1.4 Minimal Submanifolds in the Sphere

Besides minimal submanifolds in Euclidean space, minimal submanifolds in the sphere are of the most important subject. There is canonical imbedding of the sphere in Euclidean space. For minimal submanifolds in the sphere we can also study its coordinate functions. We will also see that some properties of minimal submanifold in the sphere are closely related to the properties of minimal submanifold in Euclidean space.

Let $M \to \bar{M} \subset \bar{\bar{M}}$ be isometric immersions with the Levi-Civita connections ∇, $\overline{\nabla}$, and $\overline{\overline{\nabla}}$ respectively. Denote H to be the mean curvature of M in $\bar{M}$ and $\overline{H}$ for M in $\bar{\bar{M}}$. Choose a local orthonormal frame field $\{e_1, \cdots, e_m\}$ in M. Then,

$$H = \frac{1}{m}(\sum_i \overline{\nabla}_{e_i} e_i)^N = \frac{1}{m}\left((\sum_i \overline{\overline{\nabla}}_{e_i} e_i)^{T\bar{M}}\right)^N$$
$$= \frac{1}{m}\left((\sum_i \overline{\overline{\nabla}}_{e_i} e_i)^N\right)^{T\bar{M}} = \overline{H}^{T\bar{M}}.$$

If $\bar{M} \subset \bar{\bar{M}}$ is a totally geodesic, then $\overline{\nabla} \equiv \overline{\overline{\nabla}}$ along $\bar{M}$ and $\overline{H} = H$, which means that if M is a minimal submanifold in $\bar{M}$, and $\bar{M}$ is a totally geodesic submanifold in $\bar{\bar{M}}$, then M is a minimal submanifold in $\bar{\bar{M}}$.

If $\psi : M \to \bar{M} \subset \mathbb{R}^N$, from Proposition 1.3.1 it follows that ψ is a minimal immersion if and only if $(\Delta\psi)^{T\bar{M}} = 0$, namely $\Delta\psi$ is always orthonormal to $\bar{M}$.

THEOREM 1.4.1 *For an isometric immersion $\psi : M \to S^n$ it is a minimal immersion into S^n if and only if*

$$\Delta\psi = -m\psi.$$

PROOF. From the above discussion we know that ψ is a minimal immersion if and only if for any $p \in M$, $\Delta\psi(p)$ is parallel to the normal direction to S^n in $\mathbb{R}^{n+1}$, namely, $\Delta\psi = \lambda\psi$, where $\lambda \in C^\infty(M)$. Thus,

$$0 = \Delta|\psi|^2 = \langle\psi, \Delta\psi\rangle + |\nabla\psi|^2 = \lambda|\psi|^2 + |\nabla\psi|^2 = \lambda + |\nabla\psi|^2.$$

This means that

$$\lambda = -|\nabla\psi|^2 = -\langle\nabla_{e_i}\psi, \nabla_{e_i}\psi\rangle = -\langle e_i, e_i\rangle = -m.$$

Q. E. D.

It is interesting to see that for a minimal immersion to S^n its coordinate functions in $\mathbb{R}^{n+1}$ are eigenfunctions of the Laplace operator on M with respect to the eigenvalue $-\dim M$. Conversely, we have the Takahashi theorem. Let

$$S^n(r) = \left\{(x^1, \cdots, x^n + 1) \in \mathbb{R}^{n+1}; \sum_{k=1}^{n+1} = r^2\right\}.$$

THEOREM 1.4.2 ([T]) *Let M be an m-Riemannian manifold and $\psi : M \to \mathbb{R}^{n+1}$ an isometric immersion such that for $\lambda \neq 0$ satisfying*

$$\Delta\psi = -\lambda\psi.$$

Then

(1) $\lambda > 0$;

(2) $\psi(M) \subset S^n(r)$, *where* $r^2 = \frac{m}{\lambda}$;

