PROOF. By a direct computation

$$
\begin{aligned}
\langle \bar{R}_{XY}\mu, \nu \rangle &= \langle -\overline{\nabla}_X \overline{\nabla}_Y \mu + \overline{\nabla}_Y \overline{\nabla}_X \mu + \overline{\nabla}_{[X,Y]}\mu, \nu \rangle \\
&= \langle -\overline{\nabla}_X (\nabla_Y \mu + (\overline{\nabla}_Y \mu)^T), \nu \rangle \\
&\qquad + \langle \overline{\nabla}_Y (\nabla_X \mu + (\overline{\nabla}_X \mu)^T) + \nabla_{[X,Y]}\mu, \nu \rangle \\
&= \langle -\nabla_X \nabla_Y \mu - \overline{\nabla}_X (\overline{\nabla}_Y \mu)^T, \nu \rangle \\
&\qquad + \langle \nabla_Y \nabla_X \mu + \overline{\nabla}_Y (\overline{\nabla}_X \mu)^T + \nabla_{[X,Y]}\mu, \nu \rangle \\
&= \langle R_{XY}\mu, \nu \rangle + \langle (\overline{\nabla}_Y \mu)^T, (\overline{\nabla}_X \nu)^T \rangle - \langle (\overline{\nabla}_X \mu)^T, (\overline{\nabla}_Y \nu)^T \rangle \\
&= \langle R_{XY}\mu, \nu \rangle + \langle A^\mu(Y), A^\nu(X) \rangle - \langle A^\mu(X), A^\nu(Y) \rangle \\
&= \langle R_{XY}\mu, \nu \rangle + \langle A^\mu(Y), e_i \rangle \langle A^\nu(X), e_i \rangle \\
&\qquad - \langle A^\mu(X), e_i \rangle \langle A^\nu(Y), e_i \rangle \\
&= \langle R_{XY}\mu, \nu \rangle + \langle B_{Ye_i}, \mu \rangle \langle B_{Xe_i}, \nu \rangle - \langle B_{Xe_i}, \mu \rangle \langle B_{Ye_i}, \nu \rangle ,
\end{aligned}
$$

where (1.1.1) has been used.

Q. E. D.

The equations of Gauss, Codazzi and Ricci are fundamental equations for the local theory of the immersed submanifolds. It is possible to state a generalization of the fundamental theorem of local surface theory in $\mathbb{R}^3$. We refer the readers to the book [Spi] (vol. IV, pp 64-74).

REMARK If the ambient manifold $\bar{M}$ is pseudo-Riemannian manifold which implies that the metric is not positive definite, we also can study its submanifold theory. In particular, we are interested in Riemannian submanifolds in pseudo-Euclidean space, where we can define the second fundamental form, mean curvature etc. by the parallel way. We will concentrate this situation in the last chapter.

1.2 The First Variational Formula

The notion of totally geodesic submanifolds is a higher dimensional generalization of geodesics. But, those are very few in general situation. Note that geodesics are critical points of the arc length functional.

A minimal submanifold is defined to be one with vanishing mean curvature. This definition seems to have no relation with the "minimal" terminology. In fact, Lagrange found minimal surfaces in his investigation of the calculus of variations. Now, we generalize Lagrange's study to more general setting. Consider the space $\mathcal{I}(M, \bar{M})$ of all immersions from M into $\bar{M}$. Then the volume $\mathrm{vol}(f(M))$ is a functional on the space. The critical points of the volume functional are minimal submanifolds by the following first variational formula. Thus, the notion of minimal submanifolds is an adequate generalization of that of geodesics

To obtain critical points of the functional let us derive the first variational formula. First of all we need the following algebraic result.

LEMMA 1.2.1 *Let*

$$A(t) = (a_{ij}(t)), \ |t| < \varepsilon$$

be a smooth family of $n \times n$ matrices satisfying $A(0) = I$ (the identity matrix). Then

$$\frac{d}{dt} \det A(t)\bigg|_{t=o} = \operatorname{trace} A'(0).$$

PROOF. Assume that $\varepsilon_1, \cdots, \varepsilon_n$ is a standard basis in $\mathbb{R}^n$. We have

$$\det(A(t))\varepsilon_1 \wedge \cdots \wedge \varepsilon_n = (A(t)\varepsilon_1) \wedge \cdots \wedge (A(t)\varepsilon_n).$$

Taking derivatives at both sides of the above equation, and then letting $t = 0$, we obtain

$$\begin{aligned}
(\text{R. H. S.})'|_{t=0} &= \sum_{j=1}^{n} A(0)\varepsilon_1 \wedge \cdots \wedge A'(0)\varepsilon_j \wedge \cdots \wedge A(0)\varepsilon_n \\
&= \sum_{j,k} \varepsilon_1 \wedge \cdots \wedge \langle A'(0)\varepsilon_j, \varepsilon_k \rangle \varepsilon_k \wedge \cdots \wedge \varepsilon_n \\
&= \sum_{j=1}^{n} \langle A'(0)\varepsilon_j, \varepsilon_j \rangle \varepsilon_1 \wedge \cdots \wedge \varepsilon_n \\
&= \operatorname{trace} A'(0)\, \varepsilon_1 \wedge \cdots \wedge \varepsilon_n
\end{aligned}$$

Now, we can derive the first variational formula.

