

Chapter 1

Flexible-link Manipulators: Modeling, Nonlinear Control and Observer

The interest in flexible robot manipulators has become greater in the latest years. In order to adequately exploit the advantages of this class of manipulators, accurate models and effective control schemes are necessary. This work collects a number of recent results on modeling, nonlinear control and observer for flexible-link manipulators. The equations of motion are derived on the basis of a combined Lagrange-assumed modes approach. The resulting model shows several similarities with that of a rigid manipulator, thus allowing important properties to be derived which are used to design controllers and observers. A nonlinear control scheme based on robust control techniques is proposed in order to improve the damping of the system. Since typically link coordinate rates cannot be measured, a nonlinear observer is presented which provides estimates of both joint and link coordinate rates while keeping stability of the system.

1.1 Introduction

Lightweight manipulators offer many challenges in comparison to rigid and bulky robot manipulators. Energy consumption is smaller, so that the payload-to-arm weight ratio can be increased as well as faster movements can be achieved. Due to their characteristics, this class of manipulators are specially suitable for a number of nonconventional robotic applications, including space missions. On the other hand, the study of link flexibility is enforced also for some kind of heavy manipulators such as large scale systems. In either case, it is no longer possible to assume that link deformation can be neglected. All these factors make the study of flexible robot manipulators quite interesting. The present work aims at presenting some of the latest results on modeling, nonlinear control and observer in

this field.

The importance of having an accurate model that can adequately describe the dynamics of the manipulator is obvious. A common way of modeling a flexible robot manipulator consists in using a combined Lagrange-assumed modes approach, which allows deriving a dynamic model in closed form [Book (1984); De Luca and Siciliano (1991); Yuan *et al.* (1993); Canudas de Wit *et al.* (1996); Arteaga (1998)]. Just like in the case of the dynamic model of a rigid manipulator, which possesses many helpful properties [Ortega and Spong (1989); Nicosia and Tomei (1990); Canudas de Wit *et al.* (1990)], it is possible to compute a set of properties for the dynamic model of a flexible manipulator [Arteaga (1998)], whose knowledge facilitates the design of controllers and observers for this kind of system [De Luca and Siciliano (1993a); Lammerts *et al.* (1995); Arteaga (1996a); Arteaga (1996b); Arteaga (1996c)]. Perhaps the most well-known property of (rigid and flexible) manipulators is that referring to their passive structure. With the exception of those controllers based on inverse dynamics [Canudas de Wit *et al.* (1996); De Luca and Siciliano (1993a)], this property is usually employed to prove stability of several control schemes. However, there are many other properties which have been employed to design specific control laws [Ortega and Spong (1989); Nicosia and Tomei (1990); Canudas de Wit *et al.* (1990); De Luca and Siciliano (1993a); De Luca and Siciliano (1993b); De Luca and Panzieri (1994)].

Control of flexible robot manipulators shows the difficulty that there is not an independent control input for each degree of freedom. As in the case of rigid manipulators, there are mainly two goals to be achieved: point-to-point and tracking control. For the first case, some results are given in [De Luca and Siciliano (1993b); De Luca and Panzieri (1994)], where the regulation problem under gravity is studied. In [De Luca and Siciliano (1993b)] the case of no modal damping of the links is treated. By making some assumptions on the inertia matrix, it is possible to guarantee convergence of the link coordinates to certain constant values. In [De Luca and Panzieri (1994)] a solution is proposed for the case that the gravity vector is not perfectly known.

Because an arbitrary trajectory can only be assigned for the joint coordinates, the desired trajectory for the link coordinates must be computed in such a way that the control goal can be accomplished. In [De Luca and Siciliano (1993a); Lammerts *et al.* (1995)] this problem is addressed, and in particular in [Lammerts *et al.* (1995)] not only flexible links

but also flexible joints are considered, but there is no guarantee that the computed desired trajectory remains bounded; when the model parameters are not well known, an adaptive algorithm can be used [Slotine and Li (1987)]. On the other hand, in [De Luca and Siciliano (1993a)] inverse control techniques are used [Canudas de Wit *et al.* (1996)], and it is shown that the computed desired trajectory remains bounded. In none of these works the problem of no damping is treated. In this work, the tracking control of flexible robot manipulators is studied [Arteaga (1996c); Arteaga and Siciliano (2000)]. A control law is proposed which is based on the passivity-based control approach with filtered reference velocity [Ortega and Spong (1989)]. It is proven that the desired trajectory for the link coordinates remains bounded. The no damping case is also treated and robust control techniques are used to increase the damping of the system [Dawson *et al.* (1991)].

A problem which deserves special attention regards the possible lack of measurement of link deflection rates, which typically requires the use of an observer. In addition, even though joint positions can be measured very accurately, tachometers (used to measure joint velocities) may not deliver reliable signals. That is why nonlinear observers are recommended to estimate joint speeds. In [Arteaga (1996a); Arteaga (1996b)] an observer for flexible robot manipulators is proposed. Although it is possible to measure link coordinates by using a strain gauge for each coordinate [Arteaga (1995)], the observer requires only a sensor for every flexible link. However, since it is designed independently of any control scheme, the stability of this observer together with the controller proposed in [Arteaga (1996c); Arteaga and Siciliano (2000)] and presented in this work can no longer be guaranteed. To overcome this difficulty, a new observer based on that given in [Nicosia and Tomei (1990)] is proposed [Arteaga (2000)]. In order to ensure enhancing of the damping of the system, some essential modifications are necessary.

The work is organized as follows: Section 1.2 briefly describes the kinematics of flexible robot manipulators and their dynamic modelling. Some of the most important properties of the model are listed. In Section 1.3, control of flexible manipulators is studied. By using robust control techniques, the damping of the system is increased. Since it is not always possible to measure link coordinate rates, a nonlinear observer is proposed in Section 1.4 in order to estimate them. Some simulation results are presented in Section 1.5, while Section 1.6 gives some concluding remarks.

1.2 Modeling

A common way of modeling flexible robot manipulators is using the so-called combined Lagrange–assumed modes approach [Book (1984); De Luca and Siciliano (1991); Yuan *et al.* (1993); Canudas de Wit *et al.* (1996); Arteaga (1998)]. In this case, it is necessary to describe the kinetic and potential energy of the system adequately. In order to compute them, it is advantageous to know the kinematics of the manipulator, which can be achieved by setting coordinate frames along the joint axes. In this section, the kinematics of flexible robot manipulators is briefly studied. By using Lagrange equations of motion, the dynamic model of this class of manipulators is derived in Section 1.2.2 and in Section 1.2.3 some of its most important properties are presented.

1.2.1 Kinematics

It is well known that the kinematics of a rigid robot manipulator can be described by employing the Denavit–Hartenberg representation [Denavit and Hartenberg (1955)]. The main idea is to use 4×4 transformation matrices which can be determined uniquely as a function of only 4 parameters. However, this procedure cannot be used directly to describe the kinematics of a flexible robot manipulator due to link deformation. In order to overcome this drawback, the procedure has been modified in [Book (1984); Book (1979)] by including some transformation matrices which take link elasticity into account. A description of the Denavit–Hartenberg representation for rigid manipulators is assumed to be known. Fig. 1.1 depicts a portion of the serial chain for a flexible robot manipulator. The case of revolute joints is considered.

