## 1.3 Minimal Submanifolds in Euclidean Space

The study of minimal surfaces in $\mathbb{R}^3$ is an interesting subject since Lagrange's time. Up to now the subject still attracts many mathematicians. The present section starts with its interesting feature on the coordinate functions. Then, we derive the equation for minimal graphs of codimension one in $\mathbb{R}^{n+1}$.

Let $M$ be a Riemannian manifold of dimension $m$. Consider the Laplace operator $\Delta : C^\infty(M) \to C^\infty(M)$. For $f \in C^\infty(M)$ choose a local orthonormal frame field $\{e_1, \cdots, e_m\}$ in $M$. Then

$$\Delta f = e_i e_i(f) - (\nabla_{e_i} e_i) f. \tag{1.3.1}$$

Around each point $p$, there are local coordinates $(x^1, \cdots, x^m)$, where the Riemannian metric on $M$ can be written as $ds^2 = g_{ij}dx^i dx^j$. If we denote $(g^{ij}) = (g_{ij})^{-1}$ and $g = \det(g_{ij})$, then

$$\Delta f = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^i} \left( \sqrt{g}\, g^{ij} \frac{\partial f}{\partial x^j} \right). \tag{1.3.2}$$

In general, for any differential form with values in a vector bundle we can define exterior differential operator $d$ and codifferential operator $\delta$ and the Hodge - Laplace operator $d\delta + \delta d$. The minus sign of the Hodge -Laplace operator acting on a smooth function $f$, a cross-section of the trivial bundle $M \times \mathbb{R}$, is just the ordinary Laplace operator

$$\Delta f = -\delta d f. \tag{1.3.3}$$

We omit the verification of the equivalence of those three definitions, which is left to the readers as an exercise.

Any $f \in C^\infty(M)$ satisfying $\Delta f = 0$ is called a harmonic function. We have the Hopf maximum principle for harmonic functions: any harmonic function on a Riemannian manifold has to be a constant, if it attains the local maximum in an interior point.

Now let us study the minimal submanifolds in Euclidean space.

PROPOSITION 1.3.1 *Let $\psi : M \to \mathbb{R}^n$ be an isometric immersion with the mean curvature vector $H$, then*

$$\Delta\psi = mH, \tag{1.3.4}$$

*where $\Delta\psi = (\Delta\psi^1, \cdots, \Delta\psi^n)$.*

PROOF. Note the fact $X(\psi) = \psi_* X \cong X$ for any $X \in TM$. Let $\{e_i\}$ be a local orthonormal frame field. Then

$$\begin{aligned}\Delta\psi &= e_i(e_i(\psi)) - (\nabla_{e_i} e_i)(\psi)\\ &= \overline{\nabla}_{e_i}\overline{\nabla}_{e_i}\psi - (\nabla_{e_i} e_i)(\psi)\\ &= \overline{\nabla}_{e_i} e_i - \nabla_{e_i} e_i\\ &= \left(\overline{\nabla}_{e_i} e_i\right)^N = mH.\end{aligned}$$

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COROLLARY 1.3.2 *An isometric immersion $\psi : M \to \mathbb{R}^n$ is a minimal immersion if and only if each component of $\psi$ is a harmonic function on $M$.*

REMARK 1.3.3 In this case the equation (1.3.4) reduces to $\Delta\psi = 0$. However, this is not a linear equation, since the induced metric would change when the immersion $\psi$ changes, and so does the operator $\Delta$.

From Corollary 1.3.2 and the Hopf maximum principle we have immediately:

COROLLARY 1.3.4 *There is no compact minimal submanifold in Euclidean space.*

From Corollary 1.3.4, it is natural to ask the question whether there exists a bounded but complete minimal submanifold in Euclidean space. This is the well-known Calabi-Yau problem, which has been answered positively a few years ago by [N].

From the first variational formula (1.2.1) we know that $H = 0$ is the Euler-Lagrangian equations for the volume functional of immersed submanifolds in an ambient manifold. What is the equations look like? Let us see the simplest situation.

