Chapter 1

Introduction to Stability Theory

Concept of stability in common and engineering sense reflects necessity to keep response of a disturbed system within acceptable limits. If deviations describing response of the system from a given regime (e.g. state of equilibrium) lie within prescribed limits, the system is called stable. Otherwise, the system is called unstable. Disturbances, response, and prescribed limits can be specified in each case in different ways. In this book we mostly deal with dynamical problems for multiple degrees of freedom systems, and stability of motion is understood in the Liapunov sense.

1.1 Definition of stability

Consider a dynamical system described by ordinary differential equations written in a vector form

$$\dot{\mathbf{y}} = \mathbf{f}(\mathbf{y}, t). \tag{1.1}$$

Here it is assumed that $\mathbf{y} = (y_1, y_2, \ldots, y_m)^T$ is a real state vector, the dot over a symbol means differentiation with respect to time t, and $\mathbf{f} = (f_1, \ldots, f_m)^T$ is a real vector-function smoothly dependent on its variables providing existence and uniqueness of a solution with the initial condition $\mathbf{y}(t_0) = \mathbf{y}_0$ on the semi-infinite interval of time $t \geq t_0$.

When the vector-function $\mathbf{f}$ does not depend on time explicitly, the system is called *autonomous*. Otherwise, the system is called *non-autonomous* or *non-stationary*.

Considering a partial solution $\widetilde{\mathbf{y}}(t)$ of equation (1.1) as undisturbed motion and other solutions $\mathbf{y}(t)$ as disturbed motions, we observe evolution of disturbances $y_i(t_0) - \widetilde{y}_i(t_0), i = 1, \ldots, m$, taken at the initial instant

$t = t_0$, in time. For such solutions [Liapunov (1892)] introduced the well-known definition of stability.

Definition 1.1 The undisturbed motion (solution) $\widetilde{\mathbf{y}}(t)$ of system (1.1) is called *stable* with respect to the variables $y_1, y_2, \ldots, y_m$ if for any $\varepsilon > 0$ there exists $\delta > 0$ such that for any solution $\mathbf{y}(t)$ of (1.1), satisfying the condition $\|\mathbf{y}(t_0) - \widetilde{\mathbf{y}}(t_0)\| < \delta$, the inequality

$$\|\mathbf{y}(t) - \widetilde{\mathbf{y}}(t)\| < \varepsilon \tag{1.2}$$

takes place for all $t \geq t_0$.

If, in addition,

$$\|\mathbf{y}(t) - \widetilde{\mathbf{y}}(t)\| \to 0 \quad \text{as} \quad t \to +\infty, \tag{1.3}$$

then the solution $\widetilde{\mathbf{y}}(t)$ is called *asymptotically stable.*

This definition means that small deviations of the initial conditions remain bounded in time for stable motions (solutions) and tend to zero for asymptotically stable solutions. The restrictions of the Liapunov definition of stability are that the disturbances are taken only at the initial instant of time and they are small. Besides, the state vectors for undisturbed and disturbed motions are compared at the same time, and stability is observed on the infinite interval of time. Nevertheless, the given definition of stability is very useful and practical for many physical problems.

Definition 1.2 The undisturbed motion (solution) $\widetilde{\mathbf{y}}(t)$ of system (1.1) is called *unstable* if there exists $\varepsilon > 0$ such that for any $\delta > 0$ there exists a solution $\mathbf{y}(t)$, satisfying the condition $\|\mathbf{y}(t_0) - \widetilde{\mathbf{y}}(t_0)\| < \delta$, that for some $t_* > t_0$ the inequality

$$\|\mathbf{y}(t_*) - \widetilde{\mathbf{y}}(t_*)\| > \varepsilon \tag{1.4}$$

takes place.

1.2 Equations for disturbed motion

It is convenient to write equations for disturbed motion in the deviations

$$x_i(t) = y_i(t) - \widetilde{y}_i(t). \tag{1.5}$$

Inserting (1.5) into (1.1) and expanding the right-hand side into Taylor series, we obtain

$$\dot{\widetilde{y}}_i + \dot{x}_i = f_i(\widetilde{y}_1, \dots, \widetilde{y}_m, t) + \sum_{j=1}^{m} \frac{\partial f_i}{\partial y_j} x_j + \eta_i(x_1, \dots, x_m, t), \tag{1.6}$$

where η_i are the terms of order higher than one with respect to $x_1, \dots, x_m$. Since for the undisturbed motion we have

$$\dot{\widetilde{y}}_i = f_i(\widetilde{y}_1, \dots, \widetilde{y}_m, t), \tag{1.7}$$

equation (1.6) yields

$$\dot{x}_i = \sum_{j=1}^{m} a_{ij}(t) x_j + \eta_i(x_1, \dots, x_m, t), \quad i = 1, \dots, m, \tag{1.8}$$

where the coefficients

$$a_{ij}(t) = \left(\frac{\partial f_i}{\partial y_j} \right)_{\widetilde{\mathbf{y}}(t)} \tag{1.9}$$

are evaluated at $\mathbf{y} = \widetilde{\mathbf{y}}(t)$. These are the equations for disturbed motion, which can be given in a vector form as

$$\dot{\mathbf{x}} = \mathbf{A}(t)\mathbf{x} + \boldsymbol{\eta}(\mathbf{x}, t) \tag{1.10}$$

with the real vector $\boldsymbol{\eta} = (\eta_1, \dots, \eta_m)^T$ and the matrix

$$\mathbf{A} = \begin{pmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mm} \end{pmatrix}. \tag{1.11}$$

The linear equation for the vector $\mathbf{x}$

$$\dot{\mathbf{x}} = \mathbf{A}(t)\mathbf{x} \tag{1.12}$$

is called the *equation of first approximation* or *linearized equation* for disturbed motion.

