

## Chapter 1

# Foundations of Theory of Differential Equations with Discontinuous Right-Hand Sides

This chapter is concerned with basic concepts of the theory of differential equations with a discontinuous right-hand side. In Section 1.1 we give a qualitative description of difficulties encountered when a system with a discontinuous right-hand side is considered. We also establish relations to the theory of differential equations with multiple-valued right-hand sides (differential inclusions), and offer a definition of a solution to a discontinuous system which will be accepted in this book. Furthermore, a comparison with some other definitions of a solution to a discontinuous system is given.

In Section 1.2 we present some topics from the theory of differential inclusions, and also some theorems that enable us to apply this theory to differential equations with discontinuous right-hand sides. We give a local theorem for the existence of solutions, theorems on continuability and continuous dependence on initial values, and some other results which will be used in the subsequent discussions. Moreover, sliding modes are investigated.

In Section 1.3 we formulate a number of definitions for stability and dichotomy, and prove some Lyapunov-type lemmas employed in the following chapters.

A reader who is interested only in applications can limit himself to the reading of Section 1.1 and Subsection 1.3.1.

## 1.1 Notion of Solution to Differential Equation with Discontinuous Right-Hand Side

### 1.1.1 *Difficulties encountered in the definition of a solution. Sliding modes*

The main subject of investigation in this book are the systems described by a vector differential equation of the form

$$\frac{dx}{dt} = f(x, t). \quad (1.1)$$

(Here  $x$  and  $f(x, t)$  are  $n$ -dimensional vectors.) We will be interested in the case when the right-hand side of (1.1) is of a special form. Systems of differential equations, which are generally encountered in applications, can be naturally separated into a “linear part” and a “nonlinear part.” That is, they can be written as

$$\frac{dx}{dt} = Px + q\xi, \quad \sigma = r^* x, \quad (1.2)$$

$$\xi = \varphi(\sigma, t). \quad (1.3)$$

This is exactly the form of the systems which will be discussed hereafter.

Equations (1.2) describe the linear part of the system, while equation (1.3) describes its nonlinear part, which can also be linear. Thus, equation (1.3) can be used to represent any block as deemed appropriate.

In (1.2)  $P$ ,  $q$ ,  $r$  are constant matrices of dimensions  $n \times n$ ,  $n \times m$ ,  $n \times l$  respectively, while  $\xi = \xi(t)$  and  $\sigma = \sigma(t)$  are vector functions of dimensions  $m$  and  $l$  respectively. Components of the vector  $\sigma(t)$  represent inputs of the block considered, while elements of  $\xi(t)$  represent its outputs. All the values in (1.1)–(1.3) are real.

The above separation of the system at hand into “linear” and “nonlinear” parts proves to be useful. It gives the opportunity for a more refined analysis when taking into account specific features of a nonlinear system. Any system of nonlinear differential equations can be written in the form of (1.2), (1.3). Indeed, the general form differential equation (1.1) is a specific case of system (1.2), (1.3) with a “trivial” linear part

$$n = m = l, \quad P = 0, \quad q = r = I_n.$$

When nonlinearities are continuous, the converse is also true. Namely,

system (1.2), (1.3) can be put in the form (1.1) with

$$f(x, t) = Px + q\varphi(r^*x, t). \quad (1.4)$$

When nonlinearities are discontinuous (i.e., when  $\varphi(\sigma, t)$  is discontinuous with respect to  $\sigma$ ) the problem of definition of solution for system (1.2), (1.3) arises. This problem will be discussed in detail in the following subsections. We will see later that in this case system (1.2), (1.3) is generally not equivalent to equation (1.1) with function (1.4). The set of systems (1.2), (1.3) proves to be wider than the set of equations (1.1). This fact is another argument in favor of the consideration of systems (1.2), (1.3).

We will present below a number of examples from different branches of modern engineering described by systems of the form (1.2), (1.3).

It should be particularly emphasized that many practical problems lead to systems (1.2), (1.3) with a function  $\varphi(\sigma, t)$  discontinuous with respect to  $\sigma$ .

