Chapter 1

Introduction

When we observe evolution in time of various phenomena in the macroscopic world that surrounds us, we often use the terms "chaos", or "chaotic", meaning that the changes in time are without pattern and out of control, and hence are unpredictable. The most frustrating phenomena are those, which concern long-term weather forecasting. We can never be sure about the change of weather patterns. The temperature, barometric pressure, wind direction, amount of precipitation and other important factors come as a surprise contradicting predictions made a few days ago. Sometimes we are caught in a storm, sometimes in a heat-wave.

The world stock market prices are also an example of a system that fluctuates in time in a random-like, irregular way, and the long-term prognosis does not often come true.

The two examples mentioned belong to the category of huge and complicated dynamical systems, with a huge number of variables. The unpredictability of the evolution in time of these interesting events is intuitively natural.

Simultaneously, it also seems natural that evolution of physical processes in simple systems, the systems governed by simple mathematical rules, should be predictable far into the future.

Suppose we consider a small heavy ball, which can move along a definite track, so that the position of the ball is determined by a single coordinate. Due to Newton's Second Law, the motion of the ball is governed by the second order differential equation. The well known physical system the pendulum belongs to this class of oscillators.

We were told that if the forces acting on the ball, as well as its initial position and velocity are given, one could predict the motion, i.e. the history of the system forever into the future, at least if the powers of our computers are big enough.

The scientific researchers were taken by surprise, some of them were unable to agree with the idea that even this type of system may exhibit an irregular motion, sensitive to initial conditions and though unpredictable in time, the motion is labeled as chaotic.

This book is aimed at presenting and exploring the chaotic phenomena in the single-degree-of-freedom, nonlinear driven oscillators. The oscillators considered belong to the class of dissipative deterministic dynamical systems. The term "dissipative" means that drag forces act on the ball during motion (aerodynamic forces, friction forces and others), so that the free oscillations always decay in time, and the undriven system tends to its equilibrium position. The other essential feature is that all the forces acting on the ball are determined in time. Such systems are labeled deterministic. For a long time, researchers were deeply convinced that deterministic systems always give a deterministic output.

Early discovery of chaotic output in deterministic systems came into view in the field of mathematical iteration equations of the type

$$x_{n+1} = f(x_n), \qquad n = 0, 1, 2, \ldots\ldots .$$

The formula states that the quantity x at the "instant of time" denoted n+1 can be calculated, if the previous quantity x_n is known. Interpretation of the parameter n as "instant of time" is useful in applications. One of the fundamental models of this type has its roots in ecology. Ecologists wanted to know the population growth of a given species in a controlled environment, and to predict the long-term behavior of the population. One of the simple rules used by ecologists is the logistic equation

$$x_{n+1} = kx_n(1 - x_n), \qquad n = 0, 1, 2, \ldots\ldots$$

Here, the "instants" n = 0, 1, 2,.... correspond to the end of each generation. Using this formula one can deduce the population in the succeeding generation x_{n+1} from the knowledge of only the population in the preceding generation x_n and the constant k. The results obtained for a wide range of values of the constant k were surprising. As long as k did not exceed the value of about 3.5, the behavior of the population changed in a regular way. But at higher k-values, in particular within the interval ~3.6 < k < 4.0, strange results were obtained. Namely, the consecutive values $x_0, x_1, x_2, ..., x_{n+1}$ looked like an irregular, random-like process, whose essential property was that the fluctuations were sensitive to the initial value x_0.

Dynamical systems generated by the iterative formulae belong to the category of **dynamical systems with discrete time**. In contrast, the physical systems governed by differential equations are labeled as **dynamical systems with continuous time**. In the latter case, the sought changes in time of the values of position and velocity can be found by numerical integration of the equation of motion. Indeed, we can apply a numerical procedure that enables us to obtain discrete values of position and velocity. For instance, we may record the sought values in selected instants of time, say, at intervals equal to the period of excitation T. Thus, a series of sought quantities in the discrete time

$$0, T, 2T, 3T, ..., nT, ...$$

would be obtained.

Yet, we are not able to find an analytical iterative formula for the relation between the position and velocity values at the instant $n+1$ as function of the previous values. That is why the analytical results obtained by mathematicians for dynamical systems with discrete time are not always applicable in the continuous time systems. Yet, the fundamental new concepts of nonlinear dynamics are common for both types of systems.

The book is addressed to general Readers, also to those who, although are interested in the fascinating chaotic phenomena encountered in our every day life, do not have a solid mathematical background. To make the book easily accessible, we try to reduce the mathematical approach to minimum, and to apply a simplified version of presentation of the very complex chaotic phenomena. The Reader may even skip the portions of material where equations of motion are derived, and confine his/her attention to the presented physical model.

Instead of a mathematical approach, the book is based on geometric interpretation of numerical results. The effort is focused on an explanation of both the theoretical concepts and the physical phenomena, with the aid of carefully selected examples of computer graphics.

Some portions of the material, written in small fonts, give additional remarks and refer the Reader to the literature on the problem considered. These portions might be skipped by those looking for an overview of the field.

The same simplified approach is applied to the fundamental concepts, as well as to the advanced problems recently published by the Author and her associates in international scientific journals (see references [19–24]).

Application of the simplified way of presentation of very complex problems is rather risky. The difficulty is in finding a compromise between strict mathematical accuracy and accessibility of the material. It is difficult to explain the chaotic phenomena in clear and simple language while avoiding simplifications that may lead to incorrect interpretation. In the search of compromise I asked for help from two types of Readers. I sent some parts of the manuscript to the Readers who are interested in chaotic phenomena but do not want to go deeply into mathematics of chaos, and to those who are involved in the research on chaotic problems and also give lectures for undergraduate students. The question addressed to the first group was whether the material was easy to read and understand, whereas the second group were asked whether the work was clear and exact.

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