OVERVIEW OF PHASE-SPACE QUANTIZATION

1 Introduction

There are at least three logically autonomous alternative paths to quantization. The first is the standard one utilizing operators in Hilbert space, developed by Heisenberg, Schrödinger, Dirac, and others in the 1920s. The second one relies on path integrals, and was conceived by Dirac [Dir33] and constructed by Feynman.

The third one (the bronze medal!) is the phase-space formulation, based on Wigner's (1932) quasi-distribution function [Wig32] and Weyl's (1927) correspondence [Wey27] between quantum-mechanical operators in Hilbert space and ordinary c-number functions in phase space. The crucial composition structure of these functions, which relies on the $\star$-product, was fully understood by Groenewold (1946) [Gro46], who, together with Moyal (1949) [Moy49], pulled the entire formulation together. Still, insights into interpretation and a full appreciation of its conceptual autonomy took some time to mature with the work of, among others, Takabayasi [Tak54], Baker [Bak58], and Fairlie [Fai64].

This complete formulation is based on the Wigner function (WF), which is a quasi-probability distribution function in phase space:

$$f(x,p) = \frac{1}{2\pi}\int dy\ \psi^*\left(x - \frac{\hbar}{2}y\right)\ e^{-iyp}\psi\left(x + \frac{\hbar}{2}y\right). \tag{1}$$

It is a generating function for all spatial autocorrelation functions of a given quantum-mechanical wave function $\psi(x)$. More important, it is a special representation of the density matrix (in the Weyl correspondence, as detailed in Section 12). Alternatively, in a $2n$-dimensional phase space, it amounts to

$$f(x,p) = \frac{1}{(2\pi\hbar)^n}\int d^n y\ \left\langle x + \frac{y}{2}\right| \boldsymbol{\rho} \left| x - \frac{y}{2}\right\rangle\ e^{-ip\cdot y/\hbar}, \tag{2}$$

where $\psi(x) = \langle x|\psi\rangle$ in the density operator $\boldsymbol{\rho}$,

$$\boldsymbol{\rho} = \int d^n z \int d^n x d^n p\ \left| x + \frac{z}{2}\right\rangle f(x,p)\ e^{ip\cdot z/\hbar}\left\langle x - \frac{z}{2}\right|. \tag{3}$$

There are several outstanding reviews on the subject: Refs. HOS84, Tak89, Ber80, BJ84, Lit86, deA98, Tat83, Coh95, KN91, Kub64, DeG74, KW90, Ber77, Lee95, Dah01, Sch02, DHS00, CZ83, Gad95, HH02, Str57, McD88, Leo97, Sny80, Bal63, BFF78.

Nevertheless, the central conceit of the present overview is that the above input wave functions may ultimately be bypassed, since the WFs are determined, in principle, as the solutions to suitable functional equations in phase space. Connections to the Hilbert space operator formulation of quantum mechanics may thus be ignored, in principle—even though they are provided in Section 12 for pedagogy and confirmation of the formulation's equivalence. One might then envision an imaginary world in which this formulation of quantum mechanics had preceded the conventional Hilbert-space formulation, and its own techniques and methods had arisen independently, perhaps out of generalizations of classical mechanics and statistical mechanics.

It is not only wave functions that are missing in this formulation. Beyond the ubiquitous (noncommutative, associative, pseudodifferential) operation, the $\star$-product, which encodes the entire quantum-mechanical action, there are no linear operators. Expectations of observables and transition amplitudes are phase-space integrals of c-number functions, weighted by the WF, as in statistical mechanics. Consequently, even though the WF is not positive-semidefinite (it can be, and usually is negative in parts of phase space [Wig32]), the computation of expectations and the associated concepts are evocative of classical probability theory. Still, telltale features of quantum mechanics are reflected in the noncommutative multiplication of such c-number phase-space functions through the $\star$-product, in systematic analogy to operator multiplication in Hilbert space.

This formulation of quantum mechanics is useful in describing *quantum* transport processes in phase space. Such processes are of importance in quantum optics [Sch02, Leo97, SM00], nuclear and particle physics [Bak60, SP81, MM84, CC03, BJY04], condensed matter [MMP94, DBB02, KKFR89, BP96, KL01, JBM03], the study of semiclassical limits of mesoscopic systems [Imr67, OR57, Sch69, Ber77, KW87, OM95, MS95, MOT98, Vo89, Vo78], and the transition to classical statistical mechanics [JD99, Fre87, BD98, Raj83, CV98, SM00, FZ01, Zal03].

It is the natural language to study quantum chaos and decoherence [JN90, ZP94, BC99, KZZ02, KJ99, Zu91, FBA96, Kol96, GH93, CL03, OC03] (of utility in, e.g. quantum computing [BHP02]), and provides crucial intuition in quantum mechanical interference problems [Wis97], probability flows as negative probability backflows [BM94, FMS00], and measurements of atomic systems [Smi93, Dun95, Lei96, KPM97, Lvo01, JS02, BHS02, Ber02, Cas91].

The intriguing mathematical structure of the formulation is of relevance to Lie Algebras [FFZ89], martingales in turbulence [Fan03], and string field theory [BKM03]. It has recently been retrofitted into M-theory advances linked to noncommutative geometry [SW99] (for reviews, see Refs. Cas00, Har01, DN01, HS02), and matrix models [Tay01, KS02]; these apply space–time uncertainty principles [Pei33, Yo89, JY98, SST00] reliant on the $\star$-product. (Transverse spatial dimensions act formally as momenta, and, analogously to quantum mechanics, their uncertainty is increased or decreased inversely to the uncertainty of a given direction.)

As a significant aside, the WF has extensive practical applications in signal processing, filtering, and engineering (time-frequency analysis) , since time and frequency constitute a pair of Fourier-conjugate variables just like the $\mathfrak{x}$ and $\mathfrak{p}$ pair of phase space.[a]

For simplicity, the formulation will be mostly illustrated for one coordinate and its conjugate momentum, but generalization to arbitrary-sized phase spaces is straightforward [DM86], including infinite-dimensional ones, namely scalar field theory [Dit90, Les84, Na97, CZ99, CPP01, MM94]: the respective WFs are simple products of single-particle WFs.

[a]Thus, time-varying signals are best represented in a WF as time-varying spectrograms, analogously to a music score , i.e. the changing distribution of frequencies is monitored in time [BBL80, Wok97, QC96, MH97, Coh95, Gro01]: even though the description is constrained and redundant, it gives an intuitive picture of the signal that a mere time profile or frequency spectrogram fails to convey. Applications abound [CGB91, Lou96, MH97] in bioengineering, acoustics, speech analysis, vision processing, turbulence microstructure analysis, radar imaging, seismic data analysis, and the monitoring of internal combustion engine-knocking, failing helicopter component vibrations, and so on.

2 The Wigner Function

As already indicated, the quasi-probability measure in phase space is the WF,

$$f(x,p) = \frac{1}{2\pi} \int dy \; \psi^* \left(x - \frac{\hbar}{2}y\right) \; e^{-iyp} \psi \left(x + \frac{\hbar}{2}y\right). \tag{4}$$

It is obviously normalized; $\int dpdx f(x,p) = 1$. In the classical limit, $\hbar \to 0$, it would reduce to the probability density in coordinate space x, usually highly localized, multiplied by δ-functions in momentum: the classical limit is "spiky" and certain! This expression has more $x - p$ symmetry than is apparent, as Fourier transformation to momentum-space wave-functions yields a completely symmetric expression with the roles of x and p reversed, and, upon rescaling of the arguments x and p, a symmetric classical limit.

The WF is also manifestly real.[b] In addition, it is constrained by the Schwarz inequality to be bounded, $-\frac{2}{\hbar} \leq f(x,p) \leq \frac{2}{\hbar}$. Again, this bound disappears in the spiky classical limit.

Respectively, p- or x-projection leads to marginal probability densities: a spacelike shadow, $\int dp \; f(x,p) = \rho(x)$, *or else* a momentum-space shadow, $\int dx f(x,p) = \sigma(p)$. Either is a bona-fide probability density, being positive-semidefinite. But neither can be conditioned on the other, as the uncertainty principle is fighting back: The WF $f(x,p)$ itself can, and most often is negative in some areas of phase space [Wig32, HOS84], as is illustrated below, a hallmark of QM interference in this language. (In fact, the only pure state WF which is non-negative is the Gaussian [Hud74], a state of maximum entropy [Raj83].)

The counter-intuitive "negative probability" aspects of this quasi-probability distribution have been explored and interpreted [Bar45, Fey87, BM94] (for a popular review, see LPM98), and negative probability flows amount to legitimate probability backflows in interesting settings [BM94]. Nevertheless, the WF for atomic systems can still be measured in the laboratory, albeit indirectly [Smi93, Dun95, Lei96, KPM97, Lvo01, BAD96, BHS02, Ber02, BRWK99], and reconstructed.

Smoothing f by a filter of size larger than $\hbar$ (e.g. convolving with a phase-space Gaussian) results in a positive-semidefinite function, i.e. it may be thought to have been coarsened to a classical[c] distribution [Car76, Ste80, OW81, Raj83].

Among real functions, the WFs constitute a rather small, highly constrained set. When is a real function $f(x,p)$ a bona-fide Wigner function of the form (4)? Evidently, when its

[b]In one space dimension, by virtue of nondegeneracy, ψ has the same effect as ψ^*, and f turns out to be p-even, but this is not a property used here.

[c]This one is called the Husimi distribution [Tak89, TA99], and sometimes information scientists examine it on account of its non-negative feature. Nevertheless, it comes with a heavy price, as it needs to be "dressed" back to the WF for all practical purposes when expectation values are computed with it, i.e. it does not serve as an immediate quasi-probability distribution with no further measure (see Section 13). The negative feature of the WF is, in the last analysis, an asset, not a liability, and provides an efficient description of "beats" [BBL80, Wok97, QC96, MH97, Coh95]; cf. Fig. 1. If, instead, strictly inequivalent (improper) expectation values were taken with the Husimi distribution *without* the requisite dressing of Section 13, i.e. as though it were a bona-fide probability distribution, such expectation values would reflect loss of quantum information: they would represent classically coarsened observables [WO87].