Consider two coordinate frames i and j . Their mutual position and orientation can be expressed in terms of the homogeneous transformation matrix

$${}^jT_i = \begin{bmatrix} {}^jR_i & {}^j\mathbf{d}_i \\ \mathbf{0}^T & 1 \end{bmatrix} \quad (1.1)$$

where jR_i is the 3×3 rotation matrix describing the orientation of the axes of frame i and ${}^j\mathbf{d}_i$ is the 3×1 vector describing the origin of frame i , both with respect to frame j ; also, in (1.1) $\mathbf{0}$ denotes a 3×1 vector of null elements.

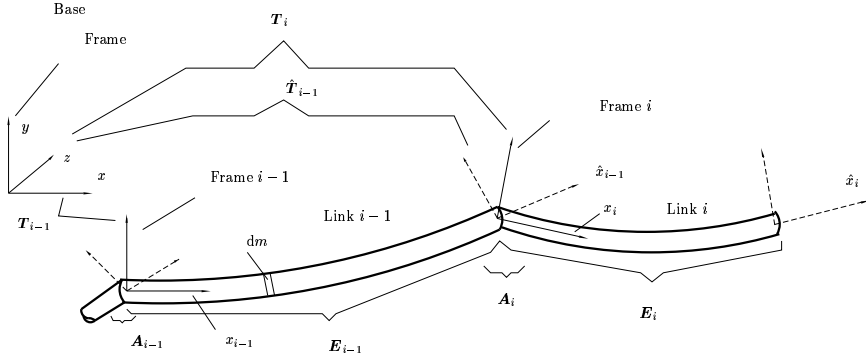


Fig. 1.1 Flexible manipulator serial chain.

The position of a point on link i with respect to frame i is given by

$${}^i \mathbf{p}_i = \begin{bmatrix} x_i \\ y_i \\ z_i \end{bmatrix}. \quad (1.2)$$

However, it is not possible to use a homogeneous transformation with a vector of the form (1.2), so that it is necessary to rewrite it as

$${}^i \mathbf{r}_i \triangleq \begin{bmatrix} {}^i \mathbf{p}_i \\ 1 \end{bmatrix}. \quad (1.3)$$

To express the position of this point in frame j , a homogeneous transformation is used, i.e.

$${}^j \mathbf{r}_i = {}^j \mathbf{T}_i {}^i \mathbf{r}_i. \quad (1.4)$$

In the case of the base frame one has

$${}^0 \mathbf{r}_i \triangleq \mathbf{r}_i = {}^0 \mathbf{T}_i {}^i \mathbf{r}_i \triangleq \mathbf{T}_i {}^i \mathbf{r}_i, \quad (1.5)$$

where the superscript 0 has been conveniently dropped.

In general, the homogeneous transformation of frame i with respect to the base frame can be characterized through the following composition of consecutive transformations:

$${}^0 \mathbf{T}_i \triangleq \mathbf{T}_i = \mathbf{A}_1 \mathbf{E}_1 \mathbf{A}_2 \mathbf{E}_2 \dots \mathbf{A}_{i-1} \mathbf{E}_{i-1} \mathbf{A}_i \triangleq \hat{\mathbf{T}}_{i-1} \mathbf{A}_i \quad (1.6)$$

$$\hat{\mathbf{T}}_{i-1} \triangleq \mathbf{T}_{i-1} \mathbf{E}_{i-1} \quad (1.7)$$

$$\mathbf{T}_1 = \mathbf{A}_1, \quad (1.8)$$

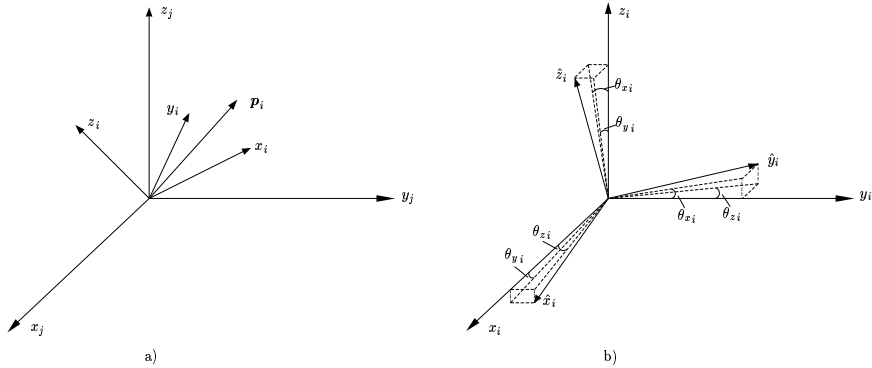


Fig. 1.2 a) Rotation of a coordinate frame; b) Rotation of a coordinate frame due to deformation of the flexible link.

where \mathbf{A}_i is the standard homogeneous transformation matrix for joint i due to rigid motion and \mathbf{E}_i is the homogeneous transformation matrix due to link i length and deflection. Notice that, even though the superscript is not explicitly indicated, each transformation matrix is referred to the frame determined by the preceding transformation.

The transformation matrix \mathbf{A}_i can be computed just like in the case of rigid robot manipulators [Sciavicco and Siciliano (2000)]. On the other hand, the transformation matrix \mathbf{E}_i deserves special attention. Firstly, consider the general form of a rotation matrix ${}^j\mathbf{R}_i$ between two coordinate frames of common origin (see Fig. 1.2 a)) [Sciavicco and Siciliano (2000)]:

$${}^j\mathbf{R}_i = \begin{bmatrix} \mathbf{x}_i^T \mathbf{x}_j & \mathbf{y}_i^T \mathbf{x}_j & \mathbf{z}_i^T \mathbf{x}_j \\ \mathbf{x}_i^T \mathbf{y}_j & \mathbf{y}_i^T \mathbf{y}_j & \mathbf{z}_i^T \mathbf{y}_j \\ \mathbf{x}_i^T \mathbf{z}_j & \mathbf{y}_i^T \mathbf{z}_j & \mathbf{z}_i^T \mathbf{z}_j \end{bmatrix} = \begin{bmatrix} \cos(\theta_{\mathbf{x}_i \mathbf{x}_j}) & \cos(\theta_{\mathbf{y}_i \mathbf{x}_j}) & \cos(\theta_{\mathbf{z}_i \mathbf{x}_j}) \\ \cos(\theta_{\mathbf{x}_i \mathbf{y}_j}) & \cos(\theta_{\mathbf{y}_i \mathbf{y}_j}) & \cos(\theta_{\mathbf{z}_i \mathbf{y}_j}) \\ \cos(\theta_{\mathbf{x}_i \mathbf{z}_j}) & \cos(\theta_{\mathbf{y}_i \mathbf{z}_j}) & \cos(\theta_{\mathbf{z}_i \mathbf{z}_j}) \end{bmatrix}, \quad (1.9)$$

