{\lambda F(x,y)} - V_x\Big) = \Phi\Big(x,\ \frac{\lambda y}{F(x,\lambda y)} - V_x\Big).$$

By the uniqueness,

$$F(x, \lambda y) = \lambda F(x, y).$$

One can show that $F_x := F|_{T_xM}$ is a Minkowski norm on T_xM. Thus $F = F(x, y)$ is a Finsler metric on M. Comparing (1.30) and (1.31), one can see that the combined force T_x has unit F-length,

$$F(x, T_x) = 1. \tag{1.32}$$

This observation leads to the following

Lemma 1.4.1 *Let (M, Φ) be a Finsler manifold and V be a vector field on M with $\Phi(x, -V_x) < 1$, $\forall x \in M$. Define $F : TM \to [0, \infty)$ by (1.31). For any piecewise C^∞ curve C in M, the F-length of C is equal to the time for which the object travels along it.*

Proof. Let $c : [0, t_o] \to M$ be the parametrization of C such that the velocity vector $\dot{c}(t) = T_{c(t)}$. Then t_o is the time for which the object travels along C. It follows from (1.32) that

$$F\Big(c(t), \dot{c}(t)\Big) = 1.$$

This implies

$$t_o = \int_0^{t_o} F\Big(c(t), \dot{c}(t)\Big)dt = \mathcal{L}_F(C).$$

Q.E.D.

For a pair $\{\Phi, V\}$ on a manifold M, where $\Phi = \Phi(x, y)$ is a Finsler metric and V is a vector field with $\Phi(x, -V_x) < 1$, we define a Finsler metric $F = F(x, y)$ by (1.31). The Finsler metric F can also be defined in the following way. First, define Φ^* and V^* on T^*M by

$$\Phi^*(x, \xi) := \sup_{y \in T_xM} \frac{\eta(y)}{\Phi(x, y)}, \qquad V^*(\xi) := \xi(V_x), \qquad \xi \in T_x^*M.$$

Then $F^* := \Phi^* + V^*$ is a co-Finsler metric on M and F is dual to F^*, i.e.,

$$F(x, y) = \sup_{\xi \in T_x^*M} \frac{\eta(y)}{F^*(x, \xi)}.$$

The proof is left to the reader.

Lemma 1.4.2 *Let $\Phi = \Phi(x, y)$ be a Finsler metric on an n-dimensional manifold M and $V = V^i(x)\frac{\partial}{\partial x^i}$ be an arbitrary vector field on M with $\Phi(x, -V_x) < 1$, $x \in M$. Let $F = F(x, y)$ denote the Finsler metric on M defined by (1.31). Then the Finsler volume forms of F and Φ are equal,*

$$dV_F = dV_\Phi. \tag{1.33}$$

Proof: Fix a basis $\{\mathbf{b}_i\}$ for T_xM and let $V_x := v^i\mathbf{b}_i$. Let

$$\mathcal{U}_\Phi := \Big\{(y^i) \in \mathrm{R}^n \mid \Phi(x, y^i\mathbf{b}_i) < 1\Big\},$$
$$\mathcal{U}_F := \Big\{(y^i) \in \mathrm{R}^n \mid F(x, y^i\mathbf{b}_i) < 1\Big\}.$$

From the definition of F, we have

$$\mathcal{U}_F = \mathcal{U}_\Phi + (v^i).$$

Since shifting does not change the Euclidean volume, $\mathrm{Vol}(\mathcal{U}_\Phi) = \mathrm{Vol}(\mathcal{U}_F)$. This implies that

$$\sigma_F(x) = \sigma_\Phi(x).$$

Thus $dV_F = dV_\Phi$. Q.E.D.

The above proposition shows that the volume of an open subset on a Finsler manifold is not disturbed by any vector field.

Example 1.4.3 Let $\phi = \phi(y)$ be a Minkowski norm on R^n and

$$\mathcal{U} := \Big\{y \in \mathrm{R}^n \;\Big|\; \phi(y) < 1\Big\}.$$

Let $\Phi(x,y) := \phi(y)$, where $y \in T_x\mathrm{V} \cong \mathrm{V}$, and $V_x := -x$, where $x \in \mathrm{V}$. Φ is a Minkowski metric on R^n and V is a radial vector field toward the origin. Observe that

$$\Phi(x, -V_x) = \phi(x) < 1, \qquad x \in \mathcal{U}.$$

For a non-zero vector $y \in T_x\mathcal{U} \setminus \{0\} \cong \mathrm{R}^n \setminus \{0\}$, define $\Theta = \Theta(x,y) > 0$ by

