# CHAPTER I

# INTRODUCTION

This chapter is an introductory material to the theory of minimal submanifolds. We begin with the notion of the second fundamental form, from which the minimal submanifolds can be neatly defined. Then they are characterized by a variational property.

In this chapter we also give important properties for minimal submanifolds in Euclidean space and specify the equation for minimal graphs of codimension one. This is a famous equation in mathematics. Some basic properties for minimal submanifolds in the sphere are described which enable us to give examples of minimal submanifolds in the sphere.

The Bochner technique is important in differential geometry. J. Simons [Si] employed the technique to prove the well-known intrinsic rigidity theorem for minimal submanifolds which is introduced in the last section.

## 1.1 The Second Fundamental Form

In the classical surface theory in $\mathbb{R}^3$ there are first and second fundamental forms. We know that a plane and a circular cylinder in $\mathbb{R}^3$ are locally isometric. But their shapes are different, since they have different second fundamental forms. The invariants determined only by the first fundamental form are intrinsic; others are extrinsic invariants which are dependent not only on the first fundamental form, but also on the second fundamental form. The shape of a surface in $\mathbb{R}^3$ is influenced by the second fundamental form.

For the general setting of the immersed submanifold we can generalize this notion. This is the aim of the present section. The second fundamental form satisfies the Gauss equation, the Codazzi equations and the Ricci equations . Those are fundamental equations.

Let $\bar{M}$ be a Riemannian manifold of dimension $\bar{n}$, $M$ be an $n$-dimensional Riemannian manifold. We assume that $\bar{n} = n + k$, $k > 0$. Let $M \to \bar{M}$ be an isometric immersion which means that the natural induced Riemannian metric on $M$ from the ambient space $\bar{M}$ coincides with the original one on $M$. The number $k$ is called codimension of $M$ in $\bar{M}$. If $k = 1$, the submanifold $M$ is called a hypersurface in $\bar{M}$.

According to the fundamental theorem in Riemannian geometry, there exists a unique Levi-Civita connection. Besides its preserving inner product it satisfies the torsion free condition.

For each $p \in M$ the tangent space $T_p\bar{M}$ can be decomposed to a direct sum of $T_pM$ and its orthogonal complement $N_pM$ in $T_p\bar{M}$. Such a decomposition is differentiable. So that we have an orthogonal decomposition of the tangent bundle $T\bar{M}$ along $M$

$$T\bar{M}|_M = TM \oplus NM.$$

Let $(\cdots)^T$ and $(\cdots)^N$ denote the orthogonal projections into the tangent bundle $TM$ and the normal bundle $NM$ respectively.

Let $\overline{\nabla}$ be the Levi-Civita connection on $\bar{M}$. As vector bundles $TM$, $NM$ over $M$, they carry the induced metrics as their fiber metrics.

DEFINITION 1.1.1 For $V, W \in \Gamma(TM)$, $\nu \in \Gamma(NM)$, the induced connections on $TM$ and $NM$ are defined by

$$\nabla_V W \overset{def.}{=} (\overline{\nabla}_V W)^T,$$

$$\nabla_V \nu \overset{def.}{=} (\overline{\nabla}_V \nu)^N.$$

PROPOSITION 1.1.2 $\nabla$ *is just the Levi - Civita connection on* $M$.

PROOF. Let $X$, $Y$ and $Z \in \Gamma(TM)$ be tangent vector fields. Then

$$\begin{aligned} \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle &= \left\langle (\overline{\nabla}_X Y)^T, Z\right\rangle + \left\langle Y, (\overline{\nabla}_X Z)^T\right\rangle \\ &= \left\langle \overline{\nabla}_X Y, Z\right\rangle + \left\langle Y, \overline{\nabla}_X Z\right\rangle \\ &= \overline{\nabla}_X \langle Y, Z\rangle = \nabla_X \langle Y, Z\rangle . \end{aligned}$$

Thus, $\nabla$ preserves inner product on $TM$. To show that $\nabla$ is torsion free we see that

$$\begin{aligned}\nabla_Y Z - \nabla_Z Y - [Y, Z] &= (\overline{\nabla}_Y Z)^T - (\overline{\nabla}_Z Y)^T - [Y, Z] \\ &= (\overline{\nabla}_Y Z)^T - (\overline{\nabla}_Z Y)^T - [Y, Z]^T \\ &= (\overline{\nabla}_Y Z - \overline{\nabla}_Z Y - [Y, Z])^T.\end{aligned}$$

Q. E. D.

As done in the above proposition, the induced connection $\nabla$ on the normal bundle also preserves the inner product.