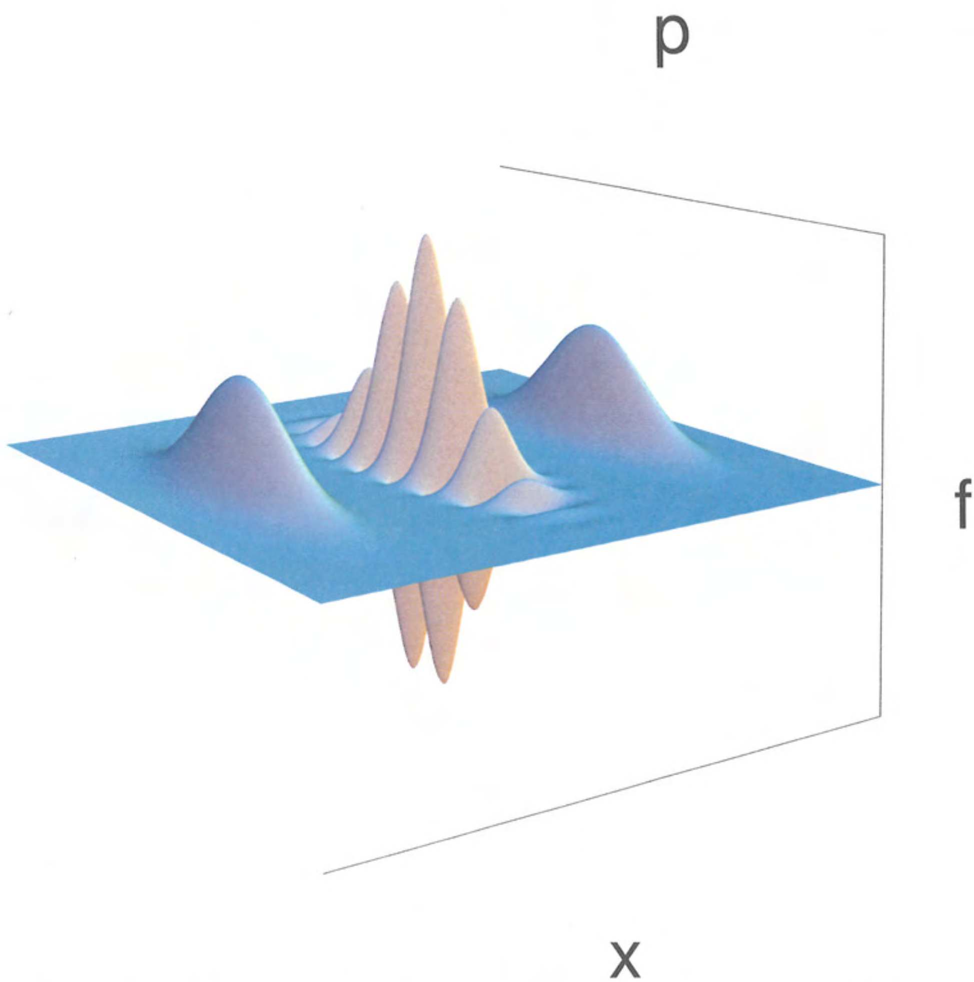

Figure 1. Wigner function of a pair of Gaussian wavepackets centered at $x = \pm a$: $f(x,p;a) = \exp[-(x^2 + p^2)](\exp(-a^2)\cosh(2ax) + \cos(2pa))]/\pi(1+e^{-a^2})$. (The corresponding wave-function is $\psi(x;a) = \exp[-(x+a)^2/2] + \exp[-(x-a)^2/2]/\pi^{1/4}\sqrt{2+2e^{-a^2}}$.) Here, $a = 6$ is chosen, quite larger than the width of the Gaussians. Note the phase-space interference structure ("beats") with negative values in the x region between the two packets where there is no wave-function support—hence vanishing probability for the presence of the particle. The oscillation frequency in the p direction is a/π.

Fourier transform (the cross-spectral density) "left–right" factorizes,

$$\tilde{f}(x,y) = \int dp\; e^{ipy} f(x,p) = g_L^*\left(x - \frac{\hbar y}{2}\right)\; g_R\left(x + \frac{\hbar y}{2}\right)\;. \tag{5}$$

That is,

$$\frac{\partial^2\;\ln \tilde{f}}{\partial(x - \hbar y/2)\;\partial(x + \hbar y/2)} = 0, \tag{6}$$

so, for real f, $g_L = g_R$.

Nevertheless, as indicated, the WF *is* a distribution function, after all: it provides the integration measure in phase space to yield expectation values from phase-space c-number functions. Such functions are often classical quantities but, in general, are uniquely associated with suitably ordered operators through Weyl's correspondence rule [Wey27]. Given an operator ordered in this prescription,

$$\mathfrak{G}(\mathfrak{x},\mathfrak{p}) = \frac{1}{(2\pi)^2}\int d\tau d\sigma dx dp\; g(x,p)\exp[i\tau(\mathfrak{p}-p)+i\sigma(\mathfrak{x}-x)]\;, \tag{7}$$

the corresponding phase-space function $g(x,p)$ (the "Weyl kernel function of the operator") is obtained by

$$\mathfrak{p}\mapsto p, \quad \mathfrak{x}\mapsto x. \tag{8}$$

That operator's expectation value is then a "phase-space average" [Gro46, Moy49],

$$\langle\mathfrak{G}\rangle = \int dx dp\; f(x,p)\; g(x,p). \tag{9}$$

The kernel function $g(x,p)$ is often the unmodified classical observable expression, such as a conventional Hamiltonian, $H = p^2/2m + V(x)$, i.e. the transition from classical mechanics is straightforward. However, it contains $\hbar$ corrections when there are quantum-mechanical ordering ambiguities, such as in the observable kernel of the square of the angular momentum $\mathfrak{L}\cdot\mathfrak{L}$: This contains a term, $-3\hbar^2/2$, introduced by the Weyl ordering [She59, DS82, DS02], beyond the mere classical expression (L^2), and accounts for the nontrivial angular momentum of the ground-state Bohr orbit. In such cases (including momentum-dependent potentials), even nontrivial $O(\hbar)$ quantum corrections in the kernel functions (which characterize different operator orderings) can be produced efficiently without direct, cumbersome consideration of operators [CZ02, Hie84]. More detailed discussion of the Weyl and alternate correspondences is provided in Sections 12 and 13.

In this sense, expectation values of the physical observables specified by kernel functions $g(x,p)$ are computed through integration with the WF, in close analogy with classical probability theory, except for the non-positive-definiteness of the distribution function. This operation corresponds to tracing an operator with the density matrix (cf. Section 12).

3 Solving for the Wigner Function

Given a specification of observables, the next step is to find the relevant WF for a given Hamiltonian. Can this be done without solving for the Schrödinger wave functions ψ, i.e. not using Schrödinger's equation directly? Indeed, the functional equations which f satisfies completely determine it.

Firstly, its dynamical evolution is specified by Moyal's equation. This is the extension of Liouville's theorem of classical mechanics, for a classical Hamiltonian $H(x,p)$, namely $\partial_t f + \{f,H\} = 0$, to quantum mechanics, in this language [Wig32, Moy49]:

$$\frac{\partial f}{\partial t} = \frac{H\star f - f\star H}{i\hbar} \equiv \{\!\{H,f\}\!\}\;, \tag{10}$$

where the $\star$-product [Gro46] is

$$\star \equiv e^{\frac{i\hbar}{2}(\overleftarrow{\partial}_x \overrightarrow{\partial}_p - \overleftarrow{\partial}_p \overrightarrow{\partial}_x)} \ . \tag{11}$$

The right-hand side of (10) is dubbed the "Moyal Bracket" (MB), and the quantum commutator is its Weyl correspondent. It is the essentially unique one-parameter ($\hbar$) associative deformation of the Poisson brackets of classical mechanics [Vey75, BFF78, FLS76, Ar83, Fle90, deW83, BCG97, TD97]. Expansion in $\hbar$ around 0 reveals that it consists of the Poisson bracket corrected by terms $O(\hbar)$.

The equation (10) also evokes Heisenberg's equation of motion for operators, except that H and f here are classical functions, and it is the $\star$-product which enforces noncommutativity. This language makes the link between quantum commutators and Poisson brackets more transparent.

Since the $\star$-product involves exponentials of derivative operators, it may be evaluated in practice through translation of function arguments ("Bopp shifts"),

$$f(x,p) \star g(x,p) = f\left(x + \frac{i\hbar}{2}\overrightarrow{\partial}_p,\ p - \frac{i\hbar}{2}\overrightarrow{\partial}_x\right)\ g(x,p). \tag{12}$$

The equivalent Fourier representation of the $\star$-product is [Neu31, Bak58]

$$\begin{aligned} f \star g = \frac{1}{\hbar^2\pi^2} \int dp' dp'' dx' dx''\ f(x',p')\ g(x'',p'') \\ \times \exp\left(\frac{-2i}{\hbar}\left[p(x'-x'') + p'(x''-x) + p''(x-x')\right]\right). \end{aligned} \tag{13}$$

An alternate integral representation of this product is [HOS84]

$$f\star g = (\hbar\pi)^{-2} \int dp' dp'' dx' dx''\ f(x+x',p+p')\ g(x+x'',p+p'')\ \exp\left[\frac{2i}{\hbar}\left(x'p'' - x''p'\right)\right], \tag{14}$$

which readily displays noncommutativity and associativity.

$\star$ multiplication of c-number phase-space functions is in complete isomorphism to Hilbert-space operator multiplication [Gro46],

$$\mathfrak{A}(\mathfrak{x},\mathfrak{p})\ \mathfrak{B}(\mathfrak{x},\mathfrak{p}) = \frac{1}{(2\pi)^2} \int d\tau d\sigma dx dp\ (a \star b)\ \exp[i\tau(\mathfrak{p}-p) + i\sigma(\mathfrak{x}-x)]. \tag{15}$$

The cyclic phase-space trace is directly seen in the representation (14) to reduce to a plain product, if there is only one $\star$ involved:

$$\int dpdx\ f \star g = \int dpdx\ fg = \int dpdx\ g \star f. \tag{16}$$

Moyal's equation is necessary, but does not suffice to specify the WF for a system. In the conventional formulation of quantum mechanics, systematic solution of time-dependent equations is usually predicated on the spectrum of stationary ones. Time-independent pure-state Wigner functions $\star$-commute with H, but clearly not every function $\star$-commuting with H can be a bona-fide WF (e.g. any $\star$ function of H will $\star$-commute with H).

Static WFs obey more powerful functional $\star$-genvalue equations [Fai64] (also see Refs. Kun67, Coh76, Dah83):

$$H(x,p)\star f(x,p) = H\left(x+\frac{i\hbar}{2}\vec{\partial}_p\ ,\ p-\frac{i\hbar}{2}\vec{\partial}_x\right)\ f(x,p)$$
$$= f(x,p)\star H(x,p) = E\ f(x,p)\ , \tag{17}$$

where E is the energy eigenvalue of $\mathfrak{H}\psi = E\psi$. These amount to a complete characterization of the WFs [CFZ98].