In the case when  $\varphi(\sigma, t)$  depends continuously on  $\sigma$  (although it can be discontinuous with respect to  $t$  under some natural assumptions) the classical existence theorem guarantees the existence of solutions (with given  $t_0$  and  $x_0 = x(t_0)$ ) at most for values of  $t$  sufficiently close to the initial value  $t_0$ . It also guarantees that a solution can be continued to any interval  $(t_0, T)$  where this solution remains bounded. (Thus, if a solution exists on some interval  $(t_0, T_0)$  and the value  $T_0$  cannot be increased then  $|x(t)|$  is an unbounded function as  $t \rightarrow T_0$ .)

If the function  $\varphi(\sigma, t)$  is discontinuous in  $\sigma$ , some well-known difficulties arise. These difficulties are not only mathematical but also of a fundamental nature, because there is a problem of adequate description of a real-world system with equations (1.2), (1.3).

It is worth noting that in the engineering literature some particular cases of system (1.2), (1.3) with a discontinuous (in  $\sigma$ ) function  $\varphi(\sigma, t)$  are often considered. Frequently, for these particular systems difficulties either do not arise, or if they do arise, specific properties of a system suggest a natural way out. Sometimes such difficulties are just ignored and not discussed.

Let us clarify the difficulties for the simplest case when  $m = l = 1$  in (1.2), (1.3) (i.e.,  $\sigma$  and  $\xi$  are scalar functions) and  $\varphi(\sigma, t) = \varphi(\sigma)$  does not depend explicitly on  $t$ . Suppose that the function  $\varphi(\sigma)$  is discontinuous at the point  $\sigma_0$ , continuous for values of  $\sigma$  sufficiently close to  $\sigma_0$ , and there

exist finite limits

$$\lim_{\sigma \rightarrow \sigma_0^-} \varphi(\sigma) = \varphi(\sigma_0 - 0), \quad \lim_{\sigma \rightarrow \sigma_0^+} \varphi(\sigma) = \varphi(\sigma_0 + 0). \quad (1.5)$$

In what follows such a point  $\sigma_0$  will be called a point of *discontinuity of the first kind*.

In the  $n$ -dimensional space  $\mathbb{R}^n = \{x\}$ , the right-hand side (1.4) of the appropriate system (1.1) has discontinuities on the surface (a hyperplane)  $r^*x = \sigma_0$ . Moreover, it is continuous in the neighborhood of this surface, and has finite limits as the point  $x$  approaches a point of the hyperplane remaining on its one side only. This means that in a vicinity of the hyperplane  $r^*x = \sigma_0$  the trajectories of the system behave in one of the three ways shown in Fig. 1.1.

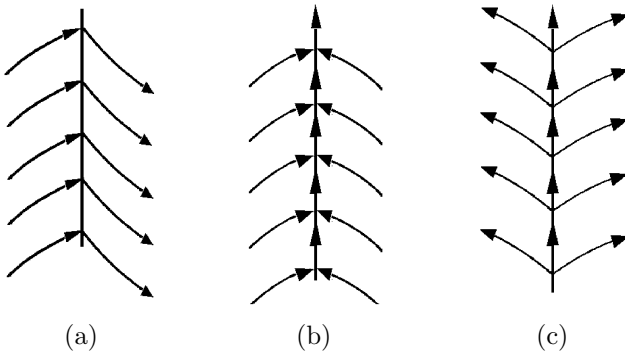


Fig. 1.1

For the case shown in Fig. 1.1 (a) no problems arise: the trajectories just pass through the hyperplane  $r^*x = \sigma_0$ , so the solution  $x(t)$ , if it reaches the discontinuity surface at  $t = t_0$ , can be uniquely continued over the values  $t > t_0$  sufficiently close to  $t_0$ .

In the case shown in Fig. 1.1 (b) the trajectories connect with each other on the discontinuity surface. So a trajectory reaching this surface at  $t = t_0$  cannot leave it and has to move along the discontinuity surface (i.e., to “slide” along it). The appropriate solution, lying on the discontinuity surface over some interval  $\Delta$  of  $t$ , is called a “sliding mode” solution (the accurate definition will be given below). In order to define trajectories on the discontinuity surface it is necessary to somehow specify a direction field

on it. This direction field will determine the sliding mode solution. Sliding modes play an important role in optimal control [Utkin (1992)] and in the study of systems with a variable structure [Emelyanov (1967)].