Generally, differential equations for disturbed motion contain time t explicitly. However, there are important cases, when these equations are independent on time. This happens when stability of an *equilibrium state* $\widetilde{\mathbf{y}}(t) \equiv \mathbf{y}_0$ of an autonomous system is studied. In this case all the functions $\widetilde{y}_i(t)$ are constant and the functions f_i do not depend explicitly on time t. That is why the equations for disturbed motion do not contain time explicitly and the coefficients a_{ij} are constant. Independence on time for the equations of disturbed motion can also occur when stability of a specific motion $\tilde{\mathbf{y}}(t)$ of an autonomous system is studied.

We will call the undisturbed motion $\tilde{\mathbf{y}}(t)$ *steady* if the corresponding equation (1.10) does not contain time t explicitly. The case of steady motion is one of the simplest for the stability study. Another rather simple case is when the coefficients a_{ij} in (1.8) are periodic functions of time t.

1.3 Linear autonomous system

In this section we consider linear autonomous systems of the form

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} \tag{1.13}$$

with a constant real $m \times m$ matrix $\mathbf{A}$. Seeking solution to this problem as

$$\mathbf{x} = \mathbf{u} \exp \lambda t, \tag{1.14}$$

we substitute (1.14) into (1.13) and come to the eigenvalue problem

$$\mathbf{A}\mathbf{u} = \lambda \mathbf{u}, \tag{1.15}$$

where λ is an eigenvalue and $\mathbf{u}$ is an eigenvector.

A non-trivial solution to (1.15) exists if and only if

$$\det(\mathbf{A} - \lambda \mathbf{I}) = 0, \tag{1.16}$$

where $\mathbf{I}$ is the $m \times m$ identity matrix. This is the *characteristic equation* for eigenvalues λ, which can be represented in a polynomial form

$$\lambda^m + a_{m-1}\lambda^{m-1} + \cdots + a_0 = 0 \tag{1.17}$$

with the coefficients $a_0, a_1, \ldots, a_{m-1}$ dependent on elements of the matrix $\mathbf{A}$. From equation (1.17) we find the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_m$, which are real or complex conjugate numbers. The eigenvalues can be simple or multiple as the roots of characteristic equation (1.17). Assuming that all the roots of equation (1.17) are simple with the corresponding eigenvectors $\mathbf{u}_1, \ldots, \mathbf{u}_m$, a general solution to (1.13) takes the form

$$\mathbf{x}(t) = c_1 \mathbf{u}_1 \exp \lambda_1 t + \cdots + c_m \mathbf{u}_m \exp \lambda_m t, \tag{1.18}$$

where $c_1, c_2, \ldots, c_m$ are constant coefficients to be found from the initial condition $\mathbf{x}(t_0) = \mathbf{x}_0$.

If we admit multiple eigenvalues as the roots of characteristic equation (1.17), the number of linearly independent eigenvectors can be less than m. The form of general solution (1.18) remains valid only for so-called *semi-simple* eigenvalues, when in spite of multiplicity the number of linearly independent eigenvectors is equal to m. But generally multiple eigenvalues lead to *secular* terms proportional to $t^{\beta} \exp \lambda t$ in the general solution, where the integer exponent β is less than the multiplicity of λ as a root of the characteristic equation.

Consider, for example, a multiple eigenvalue λ with a *Jordan chain* of vectors $\mathbf{u}_0, \ldots, \mathbf{u}_{r-1}$ satisfying the equations

$$\begin{aligned} \mathbf{A}\mathbf{u}_0 &= \lambda \mathbf{u}_0, \\ \mathbf{A}\mathbf{u}_1 &= \lambda \mathbf{u}_1 + \mathbf{u}_0, \\ &\vdots \\ \mathbf{A}\mathbf{u}_{r-1} &= \lambda \mathbf{u}_{r-1} + \mathbf{u}_{r-2}. \end{aligned} \tag{1.19}$$

The vector $\mathbf{u}_0$ is the eigenvector, and $\mathbf{u}_1, \ldots, \mathbf{u}_{r-1}$ are called *associated* vectors corresponding to λ. Then the terms in the general solution corresponding to λ are the following, see [Lancaster (1966)]:

$$\begin{aligned} & c_0 \mathbf{u}_0 \exp \lambda t + c_1 (\mathbf{u}_0 t + \mathbf{u}_1) \exp \lambda t + \\ & \cdots + c_{r-1} \left(\frac{\mathbf{u}_0 t^{r-1}}{(r-1)!} + \frac{\mathbf{u}_1 t^{r-2}}{(r-2)!} + \cdots + \mathbf{u}_{r-1} \right) \exp \lambda t. \end{aligned} \tag{1.20}$$

The general solution may contain several terms of type (1.20) with different Jordan chains corresponding to the same eigenvalue λ. This happens when there are several eigenvectors for the same λ.

From (1.18) and (1.20) it is obvious that if real parts of all the eigenvalues are negative, $\operatorname{Re} \lambda < 0$, the norm of the general solution $\|\mathbf{x}(t)\| \to 0$ as $t \to +\infty$. This property means the asymptotic stability of the trivial solution $\mathbf{x}(t) \equiv 0$. And if there exists at least one eigenvalue with a positive real part, $\operatorname{Re} \lambda > 0$, then there are infinitely growing solutions $\mathbf{x}(t)$ for arbitrarily small initial conditions, which means instability of the trivial solution. Notice that the terms in the general solution corresponding to purely imaginary or zero eigenvalues (with $\operatorname{Re} \lambda = 0$) remain bounded only for simple or semi-simple eigenvalues, otherwise due to the presence of secular terms in (1.20) we get $\|\mathbf{x}(t)\| \to \infty$ as $t \to +\infty$. Now we can formulate the statements for stability and instability of linear systems.