Lemma 1 For real functions $f(x,p)$, the Wigner form (4) for pure static eigenstates is equivalent to compliance with the $\star$-genvalue equations (17) ($\Re$ and $\Im$ parts).

Proof

$$\begin{aligned}
&H(x,p)\star f(x,p)\\
&=\frac{1}{2\pi}\left[(p-i\frac{\hbar}{2}\vec{\partial}_x)^2/2m+V(x)\right]\int dy\ e^{-iy(p+i\frac{\hbar}{2}\overleftarrow{\partial}_x)}\psi^*(x-\frac{\hbar}{2}y)\ \psi(x+\frac{\hbar}{2}y)\\
&=\frac{1}{2\pi}\int dy\ \left[(p-i\frac{\hbar}{2}\vec{\partial}_x)^2/2m+V(x+\frac{\hbar}{2}y)\right]e^{-iyp}\psi^*(x-\frac{\hbar}{2}y)\ \psi(x+\frac{\hbar}{2}y)\\
&=\frac{1}{2\pi}\int dy\ e^{-iyp}\left[(i\,\vec{\partial}_y+i\frac{\hbar}{2}\vec{\partial}_x)^2/2m+V(x+\frac{\hbar}{2}y)\right]\psi^*(x-\frac{\hbar}{2}y)\ \psi(x+\frac{\hbar}{2}y)\\
&=\frac{1}{2\pi}\int dy\ e^{-iyp}\psi^*(x-\frac{\hbar}{2}y)\ E\ \psi(x+\frac{\hbar}{2}y)\\
&=E\ f(x,p).
\end{aligned}\tag{18}$$

Action of the effective differential operators on ψ^* turns out to be null.

Symmetrically,

$$\begin{aligned}
&f\star H\\
&\quad=\frac{1}{2\pi}\int dy\ e^{-iyp}\left[-\frac{1}{2m}(\vec{\partial}_y-\frac{\hbar}{2}\vec{\partial}_x)^2+V(x-\frac{\hbar}{2}y)\right]\psi^*(x-\frac{\hbar}{2}y)\ \psi(x+\frac{\hbar}{2}y)\\
&\quad=E\ f(x,p),
\end{aligned}\tag{19}$$

where the action on ψ is now trivial.

Conversely, the pair of $\star$-eigenvalue equations dictate, for $f(x,p)=\int dy\ e^{-iyp}\tilde{f}(x,y)$,

$$\int dy\ e^{-iyp}\left[-\frac{1}{2m}(\vec{\partial}_y\pm\frac{\hbar}{2}\vec{\partial}_x)^2+V(x\pm\frac{\hbar}{2}y)-E\right]\tilde{f}(x,y)=0. \tag{20}$$

Hence, real solutions of (17) must be of the form $f=\int dy\ e^{-iyp}\psi^*(x-\frac{\hbar}{2}y)\psi(x+\frac{\hbar}{2}y)/2\pi$, such that $\mathfrak{H}\psi=E\psi$. $\square$

Equation (17) lead to spectral properties for WFs [Fai64, CFZ98], as in the Hilbert space formulation. For instance, projective orthogonality of the $\star$ genfunctions follows from associativity, which allows evaluation in two alternate groupings:

$$f\star H\star g = E_f\ f\star g = E_g\ f\star g. \tag{21}$$

Thus, for $E_g \neq E_f$, it is necessary that

$$f \star g = 0. \tag{22}$$

Moreover, precluding degeneracy (which can be treated separately), choosing $f = g$ above yields

$$f \star H \star f = E_f \; f \star f = H \star f \star f, \tag{23}$$

and hence $f \star f$ must be the stargenfunction in question,

$$f \star f \propto f. \tag{24}$$

Pure state fs then $\star$-project onto their space. In general, it can be shown [Tak54, CFZ98] that, for a pure state,

$$f_a \star f_b = \frac{1}{h} \; \delta_{a,b} \; f_a \; . \tag{25}$$

The normalization matters [Tak54]: despite linearity of the equations, it prevents superposition of solutions. (Quantum mechanical interference works differently here, in comportance with density matrix formalism.)

By virtue of (16), for different $\star$-genfunctions, the above dictates that

$$\int dpdx \; fg = 0. \tag{26}$$

Consequently, unless there is zero overlap for all such WFs, at least one of the two must go negative someplace to offset the positive overlap [HOS84, Coh95]—an illustration of the feature of negative values. This feature is an asset and not a liability.

Further, note that integrating (17) yields the expectation of the energy,

$$\int H(x,p) f(x,p) \; dxdp = E \int f \; dxdp = E. \tag{27}$$

Likewise,[d] note that integrating the above projective condition yields

$$\int dxdp \; f^2 = \frac{1}{h} \; , \tag{28}$$

i.e. the overlap increases to a divergent result in the classical limit, as the WFs grow increasingly spiky.

[d]This discussion applies to proper WFs, corresponding to pure states' density matrices. E.g. a sum of two WFs is not a pure state in general, and does not satisfy the condition (6). For such generalizations, the *impurity* is [Gro46] $1 - h\langle f \rangle = \int dxdp \; (f - hf^2) \geq 0$, where the inequality is only saturated into an equality for a pure state. For instance, for $w \equiv (f_a + f_b)/2$ with $f_a \star f_b = 0$, the impurity is nonvanishing, $\int dxdp \; (w - hw^2) = 1/2$. A pure state affords a maximum of information, while the impurity is a measure of lack of information [Fan57, Tak54]—it is the dominant term in the expansion of the quantum entropy around a pure state [Bra94].

4 The Uncertainty Principle

In classical (non-negative) probability distribution theory, expectation values of non-negative functions are likewise non-negative, and thus result in standard constraint inequalities for the constituent pieces of such functions, e.g., moments of the variables. But it was just seen that for WFs which go negative for an arbitrary function g, $\langle |g|^2 \rangle$ need not be ≥ 0. This can be easily seen by choosing the support of g to lie mostly in those regions of phase-space where the WF f is negative.

Still, such constraints are not lost for WFs. It turns out they are replaced by:

Lemma 2

$$\langle g^* \star g \rangle \geq 0. \tag{29}$$

In Hilbert space operator formalism, this relation would correspond to the positivity of the norm. This expression is non-negative because it involves a real non-negative integrand for a pure state WF satisfying the above projective condition[e],

$$\int dpdx (g^* \star g) f = h \int dxdp (g^* \star g)(f \star f) = h \int dxdp (f \star g^*) \star (g \star f) = h \int dxdp |g \star f|^2. \tag{30}$$

□

To produce Heisenberg's uncertainty relation [CZ01], one only needs to choose

$$g = a + bx + cp, \tag{31}$$

for arbitrary complex coefficients a, b, c. The resulting positive semi-definite quadratic form is then

$$a^* a + b^* b \langle x \star x \rangle + c^* c \langle p \star p \rangle + (a^* b + b^* a) \langle x \rangle + (a^* c + c^* a) \langle p \rangle + c^* b \langle p \star x \rangle + b^* c \langle x \star p \rangle \geq 0 \ , \tag{32}$$

for any a, b, c. The eigenvalues of the corresponding matrix are then non-negative, and thus so must be its determinant. Given

$$x \star x = x^2, \qquad p \star p = p^2, \qquad p \star x = px - \frac{i\hbar}{2} \ , \qquad x \star p = px + \frac{i\hbar}{2} \ , \tag{33}$$

and the usual

$$(\Delta x)^2 \equiv \langle (x - \langle x \rangle)^2 \rangle, \qquad (\Delta p)^2 \equiv \langle (p - \langle p \rangle)^2 \rangle, \tag{34}$$

this condition on the 3×3 matrix determinant amounts to

$$(\Delta x)^2 \ (\Delta p)^2 \geq \frac{\hbar^2}{4} + \Big(\langle (x - \langle x \rangle)(p - \langle p \rangle) \rangle \Big)^2, \tag{35}$$

and hence

$$\Delta x \ \Delta p \geq \frac{\hbar}{2} \ . \tag{36}$$

[e]Similarly, if f_1 and f_2 are pure state WFs, the transition probability $(|\int dx \psi_1^*(x) \psi_2(x)|^2)$ between the respective states is also non-negative [OW81], manifestly by the same argument [CZ01], namely $\int dpdx f_1 f_2 = (2\pi\hbar)^2 \int dxdp \ |f_1 \star f_2|^2 \geq 0$.

The $\hbar$ entered into the moments' constraint through the action of the $\star$ product [CZ01]. More general choices of g likewise lead to diverse expectations' inequalities in phase space; e.g. in six-dimensional phase space, the uncertainty for $g = a + bL_x + cL_y$ requires $l(l+1) \geq m(m+1)$, and hence $l \geq m$, etc. [CZ01, CZ02]. For a more extensive formal discussion of moments, cf. Ref. NO86.

5 Ehrenfest's Theorem

Moyal's equation (10),

$$\frac{\partial f}{\partial t} = \{\!\!\{ H, f \}\!\!\} \ , \tag{37}$$

serves to prove Ehrenfest's theorem for expectation values. For any phase-space function $k(x,p)$ with no explicit time dependence,

$$\begin{aligned} \frac{d\langle k\rangle}{dt} &= \int dx dp \ \frac{\partial f}{\partial t} k \\ &= \frac{1}{i\hbar}\int dx dp \ (H\star f - f\star H)\star k \\ &= \int dx dp \ f\{\!\!\{ k, H\}\!\!\} = \langle \{\!\!\{ k, H\}\!\!\}\rangle \,. \end{aligned} \tag{38}$$

(Any convective time-dependence, $\int dxdp\ [\dot{x}\partial_x\ (fk) + \dot{p}\ \partial_p(fk)]$, amounts to an ignorable surface term, $\int dxdp\ [\partial_x(\dot{x}fk) + \partial_p(\dot{p}fk)]$, by the x, p equations of motion.)

Note the ostensible sign difference between the correspondent to Heisenberg's equation,

$$\frac{dk}{dt} = \{\!\!\{ k, H\}\!\!\} \ , \tag{39}$$

and Moyal's equation above. The x, p equations of motion reduce to the classical ones of Hamilton, $\dot{x} = \partial_p H$, $\dot{p} = -\partial_x H$.