$$\Phi\Big(x,\ \frac{y}{\Theta(x,y)} - V_x\Big) = 1. \tag{1.34}$$

Then $\Theta = \Theta(x,y)$ is a Finsler metric on $\mathcal{U}$, which is called the *Funk metric* on $\mathcal{U}$. The Funk metric $\Theta = \Theta(x,y)$ can also be defined by

$$z := x + \frac{y}{\Theta(x,y)} \in \partial\mathcal{U}.$$

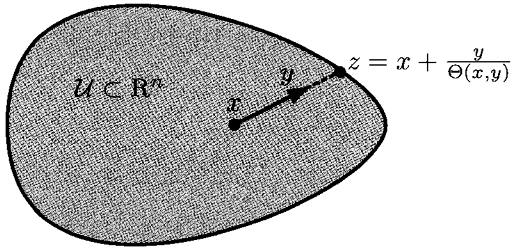

Figure 1.7

Equation (1.34) can be written as

$$\Theta(x,y) = \phi\Big(y + \Theta(x,y)x\Big). \tag{1.35}$$

Differentiating (1.35) with respect to x^k and y^k respectively, one obtains

$$\Big(1 - \phi_{w^l}(w)x^l\Big)\Theta_{x^k}(x,y) = \phi_{w^k}(w)\Theta(x,y), \tag{1.36}$$

$$\Big(1 - \phi_{w^l}(w)x^l\Big)\Theta_{y^k}(x,y) = \phi_{w^k}(w), \tag{1.37}$$

where $w := y + \Theta(x,y)x$. It follows from (1.36) and (1.37) that

$$\Theta_{x^k} = \Theta\Theta_{y^k}. \tag{1.38}$$

The above argument is given by T. Okada [77].

If $\phi = |y|$ is the standard Euclidean norm on R^n, $\mathcal{U} = \mathrm{B}^n(1)$ is the unit ball in R^n. In this case, $\Theta = F$ as defined in (1.15).

Given a Riemannian metric $h = \sqrt{h_{ij}(x)y^iy^j}$ and a vector field $V = V^i(x)\frac{\partial}{\partial x^i}$ on a manifold M with $h(x, -V_x) = \sqrt{h_{ij}(x)V^i(x)V^j(x)} < 1$, one can define a Finsler $F = F(x, y)$ by (1.31), i.e.,

$$h\Big(x, \frac{y}{F} - V\Big) = \sqrt{h_{ij}\Big(\frac{y^i}{F} - V^i\Big)\Big(\frac{y^j}{F} - V^j\Big)} = 1. \tag{1.39}$$

Solving (1.39) for F, one obtains $F = \alpha + \beta$, where $\alpha = \sqrt{a_{ij}(x)y^iy^j}$ and $\beta = b_i(x)y^i$ are given by

$$a_{ij} = \frac{(1 - h_{pq}V^pV^q)h_{ij} + h_{ip}h_{jq}V^pV^q}{(1 - h_{pq}V^pV^q)^2}, \tag{1.40}$$

$$b_i = -\frac{h_{ip}V^p}{1 - h_{pq}V^pV^q}. \tag{1.41}$$

It is easy to show that

$$\|\beta_x\|_\alpha = \sqrt{a^{ij}b_ib_j} = \sqrt{h_{ij}V^iV^j} = h(x, -V_x) < 1. \tag{1.42}$$

Thus F is a Randers metric.

Conversely, every Randers metric $F = \alpha + \beta$ on a manifold M can be constructed from a Riemannian metric h and a vector field V on M. The construction is given as follows. Let $\alpha = \sqrt{a_{ij}y^iy^j}$ and $\beta = b_iy^i$. Define

$$h_{ij} := (1 - \|\beta_x\|^2)\Big\{a_{ij} - b_ib_j\Big\}, \tag{1.43}$$

$$V^i := -\frac{a^{ij}b_j}{1 - \|\beta_x\|_\alpha^2}. \tag{1.44}$$

Then F is given by (1.39) for $h = \sqrt{h_{ij}(x)y^iy^j}$ and $V = V^i(x)\frac{\partial}{\partial x^i}$. Moreover, (1.42) holds. Thus $h(x, -V_x) < 1$ for $x \in M$. See [45] and [46] for a similar type of duality between Randers metrics defined as a function on TM and Randers co-metrics defined as a function on T^*M.