where $\mathbf{x}, \mathbf{y}, \mathbf{z}$ denote the unit vectors of the respective axes. Then, the relationship between ${}^j\mathbf{p}_i$ and ${}^i\mathbf{p}_i$ is given by

$${}^j\mathbf{p}_i = {}^j\mathbf{R}_i {}^i\mathbf{p}_i. \quad (1.10)$$

From (1.9), it can easily be understood that the knowledge of the angles $\theta_{\mathbf{x}_i \mathbf{x}_j} \cdots \theta_{\mathbf{z}_i \mathbf{z}_j}$ is enough to compute ${}^j\mathbf{R}_i$. With this background, the matrices \mathbf{E}_i can be determined as follows. Consider Fig. 1.1 again and assume that the x -axis of frame i is along the link. Assuming small link deformation [Book (1979); Meirovitch (1967); Meirovitch (1975)], \mathbf{E}_i can

be expressed as [Book (1979)]

$$\mathbf{E}_i = \begin{bmatrix} 1 & \cos(\pi/2 + \theta_{z_i}) & \cos(\pi/2 - \theta_{y_i}) & l_i + \delta_{x_i} \\ \cos(\pi/2 - \theta_{z_i}) & 1 & \cos(\pi/2 + \theta_{x_i}) & \delta_{y_i} \\ \cos(\pi/2 + \theta_{y_i}) & \cos(\pi/2 - \theta_{x_i}) & 1 & \delta_{z_i} \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad (1.11)$$

where θ_{x_i} , θ_{y_i} , θ_{z_i} are the angles of rotation, and δ_{x_i} , δ_{y_i} , δ_{z_i} represent link i deformation along x , y , z , respectively, being l_i the length of the link without deformation. The angles of rotation θ_{x_i} , θ_{y_i} , θ_{z_i} are depicted in Fig. 1.2 b). By taking into account the fact $\cos(\pi/2 + \alpha) = -\sin(\alpha)$ and assuming small angles, so that $\sin(\alpha) \approx \alpha$ is valid, the matrix \mathbf{E}_i can be approximated as

$$\mathbf{E}_i = \begin{bmatrix} 1 & -\theta_{z_i} & \theta_{y_i} & l_i + \delta_{x_i} \\ \theta_{z_i} & 1 & -\theta_{x_i} & \delta_{y_i} \\ -\theta_{y_i} & \theta_{x_i} & 1 & \delta_{z_i} \\ 0 & 0 & 0 & 1 \end{bmatrix}. \quad (1.12)$$

By using the homogeneous transformation matrices \mathbf{A}_i and \mathbf{E}_i , the position of any point along the robot manipulator can uniquely be determined from Eqs. (1.5), (1.6) and (1.12).

1.2.2 Dynamics

In order to obtain a set of differential equations of motion to adequately describe the dynamics of a flexible-link manipulator, the Lagrange's approach can be used. A system with n generalized coordinates q_i must satisfy n differential equations of the form

$$\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}_i} - \frac{\partial \mathcal{L}}{\partial q_i} + \frac{\partial \mathcal{D}}{\partial \dot{q}_i} = u_i \quad i = 1, \dots, n, \quad (1.13)$$

where \mathcal{L} is the so called Lagrangian which is given by [Wellstead (1979)]

$$\mathcal{L} = \mathcal{T} - \mathcal{U}; \quad (1.14)$$

\mathcal{T} represents the kinetic energy of the system and \mathcal{U} the potential energy. Also, in (1.13) \mathcal{D} is the Rayleigh's dissipation function which allows dissipative effects to be included, and u_i is the generalized force acting on q_i .

To compute the kinetic energy of the system, the manipulator kinematics can be described systematically as explained in the previous section.

The kinetic energy of link i link can be expressed as

$$\mathcal{T}_i = \int_{\text{link}_i} d\mathcal{T}_i = \frac{1}{2} \int_{\text{link}_i} \text{Tr} \left(\frac{d\mathbf{r}_i}{dt} \frac{d\mathbf{r}_i^T}{dt} \right) dm, \quad (1.15)$$

which implies that the kinetic energy for the whole system is

$$\mathcal{T} = \sum_{i=1}^n \int_{\text{link}_i} d\mathcal{T}_i = \frac{1}{2} \sum_{i=1}^n \int_{\text{link}_i} \text{Tr} \left(\frac{d\mathbf{r}_i}{dt} \frac{d\mathbf{r}_i^T}{dt} \right) dm; \quad (1.16)$$

$\text{Tr}(\cdot)$ represents the trace operator of a square matrix. By accounting for (1.5), the kinetic energy (1.16) can be written in the form

$$\mathcal{T} = \frac{1}{2} \dot{\mathbf{q}}^T \mathbf{H}(\mathbf{q}) \dot{\mathbf{q}}, \quad (1.17)$$

where

$$\begin{aligned} \mathbf{q}(t) &\triangleq [\theta_1(t) \cdots \theta_n(t) \quad \delta_{11}(t) \cdots \delta_{1m_1}(t) \cdots \delta_{n1}(t) \cdots \delta_{nm_n}(t)]^T \\ &= [q_{10}(t) \cdots q_{n0}(t) \quad q_{11}(t) \cdots q_{1m_1}(t) \cdots q_{n1}(t) \cdots q_{nm_n}(t)]^T \end{aligned} \quad (1.18)$$

is the vector of generalized coordinates which is formed by the rigid coordinates $\theta_1 \dots \theta_n$ ($q_{10} \dots q_{n0}$) and the flexible coordinates $\delta_{11} \dots \delta_{nm_n}$ ($q_{11} \dots q_{nm_n}$), and

$$\mathbf{H}(\mathbf{q}) = \begin{bmatrix} \mathbf{H}_{\theta\theta}(\mathbf{q}) & \mathbf{H}_{\theta\delta}(\mathbf{q}) \\ \mathbf{H}_{\theta\delta}^T(\mathbf{q}) & \mathbf{H}_{\delta\delta}(\mathbf{q}) \end{bmatrix} \quad (1.19)$$

is the inertia matrix. In particular, $\mathbf{H}_{\theta\theta}(\mathbf{q})$ is associated to the rigid coordinates, $\mathbf{H}_{\theta\delta}(\mathbf{q})$ takes into account the relationship between the flexible and rigid coordinates, and $\mathbf{H}_{\delta\delta}(\mathbf{q})$ is associated to the flexible coordinates.