A similar situation occurs in the case depicted in Fig. 1.1 (c). The difference is that a solution cannot reach the discontinuity surface (as time increases). If an initial value  $x(t_0) = x_0$  is taken on the discontinuity surface, three “local” possibilities can occur: the solution can leave the surface on either one of its sides, or it can remain on the surface. In the latter case a direction field determining a sliding mode solution also should be additionally defined. (Of course, a situation is possible when a solution remains on the surface for awhile and then leaves it.)

It is not clear a priori how to define a direction field on the discontinuity surface, i.e., how to define a sliding mode solution. The first impression is that a solution will exist for any definition of a direction field on the discontinuity surface. However, this is not the case. Frequently (but not always) the direction field on the surface is determined uniquely if we require that the solutions exist and some additional natural conditions are satisfied. For any other choice of a direction field the solution does not exist. Below, in Subsection 2.1.4, we will discuss this question in more detail. Now we present the following explanatory example. Let a direction field  $f(x)$  be defined on the plane in a neighborhood of some point  $x_0$  as shown in Fig. 1.2. That is, the trajectories converge to  $x_0$ , while in  $x_0$  itself a velocity vector

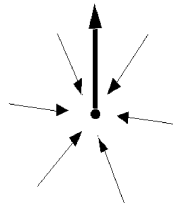


Fig. 1.2

is not defined. If we need a solution with the initial condition  $x(t_0) = x_0$  to exist, then  $f(x_0)$  is uniquely determined as  $f(x_0) = 0$ . Indeed, for any other definition of  $f(x_0)$  (see Fig. 1.2) the solution would leave  $x_0$  into its vicinity, but the direction field does not allow that. (Of course, we assume the usual properties of solutions to be satisfied.)

Something similar can occur also in the general case. It should be taken into account that a direction field can continuously transfer from one case

shown in Fig. 1.1 to another. Moreover, sliding modes can arise not only on hyperplanes  $r^*x = \sigma_0$ , but also on manifolds of lower dimension (e.g., when the dimension  $l$  of the vector  $\sigma$  is greater than one). In addition, the function  $f(x, t)$  can have discontinuities not on a hyperplane, but on a surface of a more complicated form, etc. Observe that the geometric approach, which is sometimes justified when investigating systems of low order (usually of order two, rarely of order three) with nonlinearities of a simple type (conventionally, piecewise linear), can lead to difficulties if a solution enters the discontinuity surface an infinite number of times. This situation is not rare, i.e., it occurs (under certain values of parameters) for an equation of forced oscillations with both dry and viscous friction

$$\frac{d^2y}{dt^2} + \alpha \frac{dy}{dt} + \beta \operatorname{sgn} \frac{dy}{dt} + \gamma y = e(t)$$

( $\alpha > 0$ ,  $\beta > 0$ ,  $\gamma > 0$ ,  $e(t)$  is a continuous function).

### **1.1.2 *The concept of a solution of a system with discontinuous nonlinearities accepted in this book. Connection with the theory of differential equations with multiple-valued right-hand sides***

The amount of literature on the theory of differential equations (1.1) with discontinuous right-hand sides is vast, including a review [Matrosov (1967)], papers [Aizerman and Pyatnitskii (1974a); Aizerman and Pyatnitskii (1974b); Filippov (1960)] and a monograph [Filippov (1988)]. Our approach in some ways differs from the others. We will describe it now, and concentrate only on the approaches to the definition of solutions which are similar to ours.

When developing the theory of systems with discontinuous right-hand sides, both engineering and mathematical aspects should be taken into account. On the one hand, the theory has to provide mathematical tools for study of these systems, such as conventional existence theorems, theorems on continuation of solutions, standard theorems from the qualitative theory or their counterparts. On the other hand, the theory should adequately describe physical reality.

Our considerations are based on the theory of differential equations with multiple-valued right-hand sides (the theory of differential inclusions) proposed in [Marchaud (1934); Marchaud (1936); Zaremba (1934); Zaremba (1936)] and later developed in [Filippov (1960); Ważewski (1961); Filippov

(1971); Filippov (1988); Deimling (1992); Tolstonogov (2000)] and other works.