Moyal [Moy49] stressed that his eponymous quantum evolution equation (10) contrasts to Liouville's theorem for classical phase-space densities,

$$\frac{df_{\text{cl}}}{dt} = \frac{\partial f_{\text{cl}}}{\partial t} + \dot{x}\ \partial_x f_{\text{cl}} + \dot{p}\ \partial_p f_{\text{cl}} = 0 \ . \tag{40}$$

Specifically, unlike its classical counterpart, in general, f does not flow like an incompressible fluid in phase space.

For an arbitrary region Ω about a representative point in phase space,

$$\frac{d}{dt}\int_\Omega dxdp\ f = \int_\Omega dxdp \left[\frac{\partial f}{\partial t} + \partial_x(\dot{x}f) + \partial_p(\dot{p}f)\right] = \int_\Omega dxdp\ (\{\!\!\{ H, f\}\!\!\} - \{H, f\}) \neq 0 \ . \tag{41}$$

That is, the phase-space region does not conserve in time the number of points swarming about the representative point: points diffuse away, in general, without maintaining the density of the quantum quasi-probability fluid, and, conversely, they are not prevented from coming together, in contrast to deterministic flow. For infinite Ω encompassing the entire phase space, both surface terms above vanish to yield a time-invariant normalization for the WF. The $O(\hbar^2)$ higher momentum derivatives of the WF present in the MB (but absent in the PB—higher space derivatives probing nonlinearity in the potential) modify the Liouville flow into characteristic quantum configurations [KZZ02, FBA96, ZP94].

6 Illustration: The Harmonic Oscillator

To illustrate the formalism on a simple prototype problem, one may look at the harmonic oscillator. In the spirit of this picture, one can, in fact, eschew solving the Schrödinger problem and plugging the wave functions into (4); instead, one may solve (17) directly for $H = (p^2 + x^2)/2$ (with $m = 1$, $\omega = 1$):

$$\left[(x + \frac{i\hbar}{2}\partial_p)^2 + (p - \frac{i\hbar}{2}\partial_x)^2 - 2E\right] f(x,p) = 0. \tag{42}$$

For this Hamiltonian, the equation has collapsed to two simple PDEs. The first one, the Imaginary part,

$$(x\partial_p - p\partial_x)f = 0, \tag{43}$$

restricts f to depend on only one variable, the scalar in phase space, $z = 4H/\hbar = 2(x^2 + p^2)/\hbar$. Thus the second one, the Real part, is a simple ODE,

$$\left(\frac{z}{4} - z\partial_z^2 - \partial_z - \frac{E}{\hbar}\right) f(z) = 0. \tag{44}$$

Setting $f(z) = \exp(-z/2)L(z)$ yields Laguerre's equation,

$$\left[z\partial_z^2 + (1 - z)\partial_z + \frac{E}{\hbar} - \frac{1}{2}\right] L(z) = 0. \tag{45}$$

It is solved by Laguerre polynomials,

$$L_n = \frac{1}{n!} e^z \, \partial_z^n (e^{-z} z^n) \,, \tag{46}$$

for $n = E/\hbar - 1/2 = 0, 1, 2, \ldots$, so the $\star$ gen–Wigner functions are [Gro46]

$$f_n = \frac{(-1)^n}{\pi\hbar} \, e^{-2H/\hbar} \, L_n \left(\frac{4H}{\hbar}\right) ;$$
$$L_0 = 1, \quad L_1 = 1 - \frac{4H}{\hbar}, \quad L_2 = \frac{8H^2}{\hbar^2} - \frac{8H}{\hbar} + 1, \ldots . \tag{47}$$

But for the Gaussian ground state, they all have zeros and go negative. These functions become spiky in the classical limit $\hbar \to 0$; e.g. the ground state Gaussian f_0 goes to a δ function.

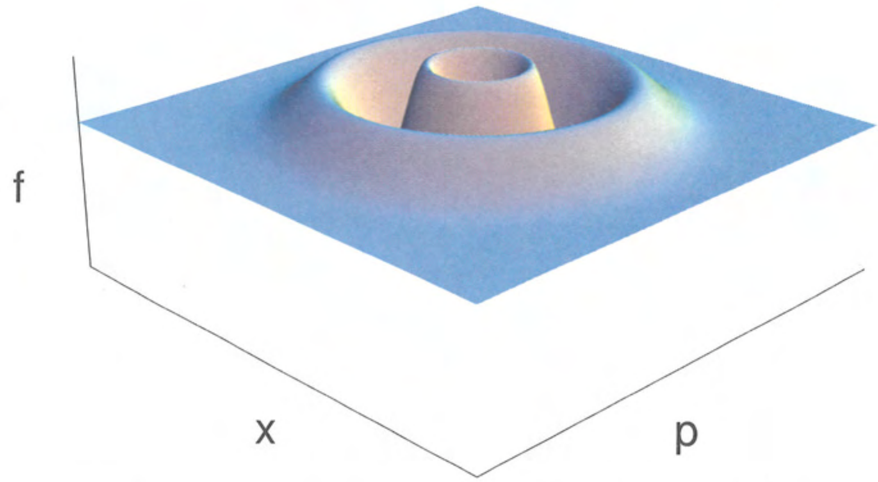

Figure 2. The oscillator WF for the third excited state. Note the negative values.

Their sum provides a resolution of the identity [Moy49],

$$\sum_n f_n = \frac{1}{h} \ . \tag{48}$$

(For the rest of this section, set $\hbar = 1$, for algebraic simplicity.)

Dirac's Hamiltonian factorization method for the alternate algebraic solution of the same problem carries through intact, with $\star$-multiplication supplanting operator multiplication. That is to say,

$$H = \frac{1}{2}(x - ip) \star (x + ip) + \frac{1}{2} \ . \tag{49}$$

This motivates definition of raising and lowering functions (not operators)

$$a \equiv \frac{1}{\sqrt{2}}(x + ip), \qquad\qquad a^\dagger \equiv \frac{1}{\sqrt{2}}(x - ip), \tag{50}$$

where

$$a \star a^\dagger - a^\dagger \star a = 1 \ . \tag{51}$$

The annihilation ones $\star$-annihilate the $\star$ Fock vacuum:

$$a \star f_0 = \frac{1}{\sqrt{2}}(x + ip) \star e^{-(x^2+p^2)} = 0 \ . \tag{52}$$

Thus, the associativity of the $\star$-product permits the customary ladder spectrum generation [CFZ98]. The $\star$-genstates for $H \star f = f \star H$ are

$$f_n = \frac{1}{n!}(a^\dagger \star)^n \ f_0 \ (\star a)^n \ . \tag{53}$$

They are manifestly real, like the Gaussian ground state, and left–right-symmetric; it is easy to see they are $\star$-orthogonal for different eigenvalues. Likewise, they can be seen by

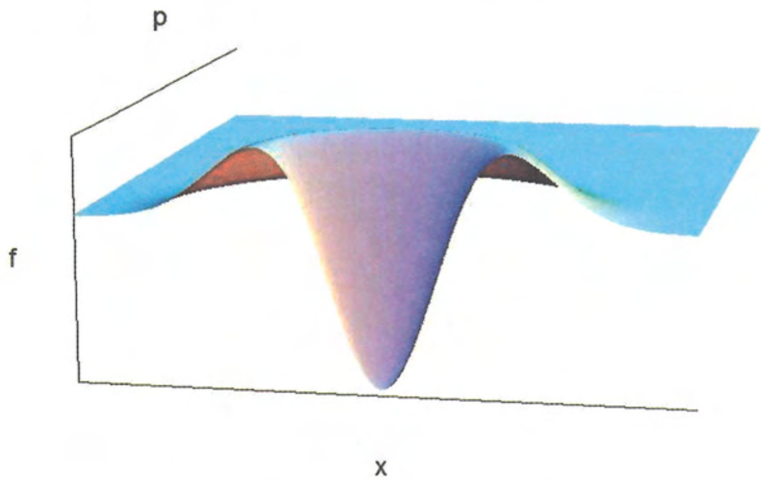

Figure 3. Section of the WF for the first excited state. Note the negative values.

the evident algebraic normal ordering to project to themselves, since the Gaussian ground state does, $f_0 \star f_0 = f_0/h$. The corresponding coherent state WFs [KL01, CUZ01, Har01, DG80] are likewise analogous to the conventional formulation.

This type of analysis carries over well to a broader class of problems [CFZ98] with "essentially isospectral" pairs of partner potentials, connected with each other through Darboux transformations relying on Witten superpotentials W (cf. the Pöschl–Teller potential [Ant01]). It closely parallels the standard differential operator structure of the recursive technique. That is, the pairs of related potentials and corresponding $\star$-genstate Wigner functions are constructed recursively [CFZ98] through ladder operations analogous to the algebraic method outlined above for the oscillator.

Beyond such recursive potentials, examples of further simple systems where the $\star$-genvalue equations can be solved on first principles are the linear potential [GM80, CFZ98, TZM96], the exponential interaction Liouville potentials, and their supersymmetric Morse generalizations [CFZ98]. (Also see Refs. Fra00, CH86, HL99, KL94.)

Further systems may be handled through the Chebyshev-polynomial numerical techniques of Ref. HMS98.

First-principles phase-space solution of the hydrogen atom is less than straightforward and complete. The reader is referred to Refs. BFF78, Bon84, DS82, CH87 for significant partial results.

Algebraic methods of generating spectra of quantum-integrable models are described in Ref. CZ02.

7 Time Evolution

Moyal's equation (10) is formally solved by virtue of associative combinatoric operations completely analogous to Hilbert space quantum mechanics, through definition of a $\star$-unitary evolution operator, a "$\star$-exponential" [BFF78]

$$U_\star(x,p;t) = e_\star^{itH/\hbar} \equiv 1 + (it/\hbar)H(x,p) + \frac{(it/\hbar)^2}{2!} H \star H + \frac{(it/\hbar)^3}{3!} H \star H \star H + ..., \quad (54)$$

for arbitrary Hamiltonians. The solution to Moyal's equation, given the WF at $t = 0$, then, is

$$f(x,p;t) = U_\star^{-1}(x,p;t) \star f(x,p;0) \star U_\star(x,p;t). \quad (55)$$

In general, just like any $\star$-function of H, the $\star$-exponential (54) resolves spectrally [Bon84]:

$$\exp_\star\left(\frac{it}{\hbar}H\right) = \exp_\star\left(\frac{it}{\hbar}H\right) \star 1 = \exp_\star\left(\frac{it}{\hbar}H\right) \star 2\pi\hbar \sum_n f_n = 2\pi\hbar \sum_n e^{itE_n/\hbar} f_n \ . \quad (56)$$

(Of course, for $t = 0$, the obvious identity resolution is recovered.) In turn, any particular $\star$-genfunction is projected out formally by

$$\int dt \exp_\star\left[\frac{it}{\hbar}(H - E_m)\right] = (2\pi\hbar)^2 \sum_n \delta(E_n - E_m)\ f_n \ \propto f_m \ , \quad (57)$$

which is manifestly seen to be a $\star$-function.