In order to find an analytical form for the inertia matrix, it is necessary to describe the deflection and torsion of each link as a function of the link coordinates. These can be expressed according to the so-called assumed modes method, i.e. [De Luca and Siciliano (1991); Yuan *et al.* (1993);

Meirovitch (1967); Meirovitch (1975)]

$$\delta_{x_i} = \sum_{j=1}^{m_i} \phi_{x_{ij}} \delta_{ij} \quad \theta_{x_i} = \sum_{j=1}^{m_i} \theta_{x_{ij}} \delta_{ij} \quad (1.20)$$

$$\delta_{y_i} = \sum_{j=1}^{m_i} \phi_{y_{ij}} \delta_{ij} \quad \theta_{y_i} = \sum_{j=1}^{m_i} \theta_{y_{ij}} \delta_{ij} \quad (1.21)$$

$$\delta_{z_i} = \sum_{j=1}^{m_i} \phi_{z_{ij}} \delta_{ij} \quad \theta_{z_i} = \sum_{j=1}^{m_i} \theta_{z_{ij}} \delta_{ij}, \quad (1.22)$$

where $\phi_{x_{ij}}, \phi_{y_{ij}}, \phi_{z_{ij}}$ ($\theta_{x_{ij}}, \theta_{y_{ij}}, \theta_{z_{ij}}$) are the spatial mode shapes used to model the deflection (torsion) of link i , being m_i the number of link coordinates.

From (1.16), the elements of $\mathbf{H}_{\theta\theta}(\mathbf{q})$ can be computed as

$$h_{\alpha 0 h 0} = \sum_{i=\max\{\alpha, h\}}^n \text{Tr} \left(\left(\hat{\mathbf{T}}_{\alpha-1} \mathbf{U}_{\alpha} {}^{\alpha} \tilde{\mathbf{T}}_i \right) \mathbf{F}_i \left(\hat{\mathbf{T}}_{h-1} \mathbf{U}_h {}^h \tilde{\mathbf{T}}_i \right)^T \right) \quad (1.23)$$

with

$${}^h \mathbf{T}_i \triangleq \mathbf{A}_{h+1} \mathbf{E}_{h+1} \mathbf{A}_{h+2} \mathbf{E}_{h+2} \cdots \mathbf{A}_{i-1} \mathbf{E}_{i-1} \mathbf{A}_i \quad (1.24)$$

$${}^h \tilde{\mathbf{T}}_i \triangleq \mathbf{E}_h {}^h \mathbf{T}_i \quad (1.25)$$

$$\mathbf{U}_h \triangleq \frac{\partial \mathbf{A}_h}{\partial q_{h0}} \quad (1.26)$$

$$\mathbf{F}_i \triangleq \mathbf{C}_i + \sum_{j=1}^{m_i} \delta_{ij} \left((\mathbf{C}_{ij} + \mathbf{C}_{ij}^T) + \sum_{k=1}^{m_i} \delta_{ik} \mathbf{C}_{ikj} \right) = \mathbf{F}_i^T \quad (1.27)$$

$$\mathbf{C}_i \triangleq \int_{\text{link}_i} [x_i \ y_i \ z_i \ 1]^T [x_i \ y_i \ z_i \ 1] dm \quad (1.28)$$

$$\mathbf{C}_{ij} \triangleq \int_{\text{link}_i} [x_i \ y_i \ z_i \ 1]^T [\phi_{x_{ij}} \ \phi_{y_{ij}} \ \phi_{z_{ij}} \ 0] dm \quad (1.29)$$

$$\mathbf{C}_{ikj} \triangleq \int_{\text{link}_i} [\phi_{x_{ik}} \ \phi_{y_{ik}} \ \phi_{z_{ik}} \ 0]^T [\phi_{x_{ij}} \ \phi_{y_{ij}} \ \phi_{z_{ij}} \ 0] dm = \mathbf{C}_{ij}^T, \quad (1.30)$$

the elements of $\mathbf{H}_{\theta\delta}(\mathbf{q})$ can be computed as

$$h_{h0\alpha\beta} = \gamma_{h\alpha} + \sum_{i=\max\{h, \alpha+1\}}^n \text{Tr} \left(\left(\hat{\mathbf{T}}_{h-1} \mathbf{U}_h {}^h \tilde{\mathbf{T}}_i \right) \mathbf{F}_i (\mathbf{T}_{\alpha} \mathbf{N}_{\alpha\beta} {}^{\alpha} \mathbf{T}_i)^T \right) \quad (1.31)$$

with

$$\gamma_{h\alpha} = \begin{cases} 0 & \text{if } h > \alpha \\ \text{Tr} \left(\left(\hat{\mathbf{T}}_{h-1} \mathbf{U}_h {}^h \tilde{\mathbf{T}}_\alpha \right) \mathbf{D}_{\alpha\beta} \mathbf{T}_\alpha^T \right) & \text{if } h \leq \alpha \end{cases}$$

$$\mathbf{N}_{\alpha\beta} \triangleq \begin{bmatrix} 0 & -\theta_{z\alpha\beta} & \theta_{y\alpha\beta} & \phi_{x\alpha\beta} \\ \theta_{z\alpha\beta} & 0 & -\theta_{x\alpha\beta} & \phi_{y\alpha\beta} \\ -\theta_{y\alpha\beta} & \theta_{x\alpha\beta} & 0 & \phi_{z\alpha\beta} \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad (1.32)$$

$$\mathbf{D}_{\alpha\beta} \triangleq \mathbf{C}_{\alpha\beta} + \sum_{k=1}^{m_\alpha} \delta_{\alpha k} \mathbf{C}_{\alpha k\beta}, \quad (1.33)$$

and the elements of $\mathbf{H}_{\delta\delta}(\mathbf{q})$ can be computed as

$$h_{hk\alpha\beta} = \eta_{h\alpha} + \sum_{i=\max\{h,\alpha\}+1}^n \text{Tr} \left(\left(\mathbf{T}_h \mathbf{N}_{hk} {}^h \mathbf{T}_i \right) \mathbf{F}_i \left(\mathbf{T}_\alpha \mathbf{N}_{\alpha\beta} {}^\alpha \mathbf{T}_i \right)^T \right) \quad (1.34)$$

with

$$\eta_{h\alpha} = \begin{cases} \text{Tr} \left(\mathbf{T}_h \mathbf{C}_{hk\beta} \mathbf{T}_h^T \right) & \text{if } h = \alpha \\ \text{Tr} \left(\left(\mathbf{T}_h \mathbf{N}_{hk} {}^h \mathbf{T}_\alpha \right) \mathbf{D}_{\alpha\beta} \mathbf{T}_\alpha^T \right) & \text{if } h < \alpha \\ \text{Tr} \left(\left(\mathbf{T}_\alpha \mathbf{N}_{\alpha\beta} {}^\alpha \mathbf{T}_h \right) \mathbf{D}_{hk} \mathbf{T}_h^T \right) & \text{if } h > \alpha. \end{cases}$$

Notice that $\mathbf{H}_{\theta\theta}(\mathbf{q})$ and $\mathbf{H}_{\delta\delta}(\mathbf{q})$ are symmetric, so that it is only necessary to compute the terms for which $h \geq \alpha$.

The next step is to compute the potential energy of the system. In a flexible-link manipulator there are two sources of potential energy: link gravity and link elasticity.