Theorem 1.1 *Linear system (1.13) is asymptotically stable if and only if all the eigenvalues of the matrix* **A** *have negative real part* $\operatorname{Re}\lambda < 0$.

System (1.13) is stable if and only if all the eigenvalues of the matrix **A** *have negative or zero real part* $\operatorname{Re}\lambda \leq 0$ *with all the purely imaginary and zero eigenvalues being simple or semi-simple.*

Finally, linear system (1.13) is unstable if and only if there exists an eigenvalue of the matrix **A** *with a positive real part* $\operatorname{Re}\lambda > 0$, *or an eigenvalue with zero real part* $\operatorname{Re}\lambda = 0$, *which is neither simple nor semi-simple.*

1.4 Introduction of parameters

We assume now that elements of the matrix **A** smoothly depend on a vector of real parameters $\mathbf{p} = (p_1, p_2, \ldots, p_n)$. With a change of parameters stability of system (1.13) can change to instability. This happens when one or several eigenvalues λ cross the imaginary axis, see Fig. 1.1.

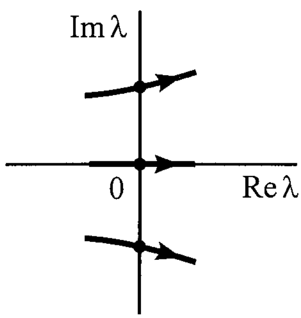

Fig. 1.1 How stability is changed to instability.

The case when a pair of complex conjugate eigenvalues crosses the imaginary axis with a frequency $\omega = \operatorname{Im}\lambda \neq 0$ is known in technical literature as *flutter instability*, and the case when a real negative eigenvalue λ crosses zero and becomes positive is called *divergence instability*. Flutter and divergence are dynamic and static forms of instability, respectively. According to Theorem 1.1, the parameter space can be divided into the stability and instability domains, see Fig. 1.2. The asymptotic stability domain is determined by the condition $\operatorname{Re}\lambda < 0$ satisfied for all the eigenvalues, and the instability domain is defined by the condition $\operatorname{Re}\lambda > 0$ for at least one eigenvalue. The boundary between the stability and instability domains is determined by the condition $\operatorname{Re}\lambda = 0$ satisfied for at least one eigenvalue

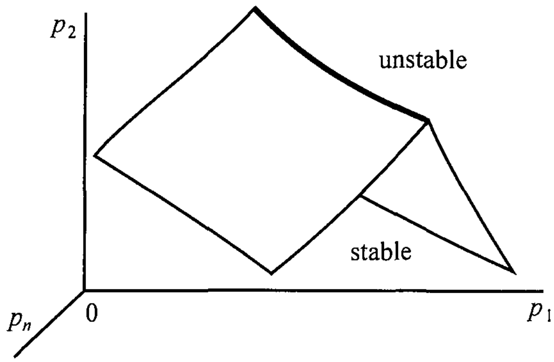

Fig. 1.2 Stability and instability domains.

while for others the condition $\mathrm{Re}\,\lambda < 0$ is fulfilled.

1.5 Stability theorems based on first approximation

Let us consider an autonomous system (1.1) and assume an equilibrium state or steady motion $\tilde{\mathbf{y}}$. Equation for disturbed motion (1.10) takes the form

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \eta(\mathbf{x}), \quad \mathbf{x} = \mathbf{y} - \tilde{\mathbf{y}}. \tag{1.21}$$

Liapunov formulated and proved two important theorems on stability of an equilibrium state or steady motion $\tilde{\mathbf{y}}$ of an autonomous system, based on linear approximation (1.13) of equation (1.21), see [Liapunov (1892); Chetayev (1961)].

Theorem 1.2 *If linearized system (1.13) is asymptotically stable, i.e., all the eigenvalues of the matrix* $\mathbf{A}$ *have negative real part* $\mathrm{Re}\,\lambda < 0$, *then the equilibrium state or steady motion* $\tilde{\mathbf{y}}$ *is asymptotically stable independently of the nonlinear term* $\eta(\mathbf{x})$ *in (1.21).*

Theorem 1.3 *If linearized system (1.13) has an eigenvalue with a positive real part* $\mathrm{Re}\,\lambda > 0$, *then the equilibrium state or steady motion* $\tilde{\mathbf{y}}$ *is unstable independently of the nonlinear term* $\eta(\mathbf{x})$ *in (1.21).*

These are the main theorems for stability and instability of non-linear systems based on the analysis of the first (linear) approximation. Notice that the case when some of the eigenvalues or all of them have zero real part is not covered by these theorems. Those cases are called *special cases* of Liapunov, see [Liapunov (1892); Chetayev (1961); Malkin (1966)]. Stability and instability of an equilibrium state or steady motion in these special cases depend on the non-linear terms.

We assume now that the right-hand side of equation for disturbed motion (1.21) smoothly depends on a vector of real parameters $\mathbf{p} = (p_1, \ldots, p_n)$. Then the parameter space can be divided into the stability and instability domains based on properties of eigenvalues of the matrix $\mathbf{A}(\mathbf{p})$. The boundary between these domains is characterized by the conditions $\operatorname{Re}\lambda = 0$ for some eigenvalues and $\operatorname{Re}\lambda < 0$ for the others. In Chapter 3 it is shown that the boundary between the stability and instability domains is a smooth surface, which can have different kind of singularities. We note that generally nothing can be said about stability or instability of the solution $\widetilde{\mathbf{y}}$ on the boundary between the stability and instability domains based on Liapunov's Theorems 1.2 and 1.3. However, in many physical problems this is not so important, since the boundary is a set of zero measure in the parameter space.