For oscillator $\star$-genfunctions, the $\star$-exponential (56) is directly seen to sum to

$$\exp_\star\left(\frac{itH}{\hbar}\right) = \left[\cos(\frac{t}{2})\right]^{-1} \exp\left[\frac{2i}{\hbar} H \tan(\frac{t}{2})\right], \quad (58)$$

which is to say, a Gaussian [BFF78] in phase space.[f]

For the variables x and p, the evolution equations collapse to mere *classical* trajectories,

$$\frac{dx}{dt} = \frac{x \star H - H \star x}{i\hbar} = \partial_p H = p \ , \quad (59)$$

$$\frac{dp}{dt} = \frac{p \star H - H \star p}{i\hbar} = -\partial_x H = -x \ , \quad (60)$$

where the concluding member of these two equations hold for the oscillator only. Thus, for the oscillator,

$$x(t) = x\cos t + p \sin t, \qquad p(t) = p\cos t - x \sin t. \quad (61)$$

As a consequence, for the oscillator, the functional form of the Wigner function is preserved along classical phase-space trajectories [Gro46]:

$$f(x,p;t) = f(x\cos t - p\sin t, p\cos t + x\sin t; 0). \quad (62)$$

[f] As an application, note that the celebrated hyperbolic tangent $\star$-composition law of Gaussians follows trivially, since these amount to $\star$-exponentials with additive time intervals, $\exp_\star(tf) \star \exp_\star(Tf) = \exp_\star[(t+T)f)]$, [BFF78]. That is,

$$\exp\left[-\frac{a}{\hbar}(x^2+p^2)\right] \ \star \ \exp\left[-\frac{b}{\hbar}(x^2+p^2)\right] = \frac{1}{1+ab} \exp\left[-\frac{a+b}{\hbar(1+ab)}(x^2+p^2)\right].$$

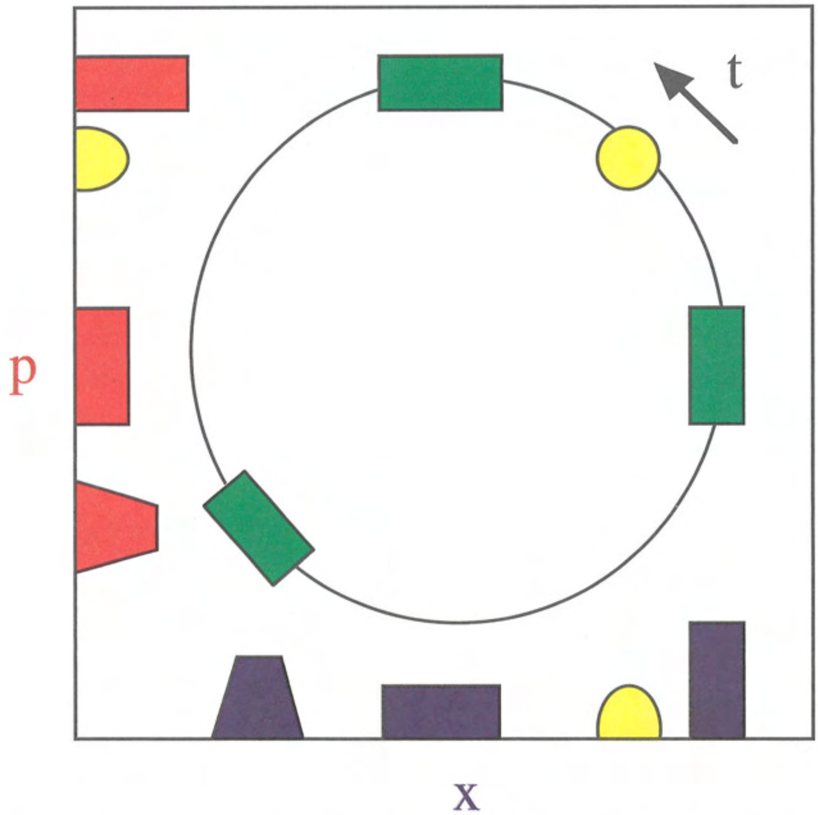

Figure 4. Time evolution of generic WF configurations driven by an oscillator Hamiltonian. The t-arrow indicates the rotation sense of x and p, and so, for fixed x and p axes, the WF shoebox configurations rotate rigidly in the opposite direction, clockwise. (The sharp angles of the WFs in the illustration are actually unphysical, and were only chosen to monitor their "spreading wavepacket" projections more conspicuously.) These x- and p-projections (shadows) are meant to be intensity profiles on those axes, but are expanded on the plane to aid visualization. The circular figure represents a coherent state, which projects on either axis identically at all times, thus without shape alteration of its wavepacket through time evolution.

Any oscillator WF configuration rotates uniformly on the phase plane around the origin, in essence classically (cf. Fig. 4), even though it provides a complete quantum-mechanical description [Gro46, BM49, Wig32, Les84, CZ99, ZC99].

Naturally, this rigid rotation in phase-space preserves areas, and thus automatically illustrates the uncertainty principle. By contrast, in general, in the conventional formulation of quantum mechanics, this result is deprived of intuitive import, or, at the very least, simplicity: upon integration in x (or p) to yield usual marginal probability densities, the rotation induces apparent complicated shape variations of the oscillating probability density profile, such as wavepacket spreading (as evident in the shadow projections on the x and p axes of Fig. 4).

Only when (as is the case for coherent states [HSD95, Sam00)] a Wigner function configuration has an *additional* axial $x-p$ symmetry around its *own* center, will it possess an invariant profile upon this rotation, and hence a shape-invariant oscillating probability density [ZC99].

In Dirac's interaction representation, a more complicated interaction Hamiltonian superposed on the oscillator one leads to shape changes of the WF configurations placed on the above "turntable," and serves to generalize to scalar field theory [CZ99].

8 Nondiagonal Wigner Functions

More generally, to represent all operators on phase space in a selected basis, one looks at the Weyl-correspondents of arbitrary $|a\rangle\langle b|$, referred to as *nondiagonal WFs* [Gro46]. These enable investigation of interference phenomena and the transition amplitudes in the formulation of quantum-mechanical perturbation theory [BM49, WO88, CUZ01].

Both the diagonal and the non-diagonal WFs are represented in (2), by replacing $\boldsymbol{\rho} \rightarrow |\psi_a\rangle\langle\psi_b|$:

$$\begin{aligned} f_{ba}(x,p) &\equiv \frac{1}{2\pi}\int dy\; e^{-iyp}\left\langle x+\frac{\hbar}{2}y\right|\psi_a\Big\rangle\;\Big\langle\psi_b\left|\,x-\frac{\hbar}{2}y\right\rangle \\ &= \frac{1}{2\pi}\int dy e^{-iyp}\psi_b^*\left(x-\frac{\hbar}{2}y\right)\psi_a\left(x+\frac{\hbar}{2}y\right) = f_{ab}^*(x,p) \\ &= \psi_a(x)\star\delta(p)\star\psi_b^*(x)\ , \end{aligned} \tag{63}$$

The representation on the last line is due to Ref. Bra94 and lends itself to a more compact and elegant proof of Lemma 1. Just as pure-state diagonal WFs obey a projection condition, so too the non-diagonals. For wave functions which are orthonormal for discrete state labels, $\int dx\,\psi_a^*(x)\psi_b(x)=\delta_{ab}$, the transition amplitude collapses to

$$\int dxdp\, f_{ab}\,(x,p) = \delta_{ab}\ . \tag{64}$$

To perform spectral operations analogous to those of Hilbert space, it is useful to note that these WFs are $\star$-orthogonal [Fai64],

$$(2\pi\hbar)\ f_{ba}\star f_{dc} = \delta_{bc} f_{da}\ , \tag{65}$$

as well as complete [Moy49] for integrable functions on phase space,

$$(2\pi\hbar)\sum_{a,b} f_{ab}\,(x_1,p_1)\ f_{ba}\,(x_2,p_2) = \delta\,(x_1-x_2)\ \delta\,(p_1-p_2)\ . \tag{66}$$

For example, for the SHO in one dimension, non-diagonal WFs are

$$f_{kn} = \frac{1}{\sqrt{n!k!}}\,(a^*\star)^n\ f_0\ (\star a)^k\,, \qquad f_0 = \frac{1}{\pi\hbar}e^{-(x^2+p^2)/\hbar} \tag{67}$$

(cf. coherent states [CUZ01, DG80)]. Explicitly, in terms of associated Laguerre polynomials, these are [Gro46, BM49, Fai64]

$$f_{kn} = \sqrt{\frac{k!}{n!}}\ e^{i(n-k)\ \arctan(p/x)}\,\frac{(-1)^k}{\pi\hbar}\left(\frac{x^2+p^2}{\hbar/2}\right)^{(n-k)/2} L_k^{n-k}\left(\frac{x^2+p^2}{\hbar/2}\right)\,e^{-(x^2+p^2)/\hbar}. \tag{68}$$

The SHO nondiagonal WFs are direct solutions to [Fai64]

$$H \star f_{kn} = E_n\, f_{kn}\ , \qquad\qquad f_{kn} \star H = E_k\, f_{kn}\ . \tag{69}$$

The energy $\star$-genvalue conditions are $\left(E_n - \frac{1}{2}\right)/\hbar = n$, an integer, and $\left(E_k - \frac{1}{2}\right)/\hbar = k$, also an integer.

The general spectral theory of WFs is covered in Refs. BFF78, FM91, Lie90, BDW99, CUZ01.

9 Stationary Perturbation Theory

Given the spectral properties summarized, the phase-space perturbation formalism is self-contained: it need not make reference to the Hilbert-space treatment [BM49, WO88, CUZ01, SS02, MS96].

For a perturbed Hamiltonian,

$$H\,(x,p) = H_0(x,p) + \lambda\ H_1(x,p)\ , \tag{70}$$

seek a formal series solution,

$$f_n\,(x,p) = \sum_{k=0}^{\infty} \lambda^k f_n^{(k)}(x,p), \qquad\qquad E_n = \sum_{k=0}^{\infty} \lambda^k E_n^{(k)}, \tag{71}$$

of the left–right-$\star$-genvalue equations (17), $H \star f_n = E_n f_n = f_n \star H$.