The differential element of gravity potential energy of link i is given by

$$d\mathcal{U}_{gi} = -\mathbf{g}_0^T \mathbf{T}_i^i \mathbf{r}_i dm \quad (1.35)$$

where

$$\mathbf{g}_0 = [g_x \ g_y \ g_z \ 0]^T \quad (1.36)$$

is the gravity vector expressed in the base frame. The total gravitational energy is

$$\mathcal{U}_g = -\mathbf{g}_0^T \sum_{i=1}^n \mathbf{T}_i \mathbf{h}_i \quad (1.37)$$

with

$$\mathbf{h}_i \triangleq M_i \mathbf{l}_i + \sum_{k=1}^{m_i} \delta_{ik} \mathbf{s}_{ik} \quad (1.38)$$

$$\mathbf{l}_i = [l_{xi} \ l_{yi} \ l_{zi} \ 1]^T \quad (1.39)$$

$$\mathbf{s}_{ik} = \int_{\text{link}_i} [\phi_{xik} \ \phi_{yik} \ \phi_{z ik} \ 0]^T dm, \quad (1.40)$$

where \mathbf{l}_i is the vector from joint i to the center of gravity when link i is undeformed and M_i is the total mass of the link.

The strain potential energy associated to the deformation of link i is given by [Yuan *et al.* (1993)]

$$\mathcal{U}_{ei} = \frac{1}{2} \int_{\text{link}_i} \left(EI_y \left(\frac{\partial^2 \delta_{yi}}{\partial x_i^2} \right)^2 + EI_z \left(\frac{\partial^2 \delta_{zi}}{\partial x_i^2} \right)^2 + E_G J_x \left(\frac{\partial \theta_{xi}}{\partial x_i} \right)^2 \right) dx_i, \quad (1.41)$$

where E is Young's modulus of elasticity, I_y (I_z) is the area moment of inertia of the link about an axis parallel to y (z) through the center of mass of the cross section, E_G is the shear modulus, and J_x is the polar area moment of inertia of the link about the center of mass. The integration in (1.41) is carried out along x -axis. Notice that in (1.41) the compression in the x direction has been assumed to be negligible. In view of (1.20)–(1.22), Eq. (1.41) can be rewritten as

$$\mathcal{U}_{ei} = \frac{1}{2} \sum_{j=1}^{m_i} \sum_{k=1}^{m_i} \delta_{ij} \delta_{ik} (k_{y_{ijk}} + k_{z_{ijk}} + k_{x_{ijk}}), \quad (1.42)$$

where $k_{x_{ijk}}$, $k_{y_{ijk}}$, $k_{z_{ijk}}$ are the stiffness coefficients given by

$$k_{x_{ijk}} = \int_{\text{link}_i} E_G J_x \frac{d\theta_{x_{ij}}}{dx_i} \frac{d\theta_{x_{ik}}}{dx_i} dx_i \quad (1.43)$$

$$k_{y_{ijk}} = \int_{\text{link}_i} EI_y \frac{d^2 \phi_{y_{ij}}}{dx_i^2} \frac{d^2 \phi_{y_{ik}}}{dx_i^2} dx_i \quad (1.44)$$

$$k_{z_{ijk}} = \int_{\text{link}_i} EI_z \frac{d^2 \phi_{z_{ij}}}{dx_i^2} \frac{d^2 \phi_{z_{ik}}}{dx_i^2} dx_i. \quad (1.45)$$

The total elastic energy is

$$\mathcal{U}_e = \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^{m_i} \sum_{k=1}^{m_i} \delta_{ij} \delta_{ik} k_{ijk} \quad (1.46)$$

with

$$k_{ijk} \triangleq k_{y_{ijk}} + k_{z_{ijk}} + k_{x_{ijk}} = k_{ikj} \quad (1.47)$$

or in matrix form:

$$\mathcal{U}_e = \frac{1}{2} \boldsymbol{\delta}^T \mathbf{K} \boldsymbol{\delta} = \frac{1}{2} \mathbf{q}^T \begin{bmatrix} \mathbf{O} & \mathbf{O} \\ \mathbf{O} & \mathbf{K} \end{bmatrix} \mathbf{q} \triangleq \frac{1}{2} \mathbf{q}^T \mathbf{K}_e \mathbf{q}, \quad (1.48)$$

where

$$\boldsymbol{\delta} \triangleq [\delta_{11} \cdots \delta_{1m_1} \cdots \delta_{n1} \cdots \delta_{nm_n}]^T \quad (1.49)$$

is the vector of flexible coordinates, and

$$\mathbf{K} = \begin{bmatrix} k_{111} & \cdots & k_{11m_1} & \cdots & \cdots & 0 \\ \vdots & \ddots & \vdots & & & \vdots \\ k_{1m_11} & \cdots & k_{1m_1m_1} & & & \\ \vdots & & & \ddots & & \vdots \\ & & & & k_{n11} & \cdots & k_{n1m_n} \\ \vdots & & & & \vdots & \ddots & \vdots \\ 0 & \cdots & \cdots & k_{nm_n1} & \cdots & k_{nm_nm_n} \end{bmatrix} \quad (1.50)$$

is called the stiffness matrix.

By taking (1.14), (1.17), (1.37) and (1.48) into account, the Lagrangian can be written as

$$\mathcal{L} = \frac{1}{2} (\dot{\mathbf{q}}^T \mathbf{H}(\mathbf{q}) \dot{\mathbf{q}} - \mathbf{q}^T \mathbf{K}_e \mathbf{q}) + \mathbf{g}_0^T \sum_{i=1}^n \mathbf{T}_i \mathbf{h}_i. \quad (1.51)$$

In order to model link modal damping and joint viscous friction, the Rayleigh's dissipation function can be employed with the use of a matrix \mathbf{D} so that [Meirovitch (1967)]:

$$\mathcal{D} = \frac{1}{2} \dot{\mathbf{q}}^T \mathbf{D} \dot{\mathbf{q}}. \quad (1.52)$$

Expressing (1.13) in matrix form yields

$$\frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{q}}} \right)^T - \left(\frac{\partial \mathcal{L}}{\partial \mathbf{q}} \right)^T + \left(\frac{\partial \mathcal{D}}{\partial \dot{\mathbf{q}}} \right)^T = \mathbf{u} \quad (1.53)$$

with

$$\mathbf{u} \triangleq \begin{bmatrix} \boldsymbol{\tau} \\ \mathbf{0} \end{bmatrix}, \quad (1.54)$$

where $\boldsymbol{\tau}$ is the $n \times 1$ vector of the joint torques, and $\mathbf{0}$ denotes an $m \times 1$ vector of null elements ($m = m_1 + \dots + m_n$) accounting for the fact that no generalized force acts on the flexible coordinates $\boldsymbol{\delta}$ as long as clamped boundary conditions at the joint side are assumed.