Example 1.1 Let us consider a rigid body moving inertially about a fixed point (the Euler case). Equations of motion are the following, see [Malkin (1966)]:

$$\begin{aligned} A\dot{\omega}_x + (C - B)\omega_y\omega_z &= 0, \\ B\dot{\omega}_y + (A - C)\omega_z\omega_x &= 0, \\ C\dot{\omega}_z + (B - A)\omega_x\omega_y &= 0, \end{aligned} \tag{1.22}$$

where ω_x, ω_y, and ω_z are the projections of the vector of angular velocity on the principal axes of inertia x, y, z of the body, and A, B, C are the moments of inertia about those axes.

System (1.22) admits a partial solution

$$\omega_x = \omega_0 = const, \quad \omega_y = \omega_z = 0, \tag{1.23}$$

which is rotation about the axis x with the constant angular velocity ω_0 (steady motion). There are also similar solutions describing rotations about the axes y and z.

Let us study stability of motion (1.23). Introducing the variables

$$x_1 = \omega_x - \omega_0, \quad x_2 = \omega_y, \quad x_3 = \omega_z \tag{1.24}$$

and substituting them into (1.22), we obtain the equations for disturbed motion

$$\begin{aligned} \dot{x}_1 &= \frac{B-C}{A} x_2 x_3, \\ \dot{x}_2 &= \frac{C-A}{B} \omega_0 x_3 + \frac{C-A}{B} x_1 x_3, \\ \dot{x}_3 &= \frac{A-B}{C} \omega_0 x_2 + \frac{A-B}{C} x_1 x_2. \end{aligned} \tag{1.25}$$

The characteristic equation for the linearized problem

$$\det \begin{pmatrix} -\lambda & 0 & 0 \\ 0 & -\lambda & \dfrac{C-A}{B}\omega_0 \\ 0 & \dfrac{A-B}{C}\omega_0 & -\lambda \end{pmatrix} = 0 \tag{1.26}$$

gives the eigenvalues

$$\lambda_{1,2} = \pm\omega_0 \sqrt{\frac{(C-A)(A-B)}{BC}}, \quad \lambda_3 = 0. \tag{1.27}$$

If $C < A < B$ or $C > A > B$, i.e., if rotation takes place about the axis corresponding to the intermediate moment of inertia, then eigenvalues (1.27) are real, one of them being positive. Thus, according to Theorem 1.3 rotation about this axis is unstable.

However, stability of rotation about the axis corresponding to the extremal (smallest or largest) moment of inertia can not be established with the use of Theorem 1.2 because in these cases two eigenvalues (1.27) are purely imaginary and the third eigenvalue is zero. Stability of rotation in those cases can be proven using integrals of motion, see [Malkin (1966)].

1.6 Mechanical systems

Consider a scleronomic mechanical system with holonomic constraints having m degrees of freedom. This means that position of the system is specified by a vector of generalized coordinates $\mathbf{q} = (q_1, \ldots, q_m)^T$, generalized forces do not depend on time t explicitly, and constraints imposed on the system depend only on generalized coordinates. Motion of the system is governed by the Lagrange equations

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} = g_i(\mathbf{q}, \dot{\mathbf{q}}), \quad i = 1, \ldots, m. \tag{1.28}$$

In these equations, the *kinetic energy* T is a quadratic form with respect to generalized velocities $\dot{q}_1, \ldots, \dot{q}_m$:

$$T = \frac{1}{2}\sum_{i,j=1}^{m} m_{ij}\dot{q}_i\dot{q}_j = \frac{1}{2}\dot{\mathbf{q}}^T\mathbf{M}\dot{\mathbf{q}} \tag{1.29}$$

with the positive definite mass matrix $\mathbf{M} = [m_{ij}] > 0$ dependent only on the generalized coordinates $q_1, \ldots, q_m$; $\mathbf{g} = (g_1, \ldots, g_m)^T$ is the vector of generalized forces.

Let $\mathbf{q}(t) \equiv 0$ be an equilibrium state of the system defined by the condition $\mathbf{g}(0, 0) = 0$. Then the linear approximation of the generalized forces near the equilibrium state yields

$$\mathbf{g} = -\mathbf{B}\dot{\mathbf{q}} - \mathbf{C}\mathbf{q} \tag{1.30}$$

Generally, the matrices $\mathbf{B}$ and $\mathbf{C}$ are non-symmetric. They can be represented as the sum of symmetric and skew-symmetric matrices

$$\mathbf{B} = \mathbf{D} + \mathbf{G}, \quad \mathbf{C} = \mathbf{P} + \mathbf{N} \tag{1.31}$$

with

$$\begin{aligned} \mathbf{D} &= \tfrac{1}{2}(\mathbf{B} + \mathbf{B}^T), \quad \mathbf{G} = \tfrac{1}{2}(\mathbf{B} - \mathbf{B}^T), \\ \mathbf{P} &= \tfrac{1}{2}(\mathbf{C} + \mathbf{C}^T), \quad \mathbf{N} = \tfrac{1}{2}(\mathbf{C} - \mathbf{C}^T). \end{aligned} \tag{1.32}$$

The force $-\mathbf{P}\mathbf{q}$ with the symmetric matrix $\mathbf{P}$ is called *potential* or *conservative*, and the quadratic form

$$\Pi(\mathbf{q}) = \frac{1}{2}\mathbf{q}^T\mathbf{P}\mathbf{q} \tag{1.33}$$

is the *potential energy.* Potential forces are widely known in mechanics, e.g. gravitational and elastic forces are conservative.