Matching powers of λ in the eigenvalue equation [CUZ01],

$$E_n^{(0)} = \int dxdp\, f_n^{(0)}(x,p)\ H_0(x,p), \qquad\qquad E_n^{(1)} = \int dxdp\, f_n^{(0)}(x,p)\ H_1(x,p), \tag{72}$$

$$\begin{aligned} f_n^{(1)}\,(x,p) = &\sum_{k\neq n} f_{kn}^{(0)}\,(x,p) \int dXdP\, f_{nk}^{(0)}\,(X,P)\ H_1\,(X,P) \\ &+ \sum_{k\neq n} \frac{1}{E_n^{(0)} - E_k^{(0)}} f_{nk}^{(0)}\,(x,p) \int dXdP\, f_{kn}^{(0)}\,(X,P)\ H_1\,(X,P)\ . \end{aligned} \tag{73}$$

For example, consider all polynomial perturbations of the harmonic oscillator in a unified treatment, by choosing

$$H_1 = e^{\gamma x + \delta p} = e_\star^{\gamma x + \delta p} = \left(e^{\gamma x} \star e^{\delta p}\right)\, e^{i\gamma\delta/2} = \left(e^{\delta p} \star e^{\gamma x}\right)\, e^{-i\gamma\delta/2}\ , \tag{74}$$

to evaluate a generating function for all the first-order corrections to the energies [CUZ01],

$$E^{(1)}(s) \equiv \sum_{n=0}^{\infty} s^n E_n^{(1)} = \int dxdp \sum_{n=0}^{\infty} s^n f_n^{(0)} H_1\ ; \tag{75}$$

hence

$$E_n^{(1)} = \frac{1}{n!}\,\frac{d^n}{ds^n}\ E^{(1)}(s)\bigg|_{s=0}\ . \tag{76}$$

From the spectral resolution (56) and the explicit form of the $\star$-exponential of the oscillator Hamiltonian (58) [with $e^{it} \to s$ and $E_n^{(0)} = \left(n + \frac{1}{2}\right)\hbar$], it follows that

$$\sum_{n=0}^{\infty} s^n f_n^{(0)} = \frac{1}{\pi\hbar(1+s)} \exp\left(\frac{x^2+p^2}{\hbar} \frac{s-1}{s+1}\right) , \tag{77}$$

and hence

$$\begin{aligned} E^{(1)}(s) &= \frac{1}{\pi\hbar\,(1+s)} \int dxdp\, e^{\gamma x + \delta p} \exp\left(-\frac{x^2+p^2}{\hbar} \frac{1-s}{1+s}\right) \\ &= \frac{1}{1-s} \exp\left[\frac{\hbar}{4}\left(\gamma^2+\delta^2\right)\frac{1+s}{1-s}\right] . \end{aligned} \tag{78}$$

For example, specifically,

$$\begin{aligned} E_0^{(1)} &= \exp\frac{\hbar}{4}\left(\gamma^2+\delta^2\right) , \qquad E_1^{(1)} = \left[1+\frac{\hbar}{2}\left(\gamma^2+\delta^2\right)\right] E_0^{(1)} , \\ E_2^{(1)} &= \left[1+\hbar\left(\gamma^2+\delta^2\right)+\frac{\hbar^2}{8}\left(\gamma^2+\delta^2\right)^2\right] E_0^{(1)} , \end{aligned} \tag{79}$$

and so on. All the first order corrections to the energies are even functions of the parameters—only even functions of x and p can contribute to first-order shifts in the oscillator energies.

First-order corrections to the WFs may be similarly calculated using generating functions for nondiagonal WFs. Higher order corrections are straightforward but tedious. Degenerate perturbation theory also has an autonomous formulation in phase-space, equivalent to Hilbert space and path-integral treatments.

10 Propagators

Time evolution of general WFs beyond the above treatment is discussed at length in Refs. BM49, Ber75, GM80, CUZ01, BR93, Wo82, Wo02, FM03. A further application of the spectral techniques outlined is the computation of the WF time-evolution operator from the propagator for wave functions, which is given as a bilinear sum of energy eigenfunctions,

$$G(x,X;t) = \sum_a \psi_a(x)\, e^{-iE_a t/\hbar}\, \psi_a^*(X) \equiv \exp\left[iA_{\text{eff}}(x,X;t)\right] , \tag{80}$$

as it may be thought of as an exponentiated effective action. (Henceforth in this section, take $\hbar = 1$.)

This leads directly to a similar bilinear double sum for the WF time-transformation kernel [Moy49],

$$T(x,p;X,P;t) = 2\pi \sum_{a,b} f_{ba}(x,p)\, e^{-i(E_a-E_b)t} f_{ab}(X,P) . \tag{81}$$

Defining a "big star" operation as a $\star$-product for the upper-case (initial) phase-space variables,

$$\bigstar \equiv e^{\frac{i\hbar}{2}(\overleftarrow{\partial}_X \overrightarrow{\partial}_P - \overleftarrow{\partial}_P \overrightarrow{\partial}_X)} , \tag{82}$$

it follows that

$$T(x,p;X,P;t)\bigstar f_{dc}(X,P) = \sum_b f_{bc}(x,p)\, e^{-i(E_c-E_b)t} f_{db}(X,P) , \tag{83}$$

and hence [cf. (55)],

$$\int dXdP\, T(x,p;X,P;t) f_{dc}(X,P) = f_{dc}(x,p)e^{-i(E_c-E_d)t} = U_\star^{-1}\star f_{dc}(x,p;0)\star U_\star = f_{dc}(x,p;t). \tag{84}$$

For example, for a free particle of unit mass in one dimension, $H = p^2/2$, WFs propagate according to

$$\begin{aligned} &T_{\text{free}}\,(x,p;X,P;t) \\ &= \frac{1}{2\pi}\int dk \int dq\, e^{i(k-q)x}\,\delta\left[p-\frac{1}{2}(k+q)\right]\, e^{-i(q^2-k^2)t/2}\, e^{-i(k-q)X}\,\delta\left[P-\frac{1}{2}(k+q)\right] \\ &= \delta\,(x-X-Pt)\,\delta\,(p-P) , \end{aligned} \tag{85}$$

amounting to "classical" motion,

$$f(x,p;t) = f(x-pt,p;0) . \tag{86}$$

11 Canonical Transformations

Canonical transformations $(x,p) \mapsto [X(x,p), P(x,p)]$ preserve the phase-space volume (area) element (again, take $\hbar = 1$) through a trivial Jacobian,

$$dXdP = dxdp\ \{X,P\} , \tag{87}$$

i.e. they preserve Poisson brackets

$$\{u,v\}_{xp} \equiv \frac{\partial u}{\partial x}\frac{\partial v}{\partial p} - \frac{\partial u}{\partial p}\frac{\partial v}{\partial x} , \tag{88}$$

$$\{X,P\}_{xp} = 1, \qquad \{x,p\}_{XP} = 1. \tag{89}$$

Upon quantization, the c-number function Hamiltonian transforms "classically," $\mathcal{H}(X,P) \equiv H(x,p)$, like a scalar. Does the $\star$-product remain invariant under this transformation?

Yes, for *linear* canonical transformations [KL01], but clearly *not for general canonical transformations*. Still, things can be put right, by devising general *covariant* transformation rules for the $\star$-product [CFZ98]: the WF transforms in comportance with Dirac's quantum canonical transformation theory [Dir33].

In conventional quantum mechanics, for classical canonical transformations generated by $F_{\rm cl}(x,X)$,

$$p=\frac{\partial F_{\rm cl}(x,X)}{\partial x}~, \qquad P=-\frac{\partial F_{\rm cl}(x,X)}{\partial X}~, \tag{90}$$

the energy eigenfunctions transform in a generalization of the "representation-changing" Fourier transform [Dir33],

$$\psi_E(x)=N_E\int dX\, e^{iF(x,X)}\,\Psi_E(X)~. \tag{91}$$

(In this expression, the generating function F may contain $\hbar$ corrections [BCT82] to the classical one, in general—but for several simple quantum-mechanical systems it manages not to [CG92, DG02].) Hence [CFZ98], there is a transformation functional for WFs, $\mathcal{T}(x,p;X,P)$, such that

$$f(x,p)=\int dXdP\,\mathcal{T}(x,p;X,P)\bigstar\mathcal{F}(X,P)=\int dXdP\,\mathcal{T}(x,p;X,P)\;\mathcal{F}(X,P)~, \tag{92}$$

where

$$\begin{aligned}&\mathcal{T}\;(x,p;X,P)\\ &=\frac{|N|^2}{2\pi}\int dYdy\;\exp\left[-iyp+iPY-iF^*(x-\frac{y}{2},X-\frac{Y}{2})+iF(x+\frac{y}{2},X+\frac{Y}{2})\right].\end{aligned} \tag{93}$$

Moreover, it can be shown that [CFZ98],

$$H(x,p)\star\mathcal{T}(x,p;X,P)=\mathcal{T}(x,p;X,P)\bigstar\;\mathcal{H}(X,P). \tag{94}$$

That is, if $\mathcal{F}$ satisfies a $\bigstar$-genvalue equation, then f satisfies a $\star$-genvalue equation with the same eigenvalue, and vice versa. This proves useful in constructing WFs for simple systems which can be trivialized classically through canonical transformations.

A thorough discussion of MB automorphisms may start from Ref. BCW02 . (Also see Refs. Hie82, DKM88, GR94, DV97, Hak99, KL99, DP01.)

Time evolution is a canonical transformation [Dir33], with the generator's role played by the effective action A of the previous section, incorporating quantum corrections to both phases and normalizations; it connects initial wave functions to those at a final time.