Then, substituting (1.52) into (1.53) and accounting for (1.51) leads to

$$\mathbf{H}(\mathbf{q})\ddot{\mathbf{q}} + \mathbf{h}_c(\mathbf{q}, \dot{\mathbf{q}}) + \mathbf{K}_e \mathbf{q} + \mathbf{D}\dot{\mathbf{q}} + \mathbf{g}(\mathbf{q}) = \mathbf{u}, \quad (1.55)$$

where

$$\mathbf{h}_c(\mathbf{q}, \dot{\mathbf{q}}) \triangleq \mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})\dot{\mathbf{q}} \quad (1.56)$$

is the vector of Coriolis and centrifugal forces with

$$c_{rs\alpha\beta} = \sum_{i=1}^n \sum_{j=0}^{m_i} c_{ij\alpha\beta rs} \dot{q}_{ij} \quad (1.57)$$

$$c_{ij\alpha\beta rs} \triangleq \frac{1}{2} \left(\frac{\partial h_{rs\alpha\beta}}{\partial q_{ij}} + \frac{\partial h_{rsij}}{\partial q_{\alpha\beta}} - \frac{\partial h_{ij\alpha\beta}}{\partial q_{rs}} \right) \quad (1.58)$$

$$c_{ij\alpha\beta rs} = c_{\alpha\beta ijr s} \quad (1.59)$$

$$g_{rs} = \begin{cases} -\mathbf{g}_0^T \sum_{i=r}^n \frac{\partial \mathbf{T}_i}{\partial q_{rs}} \mathbf{h}_i & \text{if } s = 0 \\ -\mathbf{g}_0^T \left(\sum_{i=r+1}^n \frac{\partial \mathbf{T}_i}{\partial q_{rs}} \mathbf{h}_i + \mathbf{T}_r \mathbf{s}_{rs} \right) & \text{if } s \neq 0. \end{cases} \quad (1.60)$$

1.2.3 Model Properties

In this section some properties of model (1.55) are presented. Many of them are rather physical properties while other arise from the procedure used to derive the dynamic model of the manipulator. Several of these properties are similar to those of rigid manipulators. As a matter of fact,

the properties presented in the following apply to rigid manipulators just by letting the link deformation be zero.

Hereafter, the Euclidean norm for vectors is used, i.e.

$$\|\mathbf{q}\| \triangleq \sqrt{\sum_{i=1}^n q_i^2}. \quad (1.61)$$

The norm of a matrix \mathbf{A} is the corresponding induced norm

$$\|\mathbf{A}\| \triangleq \sqrt{\lambda_{\max}(\mathbf{A}^T \mathbf{A})}, \quad (1.62)$$

where $\lambda_{\max}(\cdot)$ ($\lambda_{\min}(\cdot)$) denotes the largest (smallest) eigenvalue of a matrix. Since all norms in \mathfrak{R}^n are equivalent [Desoer and Vidyasagar (1975)], the results presented in this section are valid for any norm in \mathfrak{R}^n .

A well-known property of the dynamic model of a robot manipulator is the following one.

Property 1.1 The inertia matrix $\mathbf{H}(\mathbf{q})$ is symmetric positive definite.

Proof: It can be seen directly from (1.19), (1.23), (1.31) and (1.34) that $\mathbf{H}(\mathbf{q})$ is symmetric. Since the kinetic energy of any mechanical system can be zero if and only if the system is in a steady state, and otherwise it is always greater than zero, it follows from (1.17) that $\mathbf{H}(\mathbf{q})$ is positive definite. △

The next property is very important. It is related to the passive structure of robot manipulators and it is frequently used in the proof of many control schemes. It gives a relationship between the inertia matrix $\mathbf{H}(\mathbf{q})$ and the matrix $\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})$ employed to compute the vector of Coriolis and centrifugal torques.

Property 1.2 The matrix $\mathbf{N}(\mathbf{q}, \dot{\mathbf{q}}) \triangleq \dot{\mathbf{H}}(\mathbf{q}) - 2\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})$ is skew symmetric.

Proof: Every element of $\dot{\mathbf{H}}(\mathbf{q})$ satisfies

$$\dot{h}_{rs\alpha\beta} = \sum_{i=1}^n \sum_{j=0}^{m_i} \frac{\partial h_{rs\alpha\beta}}{\partial q_{ij}} \dot{q}_{ij}. \quad (1.63)$$

By taking (1.57) and (1.58) into account, the elements of $\mathbf{N}(\mathbf{q}, \dot{\mathbf{q}})$ can be

computed as

$$\begin{aligned}
n_{rs\alpha\beta} &\triangleq \dot{h}_{rs\alpha\beta} - 2c_{rs\alpha\beta} \\
&= \sum_{i=1}^n \sum_{j=0}^{m_i} \left(\frac{\partial h_{rs\alpha\beta}}{\partial q_{ij}} - \left(\frac{\partial h_{rs\alpha\beta}}{\partial q_{ij}} + \frac{\partial h_{rsij}}{\partial q_{\alpha\beta}} - \frac{\partial h_{ij\alpha\beta}}{\partial q_{rs}} \right) \right) \dot{q}_{ij} \\
&= \sum_{i=1}^n \sum_{j=0}^{m_i} \left(\frac{\partial h_{ij\alpha\beta}}{\partial q_{rs}} - \frac{\partial h_{rsij}}{\partial q_{\alpha\beta}} \right) \dot{q}_{ij}.
\end{aligned} \tag{1.64}$$

Since $h_{rs\alpha\beta} = h_{\alpha\beta rs}$, the property holds true. \triangle

Note that Property 1.2 has been proven using the definition of $\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})$ which is in terms of the Christoffel symbols. Since there are many possible definitions for $\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})$, it is worth pointing out that

$$\dot{\mathbf{q}}^T (\dot{\mathbf{H}}(\mathbf{q}) - 2\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})) \dot{\mathbf{q}} = 0 \tag{1.65}$$

is always true no matter what definition of $\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})$ is used [Ortega and Spong (1989)]. To show this, rewrite (1.53) and (1.55) as

$$\frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{q}}} \right)^T - \left(\frac{\partial \mathcal{L}}{\partial \mathbf{q}} \right)^T = \boldsymbol{\psi} = \mathbf{H}(\mathbf{q}) \ddot{\mathbf{q}} + \mathbf{C}(\mathbf{q}, \dot{\mathbf{q}}) \dot{\mathbf{q}} + \mathbf{K}_e \mathbf{q} + \mathbf{g}(\mathbf{q}) \tag{1.66}$$

with

$$\boldsymbol{\psi} \triangleq \mathbf{u} - \mathbf{D} \dot{\mathbf{q}}. \tag{1.67}$$

The Hamiltonian of the system is given by [Ortega and Spong (1989); Greenwood (1977)]

$$\mathcal{H} = \boldsymbol{\pi}^T \dot{\mathbf{q}} - \mathcal{L}, \tag{1.68}$$

where the generalized momentum $\boldsymbol{\pi}$ is defined as

$$\boldsymbol{\pi} = \left(\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{q}}} \right)^T. \tag{1.69}$$

On the other hand, by using (1.51), (1.68) and (1.69), the Hamiltonian can be expressed as the sum of the kinetic and the potential energy of the system, i.e.

$$\mathcal{H} = \frac{1}{2} \dot{\mathbf{q}}^T \mathbf{H}(\mathbf{q}) \dot{\mathbf{q}} + \frac{1}{2} \mathbf{q}^T \mathbf{K}_e \mathbf{q} + \mathcal{U}_g = \mathcal{T} + \mathcal{U}, \tag{1.70}$$

while the Hamilton's equations are given by

$$\dot{q}_i = \frac{\partial \mathcal{H}}{\partial p_i} \quad (1.71)$$

$$\dot{\pi}_i = -\frac{\partial \mathcal{H}}{\partial q_i} + \psi_i \quad i = 1, \dots, n+m. \quad (1.72)$$