The terms *differential equation with a multiple-valued right-hand side* or *differential inclusion* are used to define a relationship

$$\frac{dx}{dt} \in f(x, t) \quad (1.6)$$

where  $f(x, t)$  is a nonempty set depending on  $x$  and  $t$  in the  $n$ -dimensional space. Sometimes, (1.6) is also called a *contingency equation*.

The function  $f(x, t)$  is called a *multivalued function* to emphasize that its values are sets. (It would be more correct to call it a function with values in the set of all subsets of  $\mathbb{R}^n$ , but such a name is too wordy.) If for any  $(x, t)$  the set  $f(x, t)$  consists of a single point, (1.6) is an ordinary differential equation. The function  $f(x, t)$  will be called a *single-valued function at a point*  $(x_0, t_0)$ , if the set  $f(x_0, t_0)$  consists of a single point.

Usually, the theory of differential inclusions assumes that for any point  $(x_0, t_0)$  the set  $f(x_0, t_0)$  is convex, closed, and bounded and that the function  $f(x, t)$  is upper semicontinuous. (It is under these assumptions that a theorem on existence of solutions is proved below in Section 2.2.) To say it somewhat loosely, a multivalued function  $f(x, t)$  is called *upper semicontinuous at a point*  $(x_0, t_0)$  if, as  $(x, t)$  tends to  $(x_0, t_0)$ , the limit of the set  $f(x, t)$  is contained in the closure of the set  $f(x_0, t_0)$ . (Inaccuracy lies in the words “the limit of the set,” the exact definition will be given below in Section 2.2.<sup>1</sup>)

Examples of multivalued functions  $\varphi(\sigma)$  (with  $\sigma \in \mathbb{R}^1$ ,  $\varphi \in \mathbb{R}^1$ ) are shown in Fig. 1.3 and Fig. 1.4. (The values of functions are indicated with bold lines; an arrow shows the excluded end of the line; a vertical hatching in Fig. 1.4 (d) denotes a set of values.) All the functions  $\varphi(\sigma)$ , except for those shown in Fig. 1.4 (c) and Fig. 1.4 (d), are single-valued for  $\sigma \neq \sigma_0$  and multivalued for  $\sigma = \sigma_0$ ; in the case of Fig. 1.4 (c) the function is single-valued for all  $\sigma$ , while in the case of Fig. 1.4 (d) it is multivalued for all  $\sigma$ . The functions  $\varphi(\sigma)$  shown in Fig. 1.3 (a)–(d) and Fig. 1.4 (d) are upper semicontinuous for all  $\sigma$ . The functions shown in Fig. 1.4 (a)–(c) are not upper semicontinuous at the point  $\sigma_0$ . In Fig. 1.3 (c) and Fig. 1.4 (a) the set  $\varphi(\sigma_0)$  is not convex, while in Fig. 1.3 (d) it is not closed. Thus, only the functions  $f(x, t) = \varphi(\sigma)$  in Fig. 1.3 (a), (b) and Fig. 1.4 (d) satisfy

<sup>1</sup>In fact, a function  $f(x, t)$  in the theory of differential inclusions can satisfy some weaker assumptions. We will not formulate them here in order to avoid overloading the presentation with nonessential details.

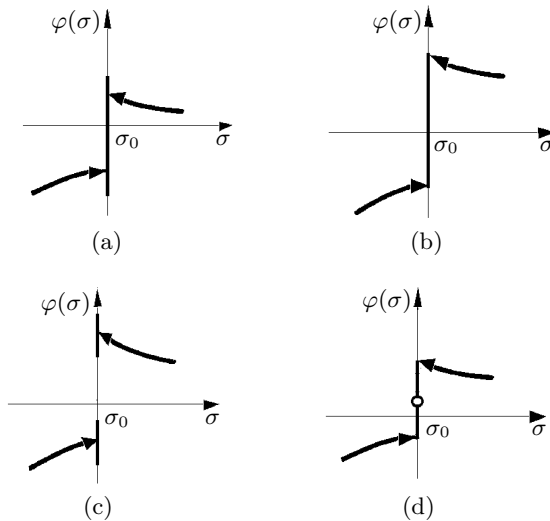


Fig. 1.3

the conditions of the existence theorem. In all other cases the existence theorem is not true. It should be noticed that Fig. 1.3 and Fig. 1.4 concern only the scalar case; the same is true for a vector differential inclusion (1.6) with the right-hand side (1.4).