For example, for the linear potential, with

$$H=p^2+x~, \tag{95}$$

wave function evolution is determined by the propagator

$$\exp\left[iA_{\rm lin}(x,X;t)\right]=\frac{1}{\sqrt{4\pi it}}\exp\left[\frac{i\,(x-X)^2}{4t}-\frac{i\,(x+X)\,t}{2}-\frac{it^3}{12}\right]. \tag{96}$$

T then evaluates to

$$\begin{aligned}&\mathcal{T}_{\rm lin}\;(x,p;X,P;t)\\ &=\frac{1}{2\pi}\int dYdy\;\exp\left[-iyp+iPY-iA^*_{\rm lin}(x-\frac{y}{2},X-\frac{Y}{2};t)+iA_{\rm lin}(x+\frac{y}{2},X+\frac{Y}{2};t)\right]\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{8\pi^2 t}\int dY dy \, \exp\left[-iyp + iPY - \frac{it}{2}(y+Y) + \frac{i}{2t}(x-X)(y-Y)\right] \\
&= \frac{1}{2t}\delta\left(p + \frac{t}{2} - \frac{x-X}{2t}\right)\delta\left(P - \frac{t}{2} - \frac{x-X}{2t}\right) \\
&= \delta\left(p+t-P\right)\delta\left(x - 2tp - t^2 - X\right) \\
&= \delta\left(x - X - (p+P)\,t\right)\delta\left(P - p - t\right). \qquad (97)
\end{aligned}
$$

The δ functions enforce exactly the classical motion for a mass= 1/2 particle subject to a negative constant force of unit magnitude (acceleration $= -2$). Thus the WF evolves "classically" as

$$
f(x,p;t) = f(x - 2pt - t^2, p + t \,; 0). \qquad (98)
$$

Note that time *in*dependence follows for $f(x,p;0)$ being any function of the energy variable, since $x + p^2 = x - 2pt - t^2 + (p+t)^2$.

The evolution kernel T propagates an arbitrary WF through just

$$
f(x,p;t) = \int dX dP \; T(x,p;X,P;t)\; f(X,P;0)\,. \qquad (99)
$$

The underlying phase-space structure, however, is more evident if one of the wave-function propagators is given in coordinate space, and the other in momentum space. Then the path integral expressions for the two propagators can be combined into a single phase-space path integral. For every time increment, phase space is integrated over to produce the new Wigner function from its immediate ancestor. The result is

$$
\begin{aligned}
&T\;(x,p;X,P;t) \qquad (100) \\
&= \frac{1}{\pi^2}\int dx_1 dp_1 \int dx_2 dp_2 e^{2i(x-x_1)(p-p_1)} e^{-ix_1 p_1} \langle x_1;t|x_2;0\rangle \, \langle p_1;t|p_2;0\rangle^* \, e^{ix_2 p_2} e^{-2i(X-x_2)(P-p_2)},
\end{aligned}
$$

where $\langle x_1;t\,|x_2;0\rangle$ and $\langle p_1;t\,|p_2;0\rangle$ are the path integral expressions in coordinate space, and in momentum space. Blending these x and p path integrals gives a genuine path integral over phase space [Ber80, DK85]. For a direct connection of $U_\star$ to this integral, see Refs. Sha79, Lea68, Sam00.

12 The Weyl Correspondence

This section summarizes the bridge and equivalence of phase-space quantization to the conventional formulation of quantum mechanics in Hilbert space. The Weyl correspondence merely provides a change of representation between phase space and Hilbert space. In itself, it does not map (commutative) classical mechanics to (non-commutative) quantum mechanics, but it makes that deformation map easier to grasp, defined within a common representation, and thus more intuitive.

Weyl [Wey27] introduced an association rule mapping invertibly c-number phase-space functions $g(x,p)$ (called phase-space kernels) to operators $\mathfrak{G}$ in a given ordering prescription.

Specifically, $p \mapsto \mathfrak{p}$, $x \mapsto \mathfrak{x}$, and, in general,

$$\mathfrak{G}(\mathfrak{x},\mathfrak{p}) = \frac{1}{(2\pi)^2} \int d\tau d\sigma dx dp \; g(x,p) \exp\Big[i\tau(\mathfrak{p}-p) + i\sigma(\mathfrak{x}-x)\Big]. \tag{101}$$

The eponymous ordering prescription requires that an arbitrary operator, regarded as a power series in $\mathfrak{x}$ and $\mathfrak{p}$, be first ordered in a completely symmetrized expression in $\mathfrak{x}$ and $\mathfrak{p}$, by use of Heisenberg's commutation relations, $[\mathfrak{x},\mathfrak{p}] = i\hbar$.

A term with m powers of $\mathfrak{p}$ and n powers of $\mathfrak{x}$ is obtained from the coefficient of $\tau^m\sigma^n$ in the expansion of $(\tau\mathfrak{p} + \sigma\mathfrak{x})^{m+n}$, which serves as a generating function of Weyl-ordered polynomials [GF91]. It is evident how the map yields a Weyl-ordered operator from a polynomial phase-space kernel. It includes every possible ordering with multiplicity one, e.g.

$$6p^2x^2 \mapsto \mathfrak{p}^2\mathfrak{x}^2 + \mathfrak{x}^2\mathfrak{p}^2 + \mathfrak{p}\mathfrak{x}\mathfrak{p}\mathfrak{x} + \mathfrak{p}\mathfrak{x}^2\mathfrak{p} + \mathfrak{x}\mathfrak{p}\mathfrak{x}\mathfrak{p} + \mathfrak{x}\mathfrak{p}^2\mathfrak{x} \; . \tag{102}$$

In general [McC32],

$$p^m x^n \mapsto \frac{1}{2^n}\sum_{r=0}^{n}\binom{n}{r}\mathfrak{x}^r\mathfrak{p}^m\mathfrak{x}^{n-r} = \frac{1}{2^m}\sum_{s=0}^{m}\binom{m}{s}\mathfrak{p}^s\mathfrak{x}^n\mathfrak{p}^{m-s}. \tag{103}$$

Phase-space constants map to the identity in Hilbert space.

In this correspondence scheme, then,

$$\mathrm{Tr}\mathfrak{G} = \int dx dp \; g. \tag{104}$$

Conversely [Gro46, Kub64, HOS84], the c-number phase-space kernels $g(x,p)$ of Weyl-ordered operators $\mathfrak{G}(\mathfrak{x},\mathfrak{p})$ are specified by $\mathfrak{p} \mapsto p$, $\mathfrak{x} \mapsto x$, or, more precisely, by the "Wigner map,"

$$\begin{aligned} g(x,p) &= \frac{1}{(2\pi)^2}\int d\tau d\sigma \; e^{i(\tau p+\sigma x)}\mathrm{Tr}\,\left(e^{-i(\tau\mathfrak{p}+\sigma\mathfrak{x})}\mathfrak{G}\right) \\ &= \frac{1}{2\pi}\int dy \; e^{-iyp}\left\langle x+\frac{\hbar}{2}y\middle|\mathfrak{G}(\mathfrak{x},\mathfrak{p})\middle|x-\frac{\hbar}{2}y\right\rangle, \end{aligned} \tag{105}$$

since the above trace reduces to

$$\int dz \; e^{i\tau\sigma\hbar/2}\langle z|e^{-i\sigma\mathfrak{x}}e^{-i\tau\mathfrak{p}}\mathfrak{G}|z\rangle = 2\pi\int dz \langle z-\hbar\tau|\mathfrak{G}|z\rangle e^{i\sigma(\tau\hbar/2-z)}. \tag{106}$$

Thus, the density matrix inserted in this expression [Moy49] yields the hermitean generalization of the Wigner function (63) encountered,

$$\begin{aligned} f_{ab}(x,p) &\equiv \frac{1}{2\pi}\int dy \; e^{-iyp}\left\langle x+\frac{\hbar}{2}y\;\middle|\psi_b\right\rangle\;\left\langle\psi_a\middle|x-\frac{\hbar}{2}y\right\rangle \\ &= \frac{1}{2\pi}\int dy e^{-iyp}\psi_a^*(x-\frac{\hbar}{2}y)\psi_b(x+\frac{\hbar}{2}y) \;\; = \;\; f_{ba}^*(x,p) \;, \end{aligned} \tag{107}$$

where the $\psi_a(x)$s are (ortho)normalized solutions to a Schrödinger problem. (Wigner [Wig32] mainly considered the diagonal elements of the pure-state density matrix, denoted above as $f_m \equiv f_{mm}$.) As a consequence, matrix elements of operators, i.e. traces of them with the density matrix, are produced through mere phase-space integrals [Moy49],

$$\langle\psi_m|\mathfrak{G}|\psi_n\rangle = \int dxdp\ g(x,p)f_{mn}(x,p), \tag{108}$$

and thus expectation values follow for $m = n$, as utilized throughout in this overview. Hence,

$$\langle\psi_m|\exp i(\sigma\mathfrak{x} + \tau\mathfrak{p})|\psi_m\rangle = \int dxdp\ f_m(x,p)\exp i(\sigma x + \tau p), \tag{109}$$

the celebrated moment-generating functional [Moy49] of the Wigner distribution.

Products of Weyl-ordered operators are not necessarily Weyl-ordered, but may be easily reordered into Weyl-ordered operators through the degenerate Campbell–Baker–Hausdorff identity. In a study of the uniqueness of the Schrödinger representation, von Neumann [Neu31] adumbrated the composition rule of kernel functions in such operator products, appreciating that Weyl's correspondence was in fact a homomorphism. (Effectively, he arrived at the Fourier space convolution representation of the star product.) Finally, Groenewold [Gro46] neatly worked out in detail how the kernel functions f and g of two operators $\mathfrak{F}$ and $\mathfrak{G}$ must compose to yield the kernel of $\mathfrak{F}\ \mathfrak{G}$,

$$\begin{aligned}\mathfrak{F}\ \mathfrak{G} = \tfrac{1}{(2\pi)^4}&\int d\xi d\eta d\xi' d\eta' dx' dx'' dp' dp'' f(x',p')g(x'',p'') \\ &\times \exp i[\xi(\mathfrak{p}-p') + \eta(\mathfrak{x}-x')]\ \exp i[\xi'(\mathfrak{p}-p'') + \eta'(\mathfrak{x}-x'')] \\ = \frac{1}{(2\pi)^4}&\int d\xi d\eta d\xi' d\eta' dx' dx'' dp' dp'' f(x',p')g(x'',p'')\exp i\Big[(\xi+\xi')\mathfrak{p} + (\eta+\eta')\mathfrak{x}\Big] \\ &\times\ \exp i\left[-\xi p' - \eta x' - \xi' p'' - \eta' x'' + \frac{\hbar}{2}(\xi\eta' - \eta\xi')\right].\end{aligned} \tag{110}$$

Changing integration variables to

$$\xi' \equiv \frac{2}{\hbar}(x-x'),\quad \xi \equiv \tau - \frac{2}{\hbar}(x-x'),\quad \eta' \equiv \frac{2}{\hbar}(p'-p),\quad \eta \equiv \sigma - \frac{2}{\hbar}(p'-p) \tag{111}$$

reduces the above integral to the fundamental

Theorem 1

$$\mathfrak{F}\ \mathfrak{G} = \frac{1}{(2\pi)^2}\int d\tau d\sigma dx dp\ \exp i\Big[\tau(\mathfrak{p}-p) + \sigma(\mathfrak{x}-x)\Big]\ (f\star g)(x,p), \tag{112}$$

where $f \star g$ is the expression (13).