The quadratic form

$$R(\dot{\mathbf{q}}) = \frac{1}{2}\dot{\mathbf{q}}^T\mathbf{D}\dot{\mathbf{q}} \tag{1.34}$$

with the symmetric positive semi-definite matrix $\mathbf{D} \geq 0$ is called the *Rayleigh's dissipative function*, and the force $-\mathbf{D}\dot{\mathbf{q}}$ is called *dissipative.* In case of positive definite matrix $\mathbf{D} > 0$ dissipation is *complete*, and the case

$\mathbf{D} \geq 0$ corresponds to *incomplete* dissipation. Dissipative forces describe viscous friction and resistance of medium appearing in physical systems.

The force $-\mathbf{G}\dot{\mathbf{q}}$ with the skew-symmetric matrix $\mathbf{G}^T = -\mathbf{G}$ is called *gyroscopic*. Power of a gyroscopic force (the work done by the force per unit time) is zero. Indeed, we have

$$(-\mathbf{G}\dot{\mathbf{q}})^T \dot{\mathbf{q}} = -\dot{\mathbf{q}}^T \mathbf{G}^T \dot{\mathbf{q}} = \dot{\mathbf{q}}^T \mathbf{G} \dot{\mathbf{q}} = 0 \tag{1.35}$$

since the vector $\dot{\mathbf{q}}$ is real and the matrix $\mathbf{G}$ is skew-symmetric. Gyroscopic forces appear in rotating systems, Coriolis and Lorentz forces are gyroscopic too.

The force $-\mathbf{N}\mathbf{q}$ is called *non-conservative positional* or *circulatory*. Notice that this force is orthogonal to the vector of generalized coordinates $\mathbf{q}$ since $\mathbf{q}^T\mathbf{N}\mathbf{q} = 0$. Circulatory forces appear as components of aerodynamic and follower forces (e.g. jet thrust). The presence of circulatory and dissipative forces means that the system can gain energy from the environment or lose energy, depending upon the ratio between the forces and their magnitudes for all the types of forces involved.

Using (1.29)–(1.32) in Lagrange equations (1.28), we obtain in a linear approximation

$$\mathbf{M}\ddot{\mathbf{q}} + (\mathbf{D} + \mathbf{G})\dot{\mathbf{q}} + (\mathbf{P} + \mathbf{N})\mathbf{q} = 0. \tag{1.36}$$

This is the linearized equation for disturbed motion near the equilibrium state $\mathbf{q} = 0$. Seeking solution to equation (1.36) in the form

$$\mathbf{q} = \mathbf{u} \exp \lambda t, \tag{1.37}$$

we come to the eigenvalue problem

$$\left(\lambda^2\mathbf{M} + \lambda(\mathbf{D} + \mathbf{G}) + \mathbf{P} + \mathbf{N}\right)\mathbf{u} = 0. \tag{1.38}$$

Here λ is an eigenvalue and $\mathbf{u}$ is an eigenvector. The eigenvalues are found from the characteristic equation

$$\det\left(\lambda^2\mathbf{M} + \lambda(\mathbf{D} + \mathbf{G}) + \mathbf{P} + \mathbf{N}\right) = 0. \tag{1.39}$$

This is an algebraic equation of $2m$th order for λ. There exist $2m$ roots $\lambda_1, \ldots, \lambda_{2m}$ (the eigenvalues), and corresponding eigenvectors should be found from equation (1.38).

Equation (1.36) can be transformed to a system of first order differential equations of double dimension:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} \tag{1.40}$$

with

$$\mathbf{x} = \begin{pmatrix} \mathbf{q} \\ \dot{\mathbf{q}} \end{pmatrix}, \quad \mathbf{A} = \begin{pmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{M}^{-1}(\mathbf{P}+\mathbf{N}) & -\mathbf{M}^{-1}(\mathbf{D}+\mathbf{G}) \end{pmatrix}. \tag{1.41}$$

It is easy to see that the characteristic equation $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$ for the matrix $\mathbf{A}$ in (1.41) is equivalent to equation (1.39) implying that the eigenvalues in these two problems coincide.

If all the eigenvalues $\lambda_1, \ldots, \lambda_{2m}$ of (1.39) are simple or semi-simple, the general solution to equation (1.36) takes the form

$$\mathbf{q}(t) = c_1\mathbf{u}_1 \exp\lambda_1 t + \cdots + c_{2m}\mathbf{u}_{2m} \exp\lambda_{2m} t. \tag{1.42}$$

If the number of eigenvectors corresponding to a multiple eigenvalue λ is less than its multiplicity (as a root of the characteristic equation), secular terms appear in the general solution. Using transformation (1.40), (1.41) and the results of Section 1.3, we find that those terms are of the form

$$\begin{aligned} & c_0\mathbf{u}_0 \exp\lambda t + c_1(\mathbf{u}_0 t + \mathbf{u}_1)\exp\lambda t + \\ & \cdots + c_{r-1}\left(\frac{\mathbf{u}_0 t^{r-1}}{(r-1)!} + \frac{\mathbf{u}_1 t^{r-2}}{(r-2)!} + \cdots + \mathbf{u}_{r-1}\right)\exp\lambda t, \end{aligned} \tag{1.43}$$

where $\mathbf{u}_0$ is the eigenvector, and $\mathbf{u}_1, \ldots, \mathbf{u}_{r-1}$ are associated vectors constituting the *Keldysh chain* of length r, see [Keldysh (1951)] and Section 2.13:

$$\begin{aligned} & \left(\lambda^2\mathbf{M} + \lambda(\mathbf{D}+\mathbf{G}) + \mathbf{P} + \mathbf{N}\right)\mathbf{u}_i + (2\lambda\mathbf{M} + \mathbf{D} + \mathbf{G})\mathbf{u}_{i-1} + \mathbf{M}\mathbf{u}_{i-2} = 0, \\ & i = 0, \ldots, r-1 \quad \text{and} \quad \mathbf{u}_{-1} = \mathbf{u}_{-2} = 0. \end{aligned} \tag{1.44}$$

From (1.42), (1.43) it is obvious that system (1.36) is asymptotically stable if all the eigenvalues have negative real part, and it is unstable if at least one eigenvalue has positive real part.