By employing (1.71) and (1.72), the derivative of \mathcal{H} can be computed as

$$\frac{d\mathcal{H}}{dt} = \sum_{i=1}^{n+m} \frac{\partial \mathcal{H}}{\partial q_i} \dot{q}_i + \sum_{i=1}^{n+m} \frac{\partial \mathcal{H}}{\partial \pi_i} \dot{\pi}_i = \dot{\mathbf{q}}^T \boldsymbol{\psi}. \quad (1.73)$$

Eqs. (1.66) and (1.70) can be used as well to obtain $d\mathcal{H}/dt$, i.e.

$$\begin{aligned} \frac{d\mathcal{H}}{dt} &= \dot{\mathbf{q}}^T \mathbf{H}(\mathbf{q}) \ddot{\mathbf{q}} + \frac{1}{2} \dot{\mathbf{q}}^T \dot{\mathbf{H}}(\mathbf{q}) \dot{\mathbf{q}} + \dot{\mathbf{q}}^T \mathbf{K}_e \mathbf{q} + \dot{\mathbf{q}}^T \left(\frac{\partial \mathcal{U}_g}{\partial \mathbf{q}} \right)^T \\ &= \dot{\mathbf{q}}^T \boldsymbol{\psi} + \frac{1}{2} \dot{\mathbf{q}}^T \left(\dot{\mathbf{H}}(\mathbf{q}) - 2\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}}) \right) \dot{\mathbf{q}}. \end{aligned} \quad (1.74)$$

By comparing (1.73) and (1.74), one can conclude that (1.65) is valid for any possible choice of $\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})$.

The following property holds for the vector $\mathbf{h}_c(\mathbf{q}, \dot{\mathbf{q}})$ of Coriolis and centrifugal torques.

Property 1.3 The vector $\mathbf{h}_c(\mathbf{q}, \dot{\mathbf{q}})$ of Coriolis and centrifugal torques satisfies the equalities:

$$\mathbf{h}_c(\mathbf{q}, \mathbf{x}, \mathbf{y}) = \mathbf{C}(\mathbf{q}, \mathbf{x})\mathbf{y} = \mathbf{C}(\mathbf{q}, \mathbf{y})\mathbf{x} = \bar{\mathbf{h}}_c(\mathbf{q}, \mathbf{y}, \mathbf{x}) \quad \forall \mathbf{x}, \mathbf{y} \in \mathfrak{R}^{n+m}. \quad (1.75)$$

Proof: The element rs of vector $\mathbf{h}_c(\mathbf{q}, \mathbf{x}, \mathbf{y})$ can be expressed as (see (1.56) and (1.57))

$$\begin{aligned} h_{c_{rs}}(\mathbf{q}, \mathbf{x}, \mathbf{y}) &= \sum_{\alpha=1}^n \sum_{\beta=0}^{m_\alpha} \left(\sum_{i=1}^n \sum_{j=0}^{m_i} c_{ij\alpha\beta rs} x_{ij} \right) y_{\alpha\beta} \\ &= \sum_{i=1}^n \sum_{j=0}^{m_i} \left(\sum_{\alpha=1}^n \sum_{\beta=0}^{m_\alpha} c_{ij\alpha\beta rs} y_{\alpha\beta} \right) x_{ij} \\ &= \sum_{i=1}^n \sum_{j=0}^{m_i} \left(\sum_{\alpha=1}^n \sum_{\beta=0}^{m_\alpha} c_{\alpha\beta ijrs} y_{\alpha\beta} \right) x_{ij} = h_{c_{rs}}(\mathbf{q}, \mathbf{y}, \mathbf{x}). \end{aligned} \quad (1.76)$$

△

Due to the orthogonality of the modes shapes of a flexible-link manipulator, the following property can be obtained.

Property 1.4 The stiffness matrix \mathbf{K} is diagonal and positive definite and satisfies

$$\lambda_{\min}(\mathbf{K})\|\mathbf{x}\|^2 \leq \mathbf{x}^T \mathbf{K} \mathbf{x} \leq \lambda_{\max}(\mathbf{K})\|\mathbf{x}\|^2 \quad \forall \mathbf{x} \in \mathfrak{R}^m. \quad (1.77)$$

Proof: To prove the positive definiteness of the stiffness matrix, the definition of its elements can be used (see (1.43)–(1.45) and (1.47)). Since the mode shapes are orthogonal [Meirovitch (1967)], the result of the integrals must be zero if $j \neq k$ and positive otherwise. Eq. (1.77) follows from the fact that \mathbf{K} is positive definite. \triangle

Regarding the matrix \mathbf{D} of link modal damping and joint viscous friction, the following property can be established.

Property 1.5 The matrix \mathbf{D} is diagonal positive semidefinite and satisfies

$$\lambda_{\min}(\mathbf{D})\|\mathbf{x}\|^2 \leq \mathbf{x}^T \mathbf{D} \mathbf{x} \leq \lambda_{\max}(\mathbf{D})\|\mathbf{x}\|^2 \quad \forall \mathbf{x} \in \mathfrak{R}^{n+m}. \quad (1.78)$$

Proof: \mathbf{D} is positive semidefinite because it is defined on the basis of the Rayleigh's dissipation function (see (1.52)). Assuming it to be diagonal is actually a special but very important and common case of the definition of the Rayleigh's dissipation function [Meirovitch (1967)]. \triangle

Finding bounds on the norms of the matrices of model (1.55) plays an important role in the control of robot manipulators because such bounds are helpful for design of many control schemes. Norm bounds are especially advantageous when Lyapunov theory is used. As a matter of fact, for any mechanical system, the vectors \mathbf{q} and $\dot{\mathbf{q}}$ are bounded. Taking only link deformation into account and in view of the small deformation assumption, the potential energy due to elasticity cannot be infinite, i.e. $U_e < \infty$ [De Luca and Siciliano (1993b); De Luca and Panzieri (1994)], so that it is possible to find a bound for the vector of link coordinates δ .

Property 1.6 The norm of δ is bounded by

$$\|\delta\| \leq \sqrt{\frac{2U_{e,\max}}{\lambda_{\min}(\mathbf{K})}} \triangleq \bar{\delta}, \quad (1.79)$$

where $U_{e,\max}$ is the maximum of the link strain potential energy.

Proof: In view of the assumption of small link deformation, there must be a maximum link strain potential energy. Directly from (1.48) it is

$$\boldsymbol{\delta}^T \mathbf{K} \boldsymbol{\delta} \leq 2U_{e,\max} < \infty, \quad (1.80)$$

from which Property 1.6 follows by taking Property 1.4 into account. \triangle

Notice that Property 1.6 means that $\boldsymbol{\delta}$ belongs to a set Δ whose elements are bounded. For simplicity, every vector $\mathbf{q} \in \mathfrak{R}^n \times \Delta \subset \mathfrak{R}^{n+m}$ will be assumed to belong to a set \mathcal{Q}^{n+m} .