□

The $\star$-product thus defines the transition from classical to quantum mechanics. In fact, the failure of Weyl-ordered operators to close under multiplication may be stood on its head [Bra02], to *define* a Weyl-symmetrizing operator product which is commutative and constitutes the Weyl transform of fg instead of the non-commutative $f \star g$. (For example, $2x \star p = 2xp + i\hbar \mapsto 2\mathfrak{x}\mathfrak{p} = \mathfrak{x}\mathfrak{p} + \mathfrak{p}\mathfrak{x} + i\hbar$. The classical piece of $2x \star p$ maps to the Weyl symmetrization of the operator product, $2xp \mapsto \mathfrak{x}\mathfrak{p} + \mathfrak{p}\mathfrak{x}$.) One may then solve for the PB in terms of the MB, and, through the Weyl correspondence, reformulate Classical Mechanics in Hilbert space as a deformation of Quantum Mechanics, instead of the other way around [Bra02].

Arbitrary operators $\mathfrak{G}(\mathfrak{x}, \mathfrak{p})$ consisting of operators $\mathfrak{x}$ and $\mathfrak{p}$, in various orderings, but with the same classical limit, could be imagined rearranged by use of Heisenberg commutations to canonical completely symmetrized Weyl-ordered forms, in general with $O(\hbar)$ terms generated in the process. Each one might then be inverse-mapped uniquely to its Weyl-correspondent c-number kernel function g in phase space. [In practice, there is the more direct Wigner transform formula (105), which bypasses a need for an actual rearrangement.] Thus, operators differing from each other by different orderings of their $\mathfrak{x}$s and $\mathfrak{p}$s correspond to kernel functions g coinciding with each other at $O(\hbar^0)$, but different at $O(\hbar)$, in general. Hence, in phase-space quantization, a survey of all alternate operator orderings in a problem with such ambiguities amounts to a survey of the "quantum correction" $O(\hbar)$ pieces of the respective kernel functions, i.e. the inverse Weyl transforms of those operators, and their study is systematized and expedited. Choice-of-ordering problems then reduce to purely $\star$-product algebraic ones, as the resulting preferred orderings are specified through particular deformations in the c-number kernel expressions resulting from the particular solution in phase space [CZ02].

13 Alternate Rules of Association

The Weyl correspondence rule (101) is not unique: there are a host of alternate *equivalent* association rules which specify corresponding representations. All these representations with equivalent formalisms are typified by characteristic quasi-distribution functions and $\star$-products, all inter-convertible among themselves. They have been surveyed comparatively and organized in Refs. Lee95, BJ84, on the basis of seminal classification work by Cohen [Coh66, Coh76], and are favored by virtue of their different characteristic properties in varying applications.

For example, instead of the operator $\exp(i\tau\mathfrak{p} + i\sigma\mathfrak{x})$ of the Weyl correspondence, one might posit, instead [Lee95, HOS84], antistandard ordering,

$$\exp(i\tau\mathfrak{p})\exp(i\sigma\mathfrak{x}) = \exp(i\tau\mathfrak{p} + i\sigma\mathfrak{x})w(\tau, \sigma), \tag{113}$$

with $w = \exp(i\hbar\tau\sigma/2)$, which specifies the Kirkwood–Rihaczek prescription; or else standard ordering, $w = \exp(-i\hbar\tau\sigma/2)$ on the right-hand side of the above, for the Mehta prescription; or normal and antinormal orderings for the Glauber–Sudarshan prescriptions, generalizing to $w = \exp[\frac{\hbar}{4}(\tau^2 + \sigma^2)]$ for the Husimi prescription [Hus40, Tak89]; or $w = \cosh[\frac{\hbar}{4}(\tau^2 + \sigma^2)]$ for the Rivier prescription; or $w = \sin(\hbar\tau\sigma/2)/(\hbar\tau\sigma/2)$, for the Born–Jordan prescription; and so on.

The corresponding quasi-distribution functions in each representation can be obtained as convolution transforms of each other [Coh76, Lee95, HOS84], and likewise the kernel function observables are convolution "dressings" of each other, as are their $\star$-products [Dun88, AW70, Ber75].

Example For instance, the Husimi distribution follows from a "Gaussian smoothing" linear conversion map [WO87, Tak89, Lee95] of the WF,

$$f_H = T(f) = \exp\left[\frac{\hbar}{4}(\partial_x^2 + \partial_p^2)\right] f \tag{114}$$
$$= \frac{1}{\pi\hbar}\int dx'dp' \exp\left[-\frac{(x'-x)^2 + (p'-p)^2)}{\hbar}\right] f(x',p'),$$

and likewise for the observables, so that

$$\langle \mathfrak{G} \rangle = \int dxdp\ g(x,p)\ \exp\left[-\frac{\hbar}{4}(\partial_x^2 + \partial_p^2)\right]\ f_H$$
$$= \int dxdp\ g_H\ e^{\hbar(\overleftarrow{\partial}_x \overrightarrow{\partial}_x + \overleftarrow{\partial}_p \overrightarrow{\partial}_p)/2}\ f_H. \tag{115}$$

Expectation values of observables now entail equivalence conversion dressings of the respective kernel functions and a corresponding $\star$-product [Ba79, OW81, Vo89, Tak89, Zac00], which now cannot be simply dropped inside integrals. For this reason, distributions such as this Husimi distribution (which is positive-semidefinite [Car76, OW81, Ste80]) cannot be automatically thought of as bona-fide probability distributions. This is often dramatized as the failure of the Husimi distribution f_H to yield the correct x- or p-marginal probabilities, upon integration by p or x, respectively [OW81, HOS84]. Since phase-space integrals are thus complicated by conversion dressing convolutions, they preclude direct applications of the Schwarz inequality and the standard inequality-based moment-constraining techniques of probability theory, as well as routine completeness and orthonormality-based functional analytic operations. (Ignoring the above equivalence dressings and, instead, simply treating the Hussimi distribution as an ordinary probability distribution in evaluating expectation values results in loss of quantum information—effectively "coarse-graining" to a classical limit.)

Similar caveats also apply to more recent symplectic tomographic representations [MMT96, MMM01, Leo97], which are positive semi-definite too, but also do not constitute conventional probability distributions.

14 The Groenewold–van Hove Theorem and the Uniqueness of MBs and $\star$-Products

Groenewold's correspondence principle theorem [Gro46] (to which van Hove's extension is often attached [vH51]) points out that, in general, there is no invertible linear map from all functions of phase space $f(x,p), g(x,p), \ldots$, to hermitean operators in Hilbert space $\mathfrak{Q}(f)$, $\mathfrak{Q}(g), \ldots$, such that the PB structure is preserved,

$$\mathfrak{Q}(\{f,g\}) = \frac{1}{i\hbar}\ [\ \mathfrak{Q}(f), \mathfrak{Q}(g)\ ]\ , \tag{116}$$

as utilized in Dirac's heuristics. Instead, the Weyl correspondence map (101) from functions to ordered operators,

$$\mathfrak{W}(f) \equiv \frac{1}{(2\pi)^2} \int d\tau d\sigma dx dp \; f(x,p) \exp[i\tau(\mathfrak{p}-p)+i\sigma(\mathfrak{x}-x)], \tag{117}$$

specifies the $\star$-product in (112), $\mathfrak{W}(f\star g)=\mathfrak{W}(f)\;\mathfrak{W}(g)$, and thus

$$\mathfrak{W}(\{\!\{f,g\}\!\}) = \frac{1}{i\hbar}\left[\mathfrak{W}(f),\mathfrak{W}(g)\right]. \tag{118}$$

It is the MB, then, instead of the PB, which maps invertibly to the quantum commutator. That is to say, the "deformation" in phase-space quantization is nontrivial: the quantum functions, in general, do not coincide with the classical ones [Gro46], and involve $O(\hbar)$ corrections, as extensively illustrated in, e.g. Refs. CZ02, DS02, CH86; also see Ref. Got99.

An alternate abstract realization of the above MB algebra in phase space (as opposed to the Hilbert space one), $\mathfrak{W}(f)$, is [FFZ89, CFZm98]

$$\mathfrak{K}(f) = f\star \;. \tag{119}$$

Realized on a toroidal phase space, with a formal identification $\hbar \mapsto 2\pi/N$, it leads to the Lie algebra of SU(N) [FFZ89], by means of Sylvester's clock-and-shift matrices [Syl82]. For generic $\hbar$, it may be thought of as a generalization of SU(N) for continuous N, allowing for taking the limit $N\to\infty$.

Essentially (up to isomorphism), the MB algebra is the unique one-parameter deformation of the Poisson bracket algebra [Vey75, BFF78, FLS76, Ar83, Fle90, deW83, BCG97, TD97], a uniqueness extending to the star product. Isomorphism allows for dressing transformations of the variables (kernel functions and WFs, as in Section 13 on alternate orderings), through linear maps $f \mapsto T(f)$, which leads to cohomologically equivalent star-product variants, i.e. [Ba79, Vo89, BFF78]

$$T(f\star g) = T(f) \circledast T(g). \tag{120}$$

Consequently, the $\star$-MB algebra is isomorphic to the algebra of $\circledast$-MB.

Computational features of $\star$-products are discussed in Refs. BFF78, Han84, RO92, Zac00, EGV89, Vo78, An97, Bra94.

15 Omitted Miscellany

Phase-space quantization extends in several interesting directions which are not covered in such a summarizing introduction.

The systematic generalization of the $\star$-product to arbitrary non-flat Poisson manifolds [Kon97], is a culmination of extensions to general symplectic and Kähler geometries [Fed94, Kis01], and varied symplectic contexts [Ber75, RT00, CPP02, BGL01]. For further work on curved spaces, cf. Refs. APW02, BF81, PT99. For extensive reviews of mathematical issues, cf. Refs. Fol89, Hor79, Wo98, AW70. For a connection to the theory of modular forms, see

Ref. Raj02. For WFs of discrete (finite systems), cf. Refs. Woo87, ACW98, RA99, RG00, BHP02.

Spin is treated in Refs. Str57, VG89, AW00; and forays into a relativistic formulation in Ref. LSU02 (also see Refs. CS75, Ran66).

Inclusion of Electromagnetic fields and gauge invariance is treated in Refs. Mue99, LF94, LF01, JVS87, ZC99, KO00. Subtleties of Berry's phase in phase space are addressed in Ref. Sam00.