Example 1.2 Let us consider vibrations of a pendulum with a linear viscous damping described by the equation

$$ml\ddot{\varphi} + \gamma l\dot{\varphi} + mg\sin\varphi = 0, \tag{1.45}$$

where φ is the angle of the pendulum measured from the vertical axis; m, l, and γ are the mass, length, and damping coefficient of the pendulum, respectively; and g is the acceleration of gravity, see Fig. 1.3.

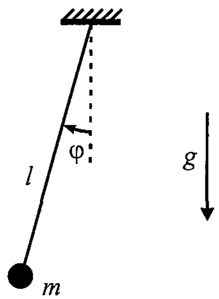

Fig. 1.3 Vibrating pendulum.

We introduce new variables $y_1 = \varphi$, $y_2 = \dot{\varphi}$, and rewrite (1.45) as the system of first order equations

$$\begin{aligned} \dot{y}_1 &= y_2, \\ \dot{y}_2 &= -\frac{g \sin y_1}{l} - \frac{\gamma y_2}{m}. \end{aligned} \tag{1.46}$$

To find stationary solutions we equate the right-hand sides of (1.46) zero and obtain two solutions

$$y_1 = 0, \quad y_2 = 0; \tag{1.47}$$

$$y_1 = \pi, \quad y_2 = 0. \tag{1.48}$$

These solutions correspond to lower and upper equilibrium states, respectively.

First, we consider equilibrium state (1.47). For this case the disturbances x_1 and x_2 coincide with the variables y_1 and y_2. Expanding the right-hand sides of equation (1.46) in Taylor series and replacing the variables, we obtain

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= -\frac{g x_1}{l} - \frac{\gamma x_2}{m} + \frac{g x_1^3}{6l} + o(x_1^3). \end{aligned} \tag{1.49}$$

Thus, the matrix $\mathbf{A}$ of the linearized system is

$$\mathbf{A} = \begin{pmatrix} 0 & 1 \\ -\frac{g}{l} & -\frac{\gamma}{m} \end{pmatrix}. \tag{1.50}$$

The characteristic equation for this matrix

$$\lambda^2 + \frac{\gamma}{m}\lambda + \frac{g}{l} = 0 \tag{1.51}$$

yields the roots

$$\lambda_{1,2} = -\frac{\gamma}{2m} \pm \sqrt{\frac{\gamma^2}{4m^2} - \frac{g}{l}}, \tag{1.52}$$

which always have negative real part for $\gamma > 0$. Thus, the lower equilibrium state is asymptotically stable.

For the case of upper equilibrium state (1.48) we have

$$x_1 = y_1 - \pi, \quad x_2 = y_2. \tag{1.53}$$

Substituting relations (1.53) into (1.46) and expanding the right-hand sides, we get the equations for disturbed motion as

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= \frac{gx_1}{l} - \frac{\gamma x_2}{m} - \frac{gx_1^3}{6l} + o(x_1^3). \end{aligned} \tag{1.54}$$

It is easy to see that the linearization of equation (1.54) gives the eigenvalues

$$\lambda_{1,2} = -\frac{\gamma}{2m} \pm \sqrt{\frac{\gamma^2}{4m^2} + \frac{g}{l}}, \tag{1.55}$$

one of them always having a positive real part. Thus, the upper vertical equilibrium state is unstable.

1.7 Asymptotic stability criteria for mechanical systems

Let us investigate stability of a linear mechanical system

$$\mathbf{M}\ddot{\mathbf{q}} + (\mathbf{D} + \mathbf{G})\dot{\mathbf{q}} + (\mathbf{P} + \mathbf{N})\mathbf{q} = 0 \tag{1.56}$$

with the corresponding eigenvalue problem

$$\left(\lambda^2\mathbf{M} + \lambda(\mathbf{D} + \mathbf{G}) + \mathbf{P} + \mathbf{N}\right)\mathbf{u} = 0. \tag{1.57}$$

System (1.56) is asymptotically stable if all the eigenvalues λ of problem (1.57) have negative real part.

We pre-multiply equation (1.57) by the complex-conjugate transposed eigenvector $\mathbf{u}^* = \overline{\mathbf{u}}^T$ and obtain the relation

$$M\lambda^2 + (D + iG)\lambda + P + iN = 0 \tag{1.58}$$

with the coefficients

$$\begin{gathered} M = \mathbf{u}^*\mathbf{M}\mathbf{u}, \quad D = \mathbf{u}^*\mathbf{D}\mathbf{u}, \quad P = \mathbf{u}^*\mathbf{P}\mathbf{u}, \\ iG = \mathbf{u}^*\mathbf{G}\mathbf{u}, \quad iN = \mathbf{u}^*\mathbf{N}\mathbf{u}, \end{gathered} \tag{1.59}$$

where M, D, P, G, and N are real quantities. Additionally, we assume that the eigenvector is normalized as

$$\mathbf{u}^*\mathbf{u} = 1. \tag{1.60}$$

Considering (1.58) as a quadratic equation for λ with complex coefficients, we demand that both roots of equation (1.58) have negative real part. Here we can use a theorem on stability properties of a polynomial with complex coefficients, see [Bilharz (1944)]. Applied to equation (1.58), the theorem states that both roots λ have negative real part if and only if the two determinants satisfy the relations

$$\det\begin{pmatrix} M & G \\ 0 & D \end{pmatrix} > 0, \quad \det\begin{pmatrix} M & G & -P & 0 \\ 0 & D & N & 0 \\ 0 & M & G & -P \\ 0 & 0 & D & N \end{pmatrix} > 0. \tag{1.61}$$