The association between the theory of equations (1.1) with discontinuous right-hand sides and the theory of differential inclusions (1.6) is obvious. Having equation (1.1) with a discontinuous function  $f(x, t)$ , we should replace the value  $f(x_0, t_0)$  at a discontinuity point  $(x_0, t_0)$  for some set. This set is to be bounded, convex, and sufficiently “large”: it has to contain all the limit values of  $f(x, t)$  as  $(x, t)$  tends to  $(x_0, t_0)$ . After such a replacement (for any discontinuous point) instead of equation (1.1), we obtain a differential inclusion (1.6) where the multivalued function  $f(x, t)$  is upper semicontinuous, the set  $f(x, t)$  is bounded, closed and convex for all  $(x, t)$ . By the existence theorem, if we take any point  $(x_0, t_0)$  from the domain of  $f(x, t)$  then a solution satisfying  $x(t_0) = x_0$  exists for all values of  $t$  sufficiently close to  $t_0$ . The theorem on the continuation of solutions and some other useful theorems (see below Section 2.2) are also valid. It is this solution that we define as a solution of the initial equation (1.1) with a discontinuous right-hand side.

At a discontinuity point  $(x_0, t_0)$  the set  $f(x_0, t_0)$  can be defined in many

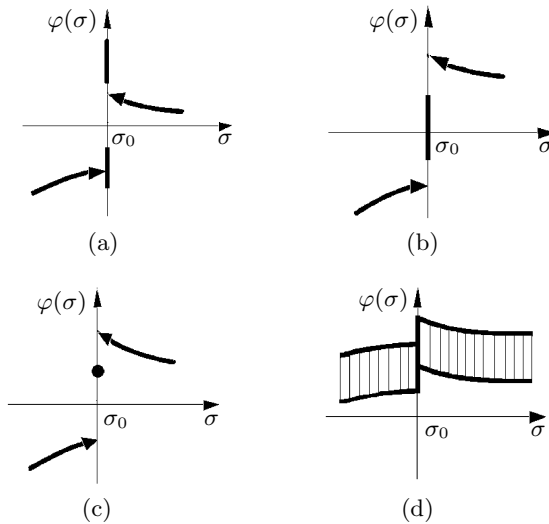


Fig. 1.4

ways (it has only to fit the above conditions) that correspond to various specifications of the direction field at the discontinuity surface. Any such specification is acceptable in the sense that a solution of the differential inclusion exists, and, by the above convention, the initial equation is also solvable. Moreover, these solutions have all the conventional properties or their reasonable counterparts.

This way of reasoning, however, runs into a difficulty. Since we start with the system (1.2), (1.3), rather than the equation (1.1) (with the right-hand side  $f(x, t) = Px + q\varphi(\sigma, t)$ ,  $\sigma = r^*x$ ), we have to find not only a solution  $x(t)$ , but also a vector function  $\xi(t)$  of the outputs of the nonlinear units. This requirement also follows from natural physical considerations. In the simplest (and most frequently occurring) case, when a function  $\xi(t)$  is uniquely determined from the first equation of (1.2) (it happens when  $\det q^*q \neq 0$ ), the detected function  $\xi(t)$  is clearly what we sought. In the general case, the existence of  $\xi(t)$  must be established by further investigation, which sometimes is not so simple. Later on, we will discuss this problem in detail.

Let us turn now to the question most important for us: How to go from a specific equation with discontinuous nonlinearities to an appropriate differential inclusion (1.6), i.e., how to define a set  $f(x_0, t_0)$  at a discontinuity

point  $(x_0, t_0)$ ; in other words, how to define a direction field on surfaces of discontinuity, with regard to the equation (1.3).