THEOREM 1.2.2 *Let M be a compact Riemannian manifold, $f : M \to \bar{M}$ an isometric immersion with mean curvature vector H. Let f_t, $|t| < \varepsilon$, $f_0 = f$, be a smooth family of immersions satisfying $f_t|_{\partial M} = f|_{\partial M}$. Denote $V = \frac{\partial f_t}{\partial t}\Big|_{t=0}$ to be the variational vector field along f. Then*

$$\frac{d}{dt}\text{vol}(f_t M)\Big|_{t=o} = -\int_M \langle n\,H, V\rangle\, d\,\text{vol}. \qquad (1.2.1)$$

PROOF. Let g_t be the induced metric of the immersion f_t, and $d\,\text{vol}_t$ its corresponding volume element. Choose a local orthonormal frame field $\{e_1, \cdots, e_n\}$ in M with respect to the metric g_0. Its dual frame field is $\{\omega^1, \cdots, \omega^n\}$. We have

$$g_{ij}(t) = \langle f_{t*}e_i, f_{t*}e_j\rangle = g_t(e_i, e_j),$$

where $g_t = g_{ij}(t)\omega^i \otimes \omega^j$, $g_{ij}(0) = \delta_{ij}$. Set $g(t) = \det((g_{ij})(t))$. Thus,

$$\begin{aligned}\text{vol}(f_t M) &= \int_M d\,\text{vot}_t = \int_M \sqrt{g(t)}\,\omega^1 \wedge \cdots \wedge \omega^n \\ &= \int_M \sqrt{g(t)}\, d\,\text{vol}.\end{aligned}$$

We have

$$\frac{d}{dt}\text{vol}(f_t M)\Big|_{t=0} = \frac{1}{2}\int_M g'(0)\, d\,\text{vol}.$$

By Lemma 1.2.1 we have at each point p in M

$$\frac{d}{dt}d\text{vol}_t\Big|_{t=0} = \frac{1}{2}\sum_{k=1}^{n} g'_{kk}(0)\, d\text{vol}.$$

Let $\{\frac{\partial}{\partial t}, e_1, \cdots, e_n\}$ be a frame field in $U \times (-\varepsilon, \varepsilon)$, where U is a small neighborhood of p in M. Let $V(t), e_1(t), \cdots, e_n(t)$ denote the images of those vector fields under the map $F : M \times (-\varepsilon, \varepsilon) \to \bar{M}$

(defined by $F(x,t) = f_t(x)$). Obviously, $e_i(0) = e_i$, $V(0) = V$ and $g_{kk}(t) = \langle e_k(t), e_k(t)\rangle$. We then have

$$\begin{aligned}\frac{d}{dt}g_{kk}(t) &= V(t)\left\langle e_k,(t), e_k(t)\right\rangle = 2\left\langle \overline{\nabla}_V e_k(t), e_k(t)\right\rangle \\ &= 2\left\langle \overline{\nabla}_{e_k(t)}V(t) + [V(t), e_k(t)], e_k(t)\right\rangle \\ &= 2\left\langle \overline{\nabla}_{e_k(t)}V(t), e_k(t)\right\rangle \\ &= 2\left[e_k(t)\left\langle V(t), e_k(t)\right\rangle - \left\langle V(t), \overline{\nabla}_{e_k(t)}e_k(t)\right\rangle\right].\end{aligned}$$

Therefore,

$$\begin{aligned}\frac{1}{2}\sum_{k=1}^{n} g'_{kk}(0) &= e_k\left\langle V, e_k\right\rangle - \left\langle V, \overline{\nabla}_{e_k}e_k\right\rangle \\ &= e_k\left\langle V^T, e_k\right\rangle - \left\langle V, (\overline{\nabla}_{e_k}e_k)^T\right\rangle - \left\langle V, n\,H\right\rangle \\ &= \operatorname{div}(V^T) - \left\langle V, n\,H\right\rangle\end{aligned}$$

and

$$\frac{d}{dt}\operatorname{vol}(f_tM)\Big|_{t=0} = \int_M \operatorname{div}(V^T)\,d\operatorname{vol} - \int_M \left\langle V, n\,H\right\rangle\,d\operatorname{vol}.$$

Then using the Stokes theorem gives (1.2.1)

Q. E. D.

REMARK 1.2.3 The first variational formula (1.2.1) shows that the $-n\,H$ represents the gradient of the volume functional. The equation $H = 0$ is the Euler - Lagrange equation for the functional.

REMARK 1.2.4 If we restrict the variation above to be normal, namely V is normal to M everywhere and $V^T = 0$, then the formula remains valid without the boundary condition.

REMARK 1.2.5 If M is not compact, then the formula can be used for compactly supported variations.