Let $\overline{H}$ and $\bar{B}$ denote the mean curvature and the second fundamental form of CM_ε in $\mathbb{R}^{n+1}$. Let H and B denote that for M. By computations we obtain

$$\nabla_{E_i} E_j = -\frac{1}{r}\delta_{ij}\tau + \frac{1}{r}h_{\alpha ij}E_\alpha, \tag{1.4.1}$$

$$\overline{H} = \frac{1}{(m+1)r}h_{\alpha ii}E_\alpha = \frac{m}{(m+1)r^2}H, \tag{1.4.2}$$

and

$$|\bar{B}|^2 = \frac{1}{r^2}|B|^2. \tag{1.4.3}$$

In summary

PROPOSITION 1.4.3 *CM_ε has parallel mean curvature in $\mathbb{R}^{n+1}$ if and only if M is a minimal submanifold in S^n.*

The detailed computation could be found in [X3]. We will see that this important property will be used extensively.

1.5 Examples

The minimal surface equation (1.3.7) is a nonlinear partial differential equation. It is hard to solve. Besides the linear functions, what are its solutions? As early as 1776 J. L. Meunier obtained two nonlinear solutions to the equation firstly. Their graphs are catenoid and helicoid (see Figure 1.1 and Figure 1.2 respectively).

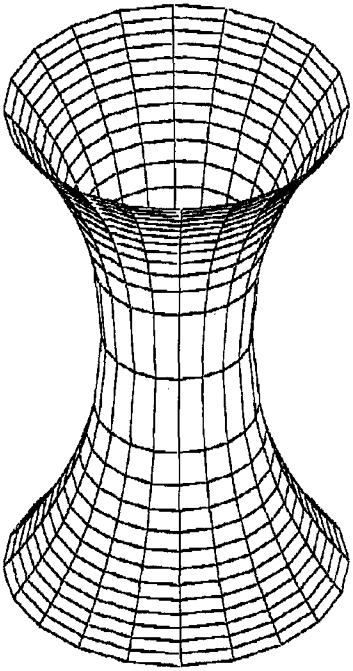

Fig. 1.1

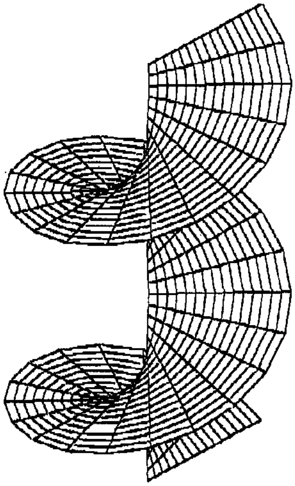

Fig. 1.2

The catenoid is defined by

$$z = \cosh^{-1} \sqrt{x^2 + y^2}, \tag{1.5.1}$$

Take a catenary in Y-Z coordinate plane. Letting it rotating about Z-axis gives the catenoid. Furthermore, we have the following result.

PROPOSITION 1.5.1 (O. BONNET 1860) *Any minimal surface which is also a surface of revolution in* $\mathbb{R}^3$ *is a catenoid or a plane up to a rigid motion in* $\mathbb{R}^3$.

We leave the proof to readers as an exercise.

The catenoid is a complete surface whose Gauss curvature is

$$K = -\frac{1}{(x^2+y^2)^2}, \tag{1.5.2}$$

and the total curvature

$$\int_M K\,dM = -4\pi. \tag{1.5.3}$$

The helicoid is defined by

$$z = \tan^{-1}\frac{x}{y}. \tag{1.5.4}$$

Let a line in $X-$axis screw about $Z-$axis. The resulting surface is a helicoid

The helicoid is also a complete surface with the Gauss curvature

$$K = -\frac{1}{(1+x^2+y^2)^2}.$$

Its total curvature is infinite. It is easy to prove the following result.

PROPOSITION 1.5.2 (E. CATALAN, 1842) *Up to a rigid motion a ruled minimal surface in* $\mathbb{R}^3$ *has to be a helicoid or a plane.*

Consider a special solution to (1.3.7) of the type

$$f(x,y) = g(x) + h(y).$$

By direct computations we obtain

$$f(x,y) = \frac{1}{a}\log\frac{\cos ax}{\cos ay}. \tag{1.5.5}$$

Its graph is called Scherk's surface which was obtained in 1835.

We now give some examples of minimal submanifolds in the sphere.

Let $\psi : M \to S^n \subset \mathbb{R}^{n+1}$ and $\psi' : M' \to S^{n'} \subset \mathbb{R}^{n'+1}$ be minimal immersions. For any constants c and c'

$$c\psi \oplus c'\psi' : M \times M' \to \mathbb{R}^{n+n'+2}$$

is also an isometric immersion of the product manifold $M \times M'$ to $\mathbb{R}^{n+n'+2}$. If we choose c and c' with $c^2 + c'^2 = 1$, then the image of $M \times M'$ under $c\psi \oplus c'\psi'$ lies in the sphere $S^{n+n'+1}$. We know that the induced metric on M under $c\psi$ is $c^2\, ds^2$, where ds^2 is the original metric on M. Then, the Laplacian on M with respect to the metric $c^2\, ds^2$ is $\frac{1}{c^2}\Delta_M$. By Theorem 1.4.1

$$\frac{1}{c^2}\Delta_M(c\psi) = \frac{1}{c^2}\, c\Delta_M\psi = -\frac{m}{c^2}(c\psi),$$

so does for $c'\psi'$ and

$$\Delta_{M\times M'}(c\psi \oplus c'\psi') = -\frac{m}{c^2}c\psi \oplus -\frac{m'}{c'^2}c'\psi'.$$