Since the matrix $\mathbf{M}$ is positive definite the quantity $M > 0$, and then (1.61) is equivalent to two inequalities

$$D > 0, \tag{1.62}$$

$$MN^2 - GDN < D^2P. \tag{1.63}$$

Metelitsyn was the first who derived inequality (1.63), assuming that (1.62) is satisfied, as a criterion for asymptotic stability of system (1.56), see [Metelitsyn (1952)].

Notice that an eigenvalue of (1.57) is one of the two roots of (1.58), the other root does not need to be an eigenvalue of (1.57). This important fact was pointed out in [Seyranian (1994b)]. Actually, it is more an exception than a rule that both roots of equation (1.58) are the eigenvalues of

(1.57). Metelitsyn made a mistake, also done in [Huseyin (1978)], believing that both roots are always eigenvalues of (1.57). This mistake led to the conclusion that inequality (1.63) is a necessary and sufficient condition for asymptotic stability. We emphasize that inequalities (1.62) and (1.63) taken for all eigenvalues λ are *sufficient* for asymptotic stability, but not necessary.

Example 1.3 Let the system be described by equation (1.56) with the 2×2 matrices

$$\mathbf{M} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \ \mathbf{D} = \begin{pmatrix} 5.8186 & 0 \\ 0 & 0.1814 \end{pmatrix}, \ \mathbf{G} = \begin{pmatrix} 0 & 3.6667 \\ -3.6667 & 0 \end{pmatrix},$$

$$\mathbf{P} = \begin{pmatrix} -0.5 & 0 \\ 0 & -0.5 \end{pmatrix}, \quad \mathbf{N} = \begin{pmatrix} 0 & 2.25 \\ -2.25 & 0 \end{pmatrix}. \tag{1.64}$$

The eigenvalues are $\lambda_{1,2} = -1 \pm 0.5i$ and $\lambda_{3,4} = -2 \pm 0.5i$, and therefore the system is asymptotically stable. Computing the corresponding eigenvectors $\mathbf{u}$, coefficients (1.59) of quadratic equation (1.58) can be determined. The roots of this equation (one equation for each eigenvalue) are, of course, the four already found eigenvalues $\lambda_{1,2}$ and $\lambda_{3,4}$, but additionally also $0.0625 \pm 0.875i$ and $0.1785 \pm 0.2859i$. Those roots have positive real part and, therefore, in spite of the system is asymptotically stable, Metelitsyn's inequality (1.63) is not satisfied since it demands that both roots of equation (1.58) have negative real part.

If we want to investigate the asymptotic stability for a given system by checking inequalities (1.62) and (1.63) as sufficient conditions, we face the following problem. The eigenvectors $\mathbf{u}$, which are used for finding coefficients (1.59) and for checking inequalities (1.62) and (1.63), are unknown. They can only be determined by solving eigenvalue problem (1.57), but then the stability analysis would be complete.

However, the statement known as the Thomson-Tait-Chetayev theorem, see [Chetayev (1961)], follows directly from inequalities (1.62) and (1.63).

Theorem 1.4 *If system (1.56) containing only potential forces is stable* ($\mathbf{P} > 0$)*, then addition of arbitrary gyroscopic forces and dissipative forces with complete dissipation* ($\mathbf{D} > 0$) *makes the system asymptotically stable.*

Indeed, the system possessing only potential forces with a positive definite matrix $\mathbf{P}$ is stable since all the eigenvalues are purely imaginary and semi-simple, see [Gantmacher (1998); Merkin (1997)]. In case of $\mathbf{P} > 0$, $\mathbf{D} > 0$, and $\mathbf{N} = 0$, i.e., $P > 0$, $D > 0$, and $N = 0$, inequality (1.63) reduces to $D^2P > 0$, guaranteeing asymptotic stability.

We are interested in obtaining a practical sufficient stability condition, which can be verified using extremal eigenvalues of the system matrices. Hermitian matrices like $\mathbf{M}$, $\mathbf{D}$, and $\mathbf{P}$ (real symmetric in our case) have only real eigenvalues. The corresponding quantities M, D, and P in (1.59), known as Rayleigh quotients, are therefore limited by the minimal and maximal eigenvalues of the matrices $\mathbf{M}$, $\mathbf{D}$, and $\mathbf{P}$, respectively, see e.g. [Lancaster and Tismenetsky (1985)]

$$\begin{aligned} M_{\min} = \lambda_{\min}(\mathbf{M}) \leq M \leq \lambda_{\max}(\mathbf{M}) = M_{\max}, \\ D_{\min} = \lambda_{\min}(\mathbf{D}) \leq D \leq \lambda_{\max}(\mathbf{D}) = D_{\max}, \\ P_{\min} = \lambda_{\min}(\mathbf{P}) \leq P \leq \lambda_{\max}(\mathbf{P}) = P_{\max}. \end{aligned} \tag{1.65}$$

We emphasize that these limits depend only on the system matrices and do not depend on the eigenvector $\mathbf{u}$.