The next four properties are related to the inertia matrix and can easily be derived from Property 1.1.

Property 1.7 The inertia matrix $\mathbf{H}(\mathbf{q})$ satisfies

$$\lambda_{\min}(\mathbf{H}(\mathbf{q}))\|\mathbf{y}\|^2 \leq \mathbf{y}^T \mathbf{H}(\mathbf{q})\mathbf{y} \leq \lambda_{\max}(\mathbf{H}(\mathbf{q}))\|\mathbf{y}\|^2 \quad \forall \mathbf{y} \in \mathfrak{R}^{n+m}. \quad (1.81)$$

Proof: Since $\mathbf{H}(\mathbf{q})$ is positive definite, each vector \mathbf{y} in \mathfrak{R}^{n+m} can be expressed in terms of an orthonormal basis $(\mathbf{y}_1, \dots, \mathbf{y}_{n+m})$ as

$$\mathbf{y} = \sum_{i=1}^{n+m} c_i \mathbf{y}_i, \quad (1.82)$$

implying that

$$\mathbf{y}^T \mathbf{H}(\mathbf{q})\mathbf{y} = c_1^2 \lambda_1(\mathbf{H}(\mathbf{q})) + \dots + c_{n+m}^2 \lambda_{n+m}(\mathbf{H}(\mathbf{q})) \quad (1.83)$$

$$\mathbf{y}^T \mathbf{y} = \|\mathbf{y}\|^2 = c_1^2 + \dots + c_{n+m}^2, \quad (1.84)$$

from which (1.81) follows. \triangle

Property 1.8 The matrix $\mathbf{H}^{-1}(\mathbf{q})$ exists and satisfies

$$\lambda_{\max}^{-1}(\mathbf{H}(\mathbf{q}))\|\mathbf{y}\|^2 \leq \mathbf{y}^T \mathbf{H}^{-1}(\mathbf{q})\mathbf{y} \leq \lambda_{\min}^{-1}(\mathbf{H}(\mathbf{q}))\|\mathbf{y}\|^2 \quad \forall \mathbf{y} \in \mathfrak{R}^{n+m} \quad (1.85)$$

Proof: This property follows directly from Property 1.7. \triangle

Property 1.9 The inertia matrix satisfies

$$\lambda_h \leq \|\mathbf{H}(\mathbf{q})\| \leq \lambda_H < \infty. \quad (1.86)$$

Proof: Since the vector of generalized coordinates is bounded, i.e. $\|\mathbf{q}\| < \infty$, it is easy to see from (1.81) that

$$\lambda_h = \min_{\mathbf{q} \in \mathcal{Q}^{n+m}} \lambda_{\min}(\mathbf{H}(\mathbf{q})) \quad (1.87)$$

$$\lambda_H = \max_{\mathbf{q} \in \mathcal{Q}^{n+m}} \lambda_{\max}(\mathbf{H}(\mathbf{q})). \quad (1.88)$$

△

Property 1.10 The inverse of the inertia matrix satisfies

$$\sigma_h \leq \|\mathbf{H}^{-1}(\mathbf{q})\| \leq \sigma_H < \infty. \quad (1.89)$$

Proof: The proof is the same as in Property 1.9 with

$$\sigma_h = \min_{\mathbf{q} \in \mathcal{Q}^{n+m}} \lambda_{\max}^{-1}(\mathbf{H}(\mathbf{q})) \quad (1.90)$$

$$\sigma_H = \max_{\mathbf{q} \in \mathcal{Q}^{n+m}} \lambda_{\min}^{-1}(\mathbf{H}(\mathbf{q})). \quad (1.91)$$

△

It is easy to recognize that Properties 1.7 to 1.10 are closely related. Of course, by taking only Property 1.7 into account, it is not difficult to develop the other three properties. These properties are very important because many Lyapunov functions employed to prove the stability of a control approach make use of the inertia matrix and its boundedness properties.

Since \mathbf{q} is bounded, it is possible to find the following bound for the matrix $\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})$.

Property 1.11 The matrix $\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})$ satisfies

$$\|\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})\| \leq k_c \|\dot{\mathbf{q}}\|. \quad (1.92)$$

Proof: From (1.57), it can be seen that matrix $\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})$ can be written as

$$\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}}) = \frac{1}{2} \sum_{i=1}^n \sum_{j=0}^{m_i} \mathbf{C}_{ij}(\mathbf{q}) \dot{q}_{ij}, \quad (1.93)$$

so that each element of matrix $\mathbf{C}_{ij}(\mathbf{q})$ is given by

$$\frac{\partial h_{rs\alpha\beta}}{\partial q_{ij}} + \frac{\partial h_{rsij}}{\partial q_{\alpha\beta}} - \frac{\partial h_{ij\alpha\beta}}{\partial q_{rs}}.$$

Computing the norm of $\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})$ leads to

$$\begin{aligned}
\|\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})\| &= \frac{1}{2} \left\| \sum_{i=1}^n \sum_{j=0}^{m_i} \mathbf{C}_{ij}(\mathbf{q}) \dot{q}_{ij} \right\| & (1.94) \\
&\leq \frac{1}{2} \sum_{i=1}^n \sum_{j=0}^{m_i} \|\mathbf{C}_{ij}(\mathbf{q}) \dot{q}_{ij}\| \\
&= \frac{1}{2} \sum_{i=1}^n \sum_{j=0}^{m_i} \|\mathbf{C}_{ij}(\mathbf{q})\| |\dot{q}_{ij}| \\
&\leq \frac{1}{2} \sum_{i=1}^n \sum_{j=0}^{m_i} \|\mathbf{C}_{ij}(\mathbf{q})\| \|\dot{\mathbf{q}}\|.
\end{aligned}$$

With

$$k_c \triangleq \frac{1}{2} \max_{\mathbf{q} \in \mathbb{Q}^{n+m}} \sum_{i=1}^n \sum_{j=0}^{m_i} \|\mathbf{C}_{ij}(\mathbf{q})\|, \quad (1.95)$$

Property 1.11 follows. \triangle

It is worth noticing that Property 1.11 applies to every vector $\mathbf{y} \in \mathbb{R}^{n+m}$.

Because the vector of gravitational torques $\mathbf{g}(\mathbf{q})$ is only a function of \mathbf{q} , one can find a bound related to it as well.

Property 1.12 The vector of gravitational torques $\mathbf{g}(\mathbf{q})$ is bounded by a constant $\sigma_g > 0$, i.e.

$$\|\mathbf{g}(\mathbf{q})\| \leq \sigma_g. \quad (1.96)$$

Proof: Since

$$\|\mathbf{g}(\mathbf{q})\| = \sqrt{\sum_{r=1}^n \sum_{s=0}^{m_r} g_{rs}^2}, \quad (1.97)$$

it should be proven that each term g_{rs} is bounded. By recalling that \mathbf{q} is bounded and taking (1.60) into account, it can easily be seen that each g_{rs} is bounded, so that (1.96) follows. \triangle