If c and c' also satisfy

$$\frac{m}{c^2} = \frac{m'}{c'^2},$$

then by Theorem 1.4.1 we obtain a minimal immersion $c\psi \oplus c'\psi' : M \times M' \to S^{n+n'+1}$. In particular, $M = S^n$ and $M' = S^{n'}$ we have the Clifford minimal hypersurface

$$S^n\left(\sqrt{\frac{n}{n+n'}}\right) \times S^{n'}\left(\sqrt{\frac{n'}{n+n'}}\right) \to S^{n+n'+1}. \tag{1.5.6}$$

A unit normal vector to the Clifford minimal hypersurface is $\nu = -c'\psi + c\psi$, because it is orthogonal to $d\psi$, $d\psi'$, and to $c\psi + c'\psi'$. Hence, its second fundamental form is

$$-\langle dx, d\nu\rangle = cc'(\langle d\psi, d\psi\rangle - \langle d\psi', d\psi'\rangle).$$

On the other hand, its induced metric is

$$ds^2 = c^2\,\langle d\psi, d\psi\rangle + c'^2\,\langle d\psi', d\psi'\rangle.$$

Noting that

$$c = \sqrt{\frac{n}{n+n'}} \quad \text{and} \quad c' = \sqrt{\frac{n'}{n+n'}},$$

it has principal curvature $\sqrt{\frac{n'}{n}}$ with the multiplicity n and $-\sqrt{\frac{n}{n'}}$ with the multiplicity n'. Therefore, the sum of the squares of the principal curvatures is $n+n'$, which is the squared norm $|B|^2$ of the second fundamental form B for the Clifford hypersurface (1.5.6). We thus have

$$|B|^2 = n + n'. \tag{1.5.7}$$

Let us consider another minimal submanifold in the sphere. Let

$$P(d) = \{\text{homogeneous polynomials of degree } d \text{ in } \mathbb{R}^{n+1}\}$$

and

$$H(d) = \{f \in P(d);\ \Delta f = 0\}.$$

Then

$$SH(d) = \{f|_{S^n(c)};\ f \in H(d)\}$$

denotes the spherical harmonic functions of degree d.

LEMMA 1.5.3 *If $f \in SH(d)$, then*

$$\Delta_{S^n(c)} f = -\frac{d(n+d-1)}{c^2} f. \tag{1.5.8}$$

PROOF. Let $\{e_i, \cdots, e_n\}$ be an orthonormal frame field in $S^n(c)$, $\nu = \frac{x}{c}$ be the unit normal vector field along S^n. Then

$$\begin{aligned}\Delta_{S^n(c)} f &= e_i e_i(f) - (\nabla_{e_i} e_i)(f)\\ &= e_i e_i(f) + \nu\nu(f) - \left(\overline{\nabla}_{e_i} e_i\right)(f) - \overline{\nabla}_\nu \nu(f)\\ &\qquad + B_{e_i e_i}(f) - \nu\nu(f) + \overline{\nabla}_\nu \nu(f)\\ &= \Delta_{\mathbb{R}^{n+1}} f + nH(f) - \nu\nu(f)\\ &= -\frac{n}{c}\nu(f) - \nu\nu(f),\end{aligned} \tag{1.5.9}$$

where $H = -\frac{1}{c}\nu$ is the mean curvature vector of $S^n(c)$ in $\mathbb{R}^{n+1}$.

Since f is a homogeneous function,

$$\nu f(x)|_{S^n(c)} = \frac{1}{c}\frac{\partial}{\partial t} f(tx)\bigg|_{t=1} = \frac{df}{c} \tag{1.5.10}$$

and

$$\nu\nu f(x)|_{S^n(c)} = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} f(tx)\Big|_{t=1} = \frac{d(d-1)\, f(x)}{c^2}. \qquad (1.5.11)$$

Substituting (1.5.10) and (1.5.11) into (1.5.9) gives (1.5.8) and the proof is completed.

Q. E. D.

Theorem 1.4.2 and Lemma 1.5.3 enable us to define minimal immersions by using homogeneous spherical harmonic functions

$$S^n\left(\sqrt{\frac{d(n+d-1)}{n}}\right) \to S^N(1),$$

where $N+1 = dimSH(d)$. In the case of $n=2$ and $d=2$ we have

$$S^2(\sqrt{3}) \to S^4,$$

which can be realized by the map

$$\begin{aligned}\Psi(x,y,z) = \Big(&\frac{1}{\sqrt{3}}xy, \frac{1}{\sqrt{3}}xz, \frac{1}{\sqrt{3}}yz,\\ &\frac{1}{2\sqrt{3}}(x^2-y^2), \frac{1}{6}(x^2+y^2-2z^2)\Big), \qquad (1.5.12)\end{aligned}$$

where $x^2+y^2+z^2=3$. It is called the Veronese surface which is an imbedding of the real projective plane of curvature $\frac{1}{3}$ into S^4.

The Clifford minimal hypersurface and the Veronese surface are important minimal submanifolds in the sphere.