Consider

$$B_{VW} \overset{def.}{=} \left(\overline{\nabla}_V W\right)^N = \overline{\nabla}_V W - \nabla_V W$$

for $V, W \in \Gamma(TM)$. First of all, for any smooth function $f$,

$$B_{fVW} = f\, B_{VW}.$$

Secondly,

$$B_{VW} - B_{WV} = \{\overline{\nabla}_V W - \overline{\nabla}_W V\}^N = \{[V, W]\}^N = 0.$$

Hence,

$$B_{VfW} = B_{fWV} = f\, B_{WV} = f\, B_{VW}.$$

Those properties show that $B$ is a symmetric bilinear form on $TM$ with values in $NM$. We call $B$ to be second fundamental form of $M$ in $\bar{M}$.

For $\nu \in \Gamma(NM)$ we define the shape operator $A^\nu : TM \to TM$ by

$$A^\nu(V) = -(\overline{\nabla}_V \nu)^T.$$

It is easy to check that $A^\nu$ is a symmetric operator on the tangent space at each point, moreover, it satisfies the Weingarten equations:

$$\langle B_{XY}, \nu\rangle = \langle A^\nu(X), Y\rangle\,. \tag{1.1.1}$$

DEFINITION 1.1.3 If $B \equiv 0$, then $M$ is called a totally geodesic submanifold in $\bar{M}$.

From the definition of the second fundamental form, we see that $M$ is a totally geodesic submanifold, if and only if any geodesic in $M$ is also a geodesic in the ambient manifold $\bar{M}$.

Suppose that $\bar{M}$ possesses an isometry $\eta$. We see that the image of any geodesic under $\eta$ in $\bar{M}$ is also a geodesic. Thus, each component of the fixed-point set of $\eta$ inherits a manifold structure and becomes a totally geodesic submanifold.

Taking the trace of $B$ gives the mean curvature vector $H$ of $M$ in $\bar{M}$ and

$$H \stackrel{def.}{=} \frac{1}{n}\text{trace}(B) = \frac{1}{n}\sum_{i=1}^{n} B_{e_i e_i},$$

where $\{e_i\}$ is a local orthonormal frame field of $M$. The mean curvature vector is a cross-section of the normal bundle.

REMARK The definition of the mean curvature in some references is different from one here by a constant factor which is equal to the dimension of the submanifold.

DEFINITION 1.1.4 If $H \equiv 0$, then $M$ is a minimal submanifold in $\bar{M}$.

DEFINITION 1.1.5 If $H$ is a parallel cross-section on the normal bundle, then $M$ is defined to be a submanifold with parallel mean curvature.

From the definitions one immediately sees that a totally geodesic submanifold $M$ in $\bar{M}$ is necessarily a minimal submanifold and any minimal submanifold is a manifold with parallel mean curvature.

Note the special case that $M$ is a hypersurface in $\bar{M}$. Fix a unit normal vector field $\nu$ locally. Then the second fundamental form is determined by

$$A \stackrel{def.}{=} A^{\nu}.$$

This is symmetric on tangent space at each point. Its eigenvalues $k_1, \cdots, k_n$ are called the principal curvatures. The product of all principal curvatures is called the Gauss - Kronecker curvature. It is easy to see that the mean curvature is the mean value of all principal curvatures. In this case there is a notion of constant mean curvature hypersurfaces instead of manifolds with parallel mean curvature.