Example 1.4.4 Let $\mathrm{B}^n \subset \mathrm{R}^n$ be the standard unit ball and let

$$h := \frac{\sqrt{1 - |a|^2}}{1 + \langle a, x\rangle}\sqrt{|y|^2 - \frac{2\langle a, y\rangle\langle x, y\rangle}{1 + \langle a, x\rangle} - \frac{(1 - |x|^2)\langle a, y\rangle^2}{(1 + \langle a, x\rangle)^2}}, \tag{1.45}$$

$$V := -\frac{1+\langle a,x\rangle}{1-|a|^2}(x+a), \tag{1.46}$$

where $y \in T_x\mathrm{B}^n = \mathrm{R}^n$ and $a \in \mathrm{R}^n$ is a constant vector with $|a| < 1$. By (1.39), one obtains

$$F = \frac{\sqrt{|y|^2 - (|x|^2|y|^2 - \langle x,y\rangle^2)}}{1-|x|^2} + \frac{\langle x,y\rangle}{1-|x|^2} + \frac{\langle a,y\rangle}{1+\langle a,x\rangle}. \tag{1.47}$$

Both $h = h(x,y)$ and $F = F(x,y)$ have some special geometric properties. See Examples 3.4.2 and 3.4.6 below. When $a = 0$, F is the Funk metric on B^n defined in (1.15).

1.5 Cartan Torsion

To characterize Euclidean norms among Minkowski norms, E. Cartan introduces a quantity for Minkowski norms [23].

Let $F = F(y)$ be a Minkowski norm on a vector space V. For a vector $y \in \mathrm{V} \setminus \{0\}$, let

$$\mathbf{C}_y(u,v,w) := \frac{1}{4}\frac{\partial^3}{\partial s \partial t \partial r}\Big[F^2(y+su+tv+rw)\Big]_{s=t=r=0},$$

where $u, v, w \in \mathrm{V}$. Each $\mathbf{C}_y$ is a symmetric trilinear form on V. We call the family $\mathbf{C} := \{\mathbf{C}_y \mid y \in \mathrm{V} \setminus \{0\}\}$ the *Cartan torsion.*

Let $\{\mathbf{b}_i\}$ be a basis for V. Let $g_{ij} := \mathbf{g}_y(\mathbf{b}_i, \mathbf{b}_j)$, $C_{ijk} := \mathbf{C}_y(\mathbf{b}_i, \mathbf{b}_j, \mathbf{b}_k)$. Then

$$g_{ij} = \frac{1}{2}[F^2]_{y^i y^j},$$
$$C_{ijk} = \frac{1}{4}[F^2]_{y^i y^j y^k} = \frac{1}{2}\frac{\partial}{\partial y^k}(g_{ij}).$$

Define the mean value of the Cartan torsion by

$$\mathbf{I}_y(u) := \sum_{i=1}^{n} g^{ij}(y)\mathbf{C}_y(u, \mathbf{b}_i, \mathbf{b}_j), \qquad u \in \mathrm{V}.$$

We call the family $\mathbf{I} := \{\mathbf{I}_y \mid y \in \mathrm{V}\setminus\{0\}\}$ the *mean Cartan torsion.* Observe that

$$\frac{\partial}{\partial y^i}\Big[\det(g_{jk})\Big] = \det(g^{jk})g^{pq}\frac{\partial g_{pq}}{\partial y^i} = 2\det(g^{jk})g^{pq}C_{ipq}.$$

We have

$$I_i = g^{jk}C_{ijk} = \frac{\partial}{\partial y^i}\Big[\ln\sqrt{\det\big(g_{jk}\big)}\Big]. \tag{1.48}$$

It follows from the homogeneity of F that

$$\mathbf{C}_y(y,v,w) = \mathbf{C}_y(u,y,w) = \mathbf{C}_y(u,v,y) = 0 \tag{1.49}$$

and

$$\mathbf{I}_y(y) = 0. \tag{1.50}$$

Moreover,

$$\mathbf{C}_{\lambda y} = \lambda^{-1}\mathbf{C}_y, \qquad \mathbf{I}_{\lambda y} = \lambda^{-1}\mathbf{I}_y, \qquad \lambda > 0. \tag{1.51}$$

From (1.49)-(1.51), one can see that $\mathbf{C}_y$ and $\mathbf{I}_y$ depend only on the geometry of the indicatrix S_F of F. Intuitively, the indicatrix of F can be viewed as a color pattern on V, then $\mathbf{C}_y$ (resp. $\mathbf{I}_y$) is the rate (resp. average rate) of tangential change of the color pattern at y.

It is obvious that F is Euclidean if and only if $\mathbf{C}_y = 0$ for any $y \in \mathrm{V}\setminus\{0\}$. In fact, Euclidean norms can be characterized by the mean Cartan torsion. The following result is due to Deicke [34].