Since the matrices $\mathbf{G}$ and $\mathbf{N}$ are real skew-symmetric, the matrices $i\mathbf{G}$ and $i\mathbf{N}$ are Hermitian. Notice that spectrum of a real skew-symmetric matrix consists of purely imaginary $\pm i\omega$ and zero eigenvalues. Therefore, the quantities G and N being real are limited by $-G_{\max}$ and $G_{\max}$, and $-N_{\max}$ and $N_{\max}$, respectively, where $G_{\max} = \lambda_{\max}(i\mathbf{G})$ and $N_{\max} = \lambda_{\max}(i\mathbf{N})$ are the maximal eigenvalues of the corresponding matrices. So, we have

$$-G_{\max} \leq G \leq G_{\max}, \quad -N_{\max} \leq N \leq N_{\max}. \tag{1.66}$$

If we assume

$$\mathbf{M} > 0, \quad \mathbf{D} > 0, \quad \mathbf{P} > 0, \tag{1.67}$$

then it is easy to see with the help of (1.65) and (1.66) that (1.63), rewritten in the form $D(DP + GN) - MN^2 > 0$, is satisfied for an arbitrary eigenvector $\mathbf{u}$ if

$$D_{\min}(D_{\min}P_{\min} - G_{\max}N_{\max}) - M_{\max}N_{\max}^2 > 0. \tag{1.68}$$

Here we took the smallest values of the first and second terms and the largest value of the third term of the inequality. Under assumption (1.67), inequality (1.68) is a practical sufficient condition for asymptotic stability of system (1.56), which can be checked knowing only the extreme eigenvalues of the system matrices $\mathbf{M}$, $\mathbf{D}$, $\mathbf{G}$, $\mathbf{P}$, and $\mathbf{N}$ [Kliem *et al.* (1998)].

From (1.68) we deduce the following stability statement.

Theorem 1.5 *Any mechanical system (1.56) can be stabilized by sufficiently large dissipative and/or potential forces* ($\mathbf{D} > 0, \mathbf{P} > 0$).

Here "large forces" means that the minimal eigenvalues D_{min} and/or P_{min} of the corresponding matrices are large enough.

Proof of the theorem follows from the observation that inequality (1.68) is satisfied by making the term $D^2_{\text{min}} P_{\text{min}}$ sufficiently large. This result was first reported in [Seyranian (1994b)]. Another consequence of inequality (1.68) is that a stable conservative system with dissipative forces with complete dissipation (assumption (1.67)) can not be destabilized by adding rather small gyroscopic and/or positional non-conservative forces.

Example 1.4 The simplest model of a rotor consists of a massless shaft of circular cross-section with an elastic coefficient k rotating with a constant angular velocity Ω and carrying a single disk of mass m, see Fig. 1.4. External and internal damping coefficients are denoted by $d_e > 0$ and $d_i > 0$, respectively. With respect to an inertial frame, the equations of motion for the center of mass of the disk moving in the plane perpendicular to the shaft are given by (1.56) with the matrices [Bolotin (1963)]

$$\mathbf{M} = \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}, \quad \mathbf{D} = \begin{pmatrix} d_e + d_i & 0 \\ 0 & d_e + d_i \end{pmatrix}, \quad \mathbf{G} = 0,$$
$$\mathbf{P} = \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}, \quad \mathbf{N} = \begin{pmatrix} 0 & d_i\Omega \\ -d_i\Omega & 0 \end{pmatrix}. \tag{1.69}$$

Quantities (1.65) and (1.66) evaluated for system matrices (1.69) are

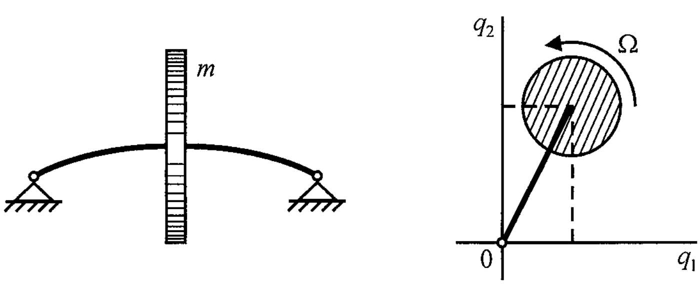

Fig. 1.4 Rotating shaft with a disk.

equal to

$$M_{\min} = M_{\max} = m, \quad D_{\min} = D_{\max} = d_e + d_i,$$
$$P_{\min} = P_{\max} = k, \quad G_{\max} = 0, \quad N_{\max} = d_i\Omega. \tag{1.70}$$

For system (1.56), (1.69) inequality (1.68) results in

$$\Omega^2 m d_i^2 < k(d_e + d_i)^2. \tag{1.71}$$

This inequality gives a lower bound for the critical angular velocity

$$\Omega_* = \sqrt{\frac{k}{m}}\left(1 + \frac{d_e}{d_i}\right). \tag{1.72}$$

Let us compare this estimate with the exact critical velocity. For this purpose, we introduce the complex variable $z = q_1 - iq_2$ and rewrite equations (1.56), (1.69) in a complex form as

$$m\ddot{z} + (d_e + d_i)\dot{z} + (k + id_i\Omega)z = 0. \tag{1.73}$$

The corresponding characteristic equation

$$m\lambda^2 + (d_e + d_i)\lambda + (k + id_i\Omega) = 0 \tag{1.74}$$

is a quadratic equation for λ with complex coefficients. Applying now inequalities (1.62) and (1.63), we obtain that the stability condition is the same as (1.71). Thus, estimate (1.72) is the exact critical velocity. This is one of those rare cases when sufficient stability condition (1.68) yields the exact stability boundary.