We can define the curvature tensors $R_{XY}Z$ and $R_{XY}\mu$, corresponding to the connections in the tangent bundle and the normal bundle respectively:

$$R_{XY}Z = -\nabla_X \nabla_Y Z + \nabla_Y \nabla_X Z + \nabla_{[X,Y]} Z,$$

$$R_{XY}\mu = -\nabla_X \nabla_Y \mu + \nabla_Y \nabla_X \mu + \nabla_{[X,Y]} \mu,$$

where $X$, $Y$, $Z$ are tangent vector fields, $\mu$ is a normal vector field. Those are related to the curvature tensor $\bar{R}$ of the ambient manifold $\bar{M}$ and the second fundamental form $B$.

PROPOSITION 1.1.6 (GAUSS EQUATION)

$$\langle R_{XY}Z, W\rangle = \langle \bar{R}_{XY}Z, W\rangle - \langle B_{XW}, B_{YZ}\rangle + \langle B_{XZ}, B_{YW}\rangle\,, \quad (1.1.2)$$

*where $X, Y, Z, W$ are tangent vector fields in $M$, their images under the isometric immersion are tangent vector fields in $\bar{M}$. For the simplicity we use the same notations.*

PROOF. Noting the definition of the curvature tensor,

$$\begin{aligned}
\langle \bar{R}_{XY}Z, W\rangle &= \langle -\overline{\nabla}_X \overline{\nabla}_Y Z + \overline{\nabla}_Y \overline{\nabla}_X Z + \overline{\nabla}_{[X,Y]} Z, W\rangle \\
&= \langle -\nabla_X \nabla_Y Z + \nabla_Y \nabla_X Z + \nabla_{[X,Y]} Z, W\rangle \\
&\quad + \langle -\overline{\nabla}_X B_{YZ} + \overline{\nabla}_Y B_{XZ}, W\rangle \\
&= \langle R_{XY}Z, W\rangle + \langle B_{YZ}, \overline{\nabla}_X W\rangle - \langle B_{XZ}, \overline{\nabla}_Y W\rangle \\
&= \langle R_{XY}Z, W\rangle + \langle B_{YZ}, B_{XW}\rangle - \langle B_{XZ}, B_{YW}\rangle
\end{aligned}$$

Q. E. D.

REMARK From the Gauss equation we obtain the famous Theorem Egiregium of Gauss: Let $M$ be a surface in $\mathbb{R}^3$. Then the sectional curvature of $M$ is equal to the Gauss - Kronecker curvature of $M$.

PROPOSITION 1.1.7 (CODAZZI EQUATIONS)

$$(\nabla_X B)_{YZ} - (\nabla_Y B)_{XZ} = -(\bar{R}_{XY} Z)^N \tag{1.1.3}$$

PROOF. By definitions

$$\begin{aligned}(\nabla_X B)_{YZ} &= \nabla_X B_{YZ} - B_{\nabla_X YZ} - B_{Y\nabla_X Z} \\ &= \left(\overline{\nabla}_X (\overline{\nabla}_Y Z)^N\right)^N - \left(\overline{\nabla}_Z (\overline{\nabla}_X Y)^T\right)^N \\ &\qquad - \left(\overline{\nabla}_Y (\overline{\nabla}_X Z)^T\right)^N,\end{aligned}$$

and similarly,

$$\begin{aligned}(\nabla_Y B)_{XZ} = \left(\overline{\nabla}_Y (\overline{\nabla}_X Z)^N\right)^N &- \left(\overline{\nabla}_Z (\overline{\nabla}_Y X)^T\right)^N \\ &- \left(\overline{\nabla}_X (\overline{\nabla}_Y Z)^T\right)^N,\end{aligned}$$

thus,

$$\begin{aligned}&(\nabla_X B)_{YZ} - (\nabla_Y B)_{XZ} \\ &\quad = \left(\overline{\nabla}_X \overline{\nabla}_Y Z\right)^N - \left(\overline{\nabla}_Y \overline{\nabla}_X Z\right)^N - \left(\overline{\nabla}_Z [X,Y]\right)^N \\ &\quad = \left(\overline{\nabla}_X \overline{\nabla}_Y Z - \overline{\nabla}_Y \overline{\nabla}_X Z - \overline{\nabla}_{[X,Y]} Z\right)^N - [Z,[X,Y]]^N \\ &\quad = -\left(\bar{R}_{XY} Z\right)^N\end{aligned}$$