In $\mathbb{R}^{n+1}$ a minimal graph $M$ is defined by

$$x^{n+1} = f(x^1, \cdots, x^n).$$

We denote $f_i = \frac{\partial f}{\partial x^i}$. The induced metric on $M$ is

$$ds^2 = g_{ij}\, dx^i\, dx^j,$$

where

$$g_{ij} = \delta_{ij} + f_i f_j.$$

Denote $w = \sqrt{1 + \sum_i f_i^2}$. We have $g^{ij} = \delta_{ij} - \frac{1}{w^2} f_i f_j$. The unit normal vector to $M$ is

$$\nu = \frac{1}{w}(f_1, \cdots, f_n, -1).$$

It is obvious that

$$\overline{\nabla}_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = \frac{\partial}{\partial x^i}\left(0, \cdots, 0, 1, 0, \cdots, 0, \frac{\partial f}{\partial x^j}\right) = (0, \cdots, f_{ij})$$

and

$$\left\langle B_{\frac{\partial}{\partial x^i} \frac{\partial}{\partial x^j}}, \nu \right\rangle = \left\langle \overline{\nabla}_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j}, \nu \right\rangle = -\frac{1}{w} f_{ij}.$$

From $H = 0$ it follows that $g^{ij} f_{ij} = 0$. Thus, we obtain the minimal hypersurface equation

$$(1 + \sum_i f_i^2)\, f_{jj} - f_i f_j f_{ij} = 0, \tag{1.3.5}$$

which is equivalent to

$$\frac{\partial}{\partial x^i}\left(\frac{1}{w} \frac{\partial f}{\partial x^i}\right) = 0. \tag{1.3.6}$$

When $n = 2$ (1.3.5) reduces to

$$(1 + f_y^2) f_{xx} - 2 f_x f_y f_{xy} + (1 + f_x^2) f_{yy} = 0, \tag{1.3.7}$$

where we denote $x = x^1$, $y = x^2$.

It is a nonlinear elliptic PDE. On a minimal submanifold in $\mathbb{R}^n$ there is another important equation. In fact, we have

PROPOSITION 1.3.5 *Let $M$ be an oriented hypersurface with constant mean curvature in $\mathbb{R}^{n+1}$ and with second fundamental form $B$. Let $\nu$ be the unit normal vector to $M$. Then for any fixed vector $a \in \mathbb{R}^{n+1}$,*

$$\Delta \langle a, \nu \rangle + |B|^2 \langle a, \nu \rangle = 0. \tag{1.3.8}$$

PROOF. Choose a local orthonormal frame field $\{e_i\}$ with $\nabla_{e_j} e_i = 0$ at the considered point. Then

$$\begin{aligned}
\Delta \langle a, \nu \rangle &= \nabla_{e_i} \nabla_{e_i} \langle a, \nu \rangle \\
&= \nabla_{e_i} \langle a, \overline{\nabla}_{e_i} \nu \rangle \\
&= \langle a, \overline{\nabla}_{e_i} \overline{\nabla}_{e_i} \nu \rangle \\
&= \langle a, \overline{\nabla}_{e_i} (\nabla_{e_i} \nu - A^\nu(e_i)) \rangle \\
&= - \langle a, \overline{\nabla}_{e_i} A^\nu(e_i) \rangle \\
&= - \langle a, \nabla_{e_i} A^\nu(e_i) + (\overline{\nabla}_{e_i} A^\nu(e_i))^N \rangle .
\end{aligned}$$

Noting that the ambient Euclidean space has vanishing curvature and the unit normal vector field $\nu$ is parallel in the normal bundle,

$$\begin{aligned}
\nabla_{e_i} A^\nu(e_i) &= \nabla_{e_i} \langle B_{e_i e_j}, \nu \rangle e_j \\
&= \nabla_{e_i} \langle \overline{\nabla}_{e_j} e_i, \nu \rangle e_j \\
&= \left( \langle \overline{\nabla}_{e_i} \overline{\nabla}_{e_j} e_i, \nu \rangle + \langle \overline{\nabla}_{e_j} e_i, \overline{\nabla}_{e_i} \nu \rangle \right) e_j \\
&= \left( \langle \overline{\nabla}_{e_j} \overline{\nabla}_{e_i} e_i, \nu \rangle + \langle \overline{\nabla}_{e_j} e_i, (\overline{\nabla}_{e_i} \nu)^T \rangle \right) e_j \\
&= \left( \langle \overline{\nabla}_{e_j} (\nabla_{e_i} e_i + B_{e_i e_i}), \nu \rangle + \langle \nabla_{e_j} e_i, (\overline{\nabla}_{e_i} \nu)^T \rangle \right) e_j \\
&= \langle B_{e_j \nabla_{e_i} e_i}, \nu \rangle e_j + \langle n \nabla_{e_j} H, \nu \rangle = 0.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\Delta \langle a, \nu \rangle &= - \langle a, (\overline{\nabla}_{e_i} A^\nu(e_i))^N \rangle \\
&= - \langle a, B_{e_i A^\nu(e_i)} \rangle = - \langle a, \nu \rangle |B|^2.
\end{aligned}$$

Q.E.D.