(3) $\psi : M \to S^n(r)$ *is a minimal immersion.*

PROOF. Let $\overline{H}$ be the mean curvature vector of M in $\mathbb{R}^{n+1}$. Combining Theorem 1.3.1 and the condition of the present theorem gives $-\lambda\psi = m\overline{H}$, which implies that ψ is a normal vector of M in $\mathbb{R}^{n+1}$. For any tangent vector X to M

$$X\langle\psi, \psi\rangle = 2\left\langle\overline{\overline{\nabla}}_X\psi, \psi\right\rangle = 2\langle X, \psi\rangle = 0.$$

Therefore,

$$|\psi|^2 \overset{def.}{=} r^2 = \text{const.}.$$

Furthermore,

$$0 = \frac{1}{2}\Delta|\psi|^2 = \langle \Delta\psi, \psi\rangle + |\nabla\psi|^2 = -\lambda r^2 + m,$$

namely,

$$\lambda = \frac{m}{r^2} > 0.$$

This proves the first and second conclusions of the theorem. We also have

$$H = (\overline{H})^{TS^n(r)} = (-\frac{1}{m}\lambda\psi)^{TS^n(r)} = 0.$$

The proof has be completed.

Q. E. D.

Any submanifold M in S^n is naturally a submanifold in $\mathbb{R}^{n+1}$. Using this relationship we already obtained some properties for minimal submanifolds in the sphere. Vice versa, the properties of a minimal submanifold M in the sphere also indicate certain properties of minimal submanifolds in Euclidean space, via cone CM over M.

Let $M \to S^n \subset \mathbb{R}^{n+1}$ be submanifold in the sphere. The cone CM over M is the image under the map of $M \times [0,1] \to \mathbb{R}^{n+1}$ defined by $(x,t) \to tx$, where $x \in M$, $t \in [0,1]$. Namely,

$$CM = \left\{tx \in \mathbb{R}^{n+1}; \quad x \in M,\ t \in [0,1]\right\}.$$

CM has a singularity $t = 0$, if M is not totally geodesic in S^n. To avoid the singularity we consider the truncated cone CM_ε, which is the image of $M \times [\varepsilon, 1]$ under the same map, where ε is any positive number. Any submanifold M in the sphere and the cone CM over M are closely related objects. We choose a local orthonormal frame field $\{e_i,\ e_\alpha\}$ of S^n along M. Then by parallel translating along rays issuing from the origin we obtain local vector fields E_i and E_α in $\mathbb{R}^{n+1}$. Obviously, $E_i = \frac{1}{r}e_i$, $E_\alpha = \frac{1}{r}e_\alpha$, where r is the distance of the corresponding point from the origin. Let τ denote the unit tangent vector along the rays, $\tau = \frac{\partial}{\partial r}$. Obviously, $\nabla_\tau \tau = 0$. Thus, $\{E_i, E_\alpha, \tau\}$ forms a local orthonormal frame field in $\mathbb{R}^{n+1}$ and $\{E_i, \tau\}$ is a frame field in CM_ε.

Let $\overline{H}$ and $\bar{B}$ denote the mean curvature and the second fundamental form of CM_ε in $\mathbb{R}^{n+1}$. Let H and B denote that for M. By computations we obtain

$$\nabla_{E_i} E_j = -\frac{1}{r}\delta_{ij}\tau + \frac{1}{r}h_{\alpha ij}E_\alpha, \tag{1.4.1}$$

$$\overline{H} = \frac{1}{(m+1)r}h_{\alpha ii}E_\alpha = \frac{m}{(m+1)r^2}H, \tag{1.4.2}$$

and

$$|\bar{B}|^2 = \frac{1}{r^2}|B|^2. \tag{1.4.3}$$

In summary

PROPOSITION 1.4.3 *CM_ε has parallel mean curvature in $\mathbb{R}^{n+1}$ if and only if M is a minimal submanifold in S^n.*

The detailed computation could be found in [X3]. We will see that this important property will be used extensively.

1.5 Examples

The minimal surface equation (1.3.7) is a nonlinear partial differential equation. It is hard to solve. Besides the linear functions, what are its solutions? As early as 1776 J. L. Meunier obtained two nonlinear solutions to the equation firstly. Their graphs are catenoid and helicoid (see Figure 1.1 and Figure 1.2 respectively).