Q. E. D.

PROPOSITION 1.1.8 (RICCI EQUATIONS)

$$\begin{aligned}\langle R_{XY}\mu, \nu\rangle = \left\langle \bar{R}_{XY}\mu, \nu\right\rangle &+ \langle B_{Xe_i}, \mu\rangle \langle B_{Ye_i}, \nu\rangle \\ &- \langle B_{Xe_i}, \nu\rangle \langle B_{Ye_i}, \mu\rangle,\end{aligned} \tag{1.1.4}$$

*where $\{e_i\}$ is a local orthonormal frame field, $\mu$, $\nu$ are normal vector fields in $M$. Here and in the sequel we use the summation convention.*

PROOF. By a direct computation

$$\begin{aligned}
\langle \bar{R}_{XY}\mu, \nu\rangle &= \langle -\overline{\nabla}_X\overline{\nabla}_Y\mu + \overline{\nabla}_Y\overline{\nabla}_X\mu + \overline{\nabla}_{[X,Y]}\mu, \nu\rangle \\
&= \langle -\overline{\nabla}_X(\nabla_Y\mu + (\overline{\nabla}_Y\mu)^T), \nu\rangle \\
&\qquad + \langle \overline{\nabla}_Y(\nabla_X\mu + (\overline{\nabla}_X\mu)^T) + \nabla_{[X,Y]}\mu, \nu\rangle \\
&= \langle -\nabla_X\nabla_Y\mu - \overline{\nabla}_X(\overline{\nabla}_Y\mu)^T, \nu\rangle \\
&\qquad + \langle \nabla_Y\nabla_X\mu + \overline{\nabla}_Y(\overline{\nabla}_X\mu)^T + \nabla_{[X,Y]}\mu, \nu\rangle \\
&= \langle R_{XY}\mu, \nu\rangle + \langle (\overline{\nabla}_Y\mu)^T, (\overline{\nabla}_X\nu)^T\rangle - \langle (\overline{\nabla}_X\mu)^T, (\overline{\nabla}_Y\nu)^T\rangle \\
&= \langle R_{XY}\mu, \nu\rangle + \langle A^\mu(Y), A^\nu(X)\rangle - \langle A^\mu(X), A^\nu(Y)\rangle \\
&= \langle R_{XY}\mu, \nu\rangle + \langle A^\mu(Y), e_i\rangle \langle A^\nu(X), e_i\rangle \\
&\qquad - \langle A^\mu(X), e_i\rangle \langle A^\nu(Y), e_i\rangle \\
&= \langle R_{XY}\mu, \nu\rangle + \langle B_{Ye_i}, \mu\rangle \langle B_{Xe_i}, \nu\rangle - \langle B_{Xe_i}, \mu\rangle \langle B_{Ye_i}, \nu\rangle,
\end{aligned}$$

where (1.1.1) has been used.

Q. E. D.

The equations of Gauss, Codazzi and Ricci are fundamental equations for the local theory of the immersed submanifolds. It is possible to state a generalization of the fundamental theorem of local surface theory in $\mathbb{R}^3$. We refer the readers to the book [Spi] (vol. IV, pp 64-74).

REMARK If the ambient manifold $\bar{M}$ is pseudo-Riemannian manifold which implies that the metric is not positive definite, we also can study its submanifold theory. In particular, we are interested in Riemannian submanifolds in pseudo-Euclidean space, where we can define the second fundamental form, mean curvature etc. by the parallel way. We will concentrate this situation in the last chapter.

## 1.2 The First Variational Formula

The notion of totally geodesic submanifolds is a higher dimensional generalization of geodesics. But, those are very few in general