First, consider the case when actually existing nonlinear (or linear non-stationary) units have scalar outputs  $\xi_j$  and hence are given by scalar equations

$$\xi_j = \varphi_j(\sigma, t), \quad j = 1, \dots, m. \quad (1.7)$$

Here  $\sigma = \|\sigma_1, \dots, \sigma_m\|$  is a vector of all their inputs. In (1.7) the function  $\varphi_j$  can actually depend only on one or several components of the vector  $\sigma$ , notation (1.7) allows for such a possibility<sup>2</sup>. We think that some engineering or physical considerations should dictate how to define at a discontinuity point  $(\sigma_0, t_0)$  the admissible values of the inputs  $\xi_j$ , whose ranges we denote  $\varphi_j(\sigma_0, t_0)$ . Thus, we agree that *a mathematical description of a nonlinear unit by means of a single-valued discontinuous function  $\varphi_j(\sigma, t)$  is not sufficient; we have to define additionally its range at any discontinuity point.* The mathematical considerations, discussed above, require the following conditions to be satisfied<sup>3</sup>: (i) *the set  $\varphi_j(\sigma_0, t_0)$  is bounded, closed, and convex;* (ii) *the multivalued function  $\varphi_j(\sigma, t)$  is upper semicontinuous.*

Since the right-hand sides of (1.7) are sets, and  $\xi_j$  are numbers, notation (1.7) is inadequate; the correct notation is

$$\xi_j \in \varphi_j(\sigma, t), \quad j = 1, \dots, m. \quad (1.8)$$

Keeping in mind that the notation (1.8) is not customary in the engineering literature, we will write (1.7) instead of (1.8). However, (1.7) is considered as a symbolic form of (1.8). (Naturally, notation (1.7) is correct for the points  $(\sigma, t)$  where  $\varphi_j(\sigma, t)$  is continuous.) As before, we assume the linear part of the system to be described by the equations (1.2), which can be conveniently written in the form

$$\frac{dx}{dt} = Px + \sum_{j=1}^m q_j \xi_j, \quad \sigma = r^* x, \quad (1.9)$$

where  $q_j$  are column vectors and  $\xi_j$  are components of  $\xi$ .

It is important to note that formulas (1.7), (1.9) (or (1.2), (1.3)) with discontinuous functions  $\varphi_j(\sigma, t)$  are not equivalent to the representation of

<sup>2</sup>It may be that some of the functions  $\varphi_j(\sigma, t)$  do not depend on  $\sigma$  at all; the corresponding units describe external actions on the system.

<sup>3</sup>These requirements can be somewhat (but insignificantly from the practical point of view) weakened; we will limit ourselves to these conditions in order to avoid overloading our presentation.

the system in the form of (1.1) with

$$f(x, t) = Px + q_1\varphi_1(\sigma, t) + \dots + q_m\varphi_m(\sigma, t). \tag{1.10}$$

Formulas (1.7), (1.9) are more informative, because, by our convention, (1.7) is a mathematical description of specific blocks. The function  $f(x, t)$  alone does not completely determine these blocks, since the same function  $f(x, t)$  can be represented in the form (1.10) in more than one way. (The numbers  $m$ , the vectors  $q_j$  and the functions  $\varphi_j(\sigma, t)$  can differ in different representations.) Thus, the mathematical description of the given system in the form of (1.1) with function (1.10) is generally inadequate; to be more precise, it is insufficient, because such a notation may result in a loss of some essential information about a system.

Let us illustrate this assertion with an example. Let  $m = 2, l = 1$ , and there are two physically different nonlinear units with a common input and similar nonlinear characteristics

$$\xi_1 = \operatorname{sgn} \sigma, \quad \xi_2 = \operatorname{sgn} \sigma. \tag{1.11}$$

A linear part of the system is described by the equations<sup>4</sup>

$$\frac{dx}{dt} = Px + \|q_1, q_2\| \left\| \begin{matrix} \xi_1 \\ \xi_2 \end{matrix} \right\| = Px + q_1\xi_1 + q_2\xi_2, \quad \sigma = r^*x. \tag{1.12}$$

The expression  $\operatorname{sgn} 0$  in (1.11) means the segment  $[-1, 1]$ . (Then conditions (i) and (ii) are clearly satisfied.) In this case, equations (1.7), (1.9) take the form of (1.11), (1.12) with

$$f(x, t) = Px + q_1\xi_1 + q_2\xi_2 = Px + (q_1 + q_2) \operatorname{sgn} \sigma, \quad \sigma = r^*x.$$