Theorem 1.5.1 ([34]) *A Minkowski norm on a vector space* V *is Euclidean if and only if* $\mathbf{I} = 0$.

The proof does not fit in this book, so it is omitted. One can see [5] for a proof.

To characterize Randers norms among Minkowski norms, M. Matsumoto introduces the following quantity [64] [66]. For $y = y^i\mathbf{b}_i \in \mathrm{V}$, define

$$M_{ijk} := C_{ijk} - \frac{1}{n+1}\Big\{I_i h_{jk} + I_j h_{ik} + I_k h_{ij}\Big\}, \tag{1.52}$$

where $h_{ij} := FF_{y^iy^j} = g_{ij} - \frac{1}{F^2}g_{ip}y^p g_{jq}y^q$. Let

$$\mathbf{M}_y(u,v,w) := M_{ijk}(x,y)u^iv^jw^k, \tag{1.53}$$

where $u = u^i\mathbf{b}_i$, $v = v^j\mathbf{b}_j$ and $w = w^k\mathbf{b}_k$. Each $\mathbf{M}_y$ is a symmetric trilinear form on V. We call the family $\mathbf{M} := \{\mathbf{M}_y \mid y \in \mathrm{V}\setminus\{0\}\}$ the *Matsumoto torsion.* Clearly, $\mathbf{M} = 0$ for all two-dimensional Minkowski norms.

Example 1.5.2 ([64]) Let $F = \alpha + \beta$ be a Randers norm on a vector space V, where $\alpha = \sqrt{a_{ij}y^iy^j}$ and $\beta = b_iy^i$ with $\|\beta\|_\alpha < 1$. Then $g_{ij} := \frac{1}{2}[F^2]_{y^iy^j}(y)$ are given by (1.3) and $\det(g_{ij})$ is given by (1.4). Note that $\det(a_{ij})$ is independent of y. By (1.48), one obtains

$$
\begin{aligned}
I_i &= \frac{\partial}{\partial y^i} \ln \sqrt{\left(\frac{\alpha+\beta}{\alpha}\right)^{n+1} \det(a_{ij})} \\
&= \frac{n+1}{2(\alpha+\beta)} \cdot \left(b_i - \frac{y_i}{\alpha}\frac{\beta}{\alpha}\right).
\end{aligned} \tag{1.54}
$$

Differentiating (1.3) with respect to y^k yields

$$
C_{ijk} = \frac{1}{n+1}\Big\{I_i h_{jk} + I_j h_{ik} + I_k h_{ij}\Big\}, \tag{1.55}
$$

where $h_{ij} := FF_{y^iy^j}$ are given by

$$
h_{ij} = \frac{\alpha+\beta}{\alpha}\left(a_{ij} - \frac{y_iy_j}{\alpha^2}\right).
$$

This implies that $M_{ijk} = 0$.

Minkowski norms with $\mathbf{M} = 0$ are said to be *C-reducible*. It turns out that every C-reducible Minkowski norm is a Randers norm in dimension $n \geq 3$.

Proposition 1.5.3 ([64], [70]) *Let F be a Minkowski norm on a vector space* V *of dimension $n \geq 3$. The Matsumoto torsion $\mathbf{M} = 0$ if and only if F is a Randers norm.*

The proof does not fit in this book, and so is omitted. See [70] for more details.

Given a Minkowski space (V, F), using the family of inner products $\mathbf{g}_y$ on V, one can define the norm of $\mathbf{I}, \mathbf{C}$ and $\mathbf{M}$ in a natural way.

$$
\begin{aligned}
\|\mathbf{I}\| &:= \sup_{y,u\in \mathrm{V}\setminus\{0\}} \frac{F(y)|\mathbf{I}_y(u)|}{\sqrt{\mathbf{g}_y(u,u)}}, \\
\|\mathbf{C}\| &:= \sup_{y,u,v,w\in \mathrm{V}\setminus\{0\}} \frac{F(y)|\mathbf{C}_y(u,v,w)|}{\sqrt{\mathbf{g}_y(u,u)\mathbf{g}_y(v,v)\mathbf{g}_y(w,w)}}, \\
\|\mathbf{M}\| &:= \sup_{y,u,v,w\in \mathrm{V}\setminus\{0\}} \frac{F(y)|\mathbf{M}_y(u,v,w)|}{\sqrt{\mathbf{g}_y(u,u)\mathbf{g}_y(v,v)\mathbf{g}_y(w,w)}}.
\end{aligned}
$$

By (1.52), $\|\mathbf{M}\|$ is bounded by $\|\mathbf{C}\|$. It is easy to construct a family of Minkowski norms F_i on R^n with $\|\mathbf{C}_i\| \to +\infty$ as $i \to +\infty$, where $\mathbf{C}_i$ denotes the Cartan torsion of F_i.

The (mean) Cartan torsion of any Randers norm is bounded from above by a number depending only on the dimension.

Lemma 1.5.4 ([49]) *Let $F = \alpha+\beta$ be a Randers norm on an n-dimensional vector space* V. *Then*

$$\|\mathbf{I}\| = \frac{n+1}{\sqrt{2}}\sqrt{1-\sqrt{1-\|\beta\|_\alpha^2}} < \frac{n+1}{\sqrt{2}}. \tag{1.56}$$

Proof. Let $\alpha = \sqrt{a_{ij}y^iy^j}$ and $\beta = b_iy^i$. Then $g_{ij} := \frac{1}{2}[F^2]_{y^iy^j}$ are given by (1.3). By (1.3) and Lemma 1.1.1, one can find the inverse matrix $(g^{ij}) = (g_{ij})^{-1}$.

$$g^{ij} = \frac{\alpha}{F}a^{ij} - \frac{\alpha}{F^2}(b^iy^j + b^jy^i) + \frac{\alpha\|\beta\|_\alpha^2 + \beta}{\alpha^3}y^iy^j. \tag{1.57}$$

By (1.54) and (1.57), one obtains

$$I_iI_jg^{ij} = \Big(\frac{n+1}{2F(y)}\Big)^2\frac{\alpha(y)}{F(y)}\Big\{\|\beta\|_\alpha^2 - \Big(\frac{\beta(y)}{\alpha(y)}\Big)^2\Big\}. \tag{1.58}$$

Since $|\beta(y)| \leq \|\beta\|_\alpha\alpha(y)$, we can write $\beta(y) = \|\beta\|_\alpha\alpha(y)\cos\theta$, where $0 \leq \theta \leq 2\pi$. Assume that y is a unit vector, i.e., $F(y) = \alpha(y) + \beta(y) = 1$. Then

$$\alpha(y) = 1 - \beta(y) = 1 - \|\beta\|_\alpha\alpha(y)\cos\theta.$$

Thus

$$\alpha(y) = \frac{1}{1 + \|\beta\|_\alpha\cos\theta}.$$

Plugging it into (1.58) yields

$$I_iI_jg^{ij} = \Big(\frac{n+1}{2}\Big)^2\frac{\|\beta\|_\alpha^2\sin^2\theta}{1+\|\beta\|_\alpha\cos\theta},$$
$$\|\mathbf{I}\|^2 = \frac{(n+1)^2}{2}\Big(1-\sqrt{1-\|\beta\|_\alpha^2}\Big).$$

This gives the upper bound (1.56) immediately. Q.E.D.

It follows from (1.55) and (1.56) that

$$\|\mathbf{C}\| \leq \frac{3}{\sqrt{2}}\sqrt{1-\sqrt{1-\|\beta\|_\alpha^2}} < \frac{3}{\sqrt{2}}. \tag{1.59}$$

Namely, the Cartan torsion is uniformly bounded by $3/\sqrt{2}$. The bound (1.59) for two-dimensional Randers norms is given in Exercise 11.2.6 in [5] which is suggested by Brad Lackey.

Example 1.5.5 Consider the generalized Funk metric $F = \alpha + \beta$ on the unit ball $\mathrm{B}^n(1) \subset \mathrm{R}^n$,

$$F = \frac{\sqrt{|y|^2 - (|x|^2|y|^2 - \langle x, y\rangle^2)}}{1-|x|^2} + \frac{\langle x, y\rangle}{1-|x|^2} + \frac{\langle a, y\rangle}{1+\langle a, x\rangle}.$$

Let $\|\mathbf{I}\|_x$ denote the norm of the mean Cartan torsion at $x \in \mathrm{B}^n(1)$. By (1.29) and (1.56), one obtains

$$\|\mathbf{I}\|_x = \frac{n+1}{\sqrt{2}}\left\{1 - \frac{\sqrt{(1-|x|^2)(1-|a|^2)}}{1+\langle a, x\rangle}\right\}.$$

Note that at $x = -a$, $\mathbf{I}_x = 0$, namely, F_x is Euclidean. However, as $x \to \partial\mathrm{B}^n(1)$, $\|\mathbf{I}\|_x \to (n+1)/\sqrt{2}$. The point $x = -a$ can be regarded as the *Euclidean center* of F.