The same function  $f(x, t)$  can be obtained for the system

$$\frac{dx}{dt} = Px + (q_1 + q_2)\xi, \quad \sigma = r^*x, \quad \xi = \operatorname{sgn} \sigma. \tag{1.13}$$

At the same time, equations (1.11), (1.12) and equation (1.13) describe physically different systems: system (1.11), (1.12) has two nonlinear blocks with the same input, while system (1.13) has only one nonlinear block. Hence, it should not be surprising that the solutions of these systems differ. (From what follows, we shall see that not every solution of (1.11), (1.12) satisfies system (1.13).) Therefore, it is natural that systems (1.11),

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<sup>4</sup>In other words, with an equation  $\sigma = \chi_1(p)\xi_1 + \chi_2(p)\xi_2$ , where  $p = \frac{d}{dt}$  and  $\chi_1(p), \chi_2(p)$  are, generally speaking, different transfer functions.

(1.12), and (1.13) may have different dynamical properties (see Example 3 in Subsection 3.1.2).

We could go from system (1.11), (1.12) to system (1.13) by substituting  $\xi_1, \xi_2$  from (1.11) in (1.12) and denoting  $\text{sgn } \sigma = \xi$ . However, the adopted convention disallows such transformations, since we agreed that every equation (1.7) describes an actually existing block. Now, incorrectness of these transformations is obvious. When writing (1.13) we lose some information on the outputs  $\xi_1 = \xi_1(t)$  and  $\xi_2 = \xi_2(t)$  of actually existing blocks. Evidently,  $\xi_1(t) = \xi_2(t)$  for  $\sigma(t) \neq 0$ , so, when  $\sigma(t) \neq 0$ , we can set  $\xi_1(t) = \xi_2(t) = \xi(t)$ . However, for system (1.11), (1.12) the equality  $\sigma(t) = 0$  may hold for all  $t \in \Delta$ , with  $\Delta$  being some interval (i.e., a sliding mode may exist). Possibly,  $\xi_1(t) \neq \xi_2(t)$  for some  $t \in \Delta$ , and so  $\xi(t)$  is defined incorrectly in this case.

Incorrectness of the indicated formal (and seemingly so obvious) transformation of the system (1.11), (1.12) into the system (1.13) follows, in fact, from the incorrectness of the conventional notation (1.11) (or, more generally, (1.7)) for equations of the blocks with discontinuous nonlinearities. Indeed, if we replace (1.11) with the correct notation

$$\xi_1(t) \in \text{sgn } \sigma(t), \quad \xi_2(t) \in \text{sgn } \sigma(t), \tag{1.14}$$

then the above incorrect transformations would be impossible, because the inclusion  $q_1\xi_1(t) + q_2\xi_2(t) \in (q_1 + q_2)\text{sgn } \sigma(t)$  is violated for  $t \in \Delta$ . (E.g., it is wrong that

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} \xi_1 + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \xi_2 \in \begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{sgn } 0$$

follows from  $\xi_1 \in \text{sgn } 0, \xi_2 \in \text{sgn } 0$ .)

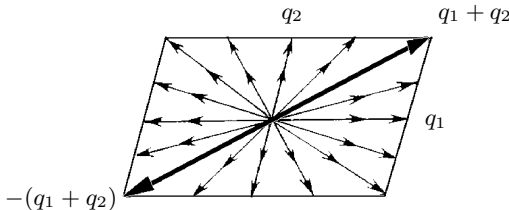


Fig. 1.5

Fig. 1.5 illustrates the above considerations in the two-dimensional case. If the vectors  $q_1$  and  $q_2$  are linearly independent, then the set  $q_1 \operatorname{sgn} 0 + q_2 \operatorname{sgn} 0$  is a parallelogram, while the set  $(q_1 + q_2) \operatorname{sgn} 0$  is a segment with the ends  $q_1 + q_2$  and  $-(q_1 + q_2)$  (a bold line). So the range of  $dx/dt$  for equations (1.11), (1.12) is wider than the similar range for equations (1.13). Hence, not all the solutions of (1.11), (1.12) satisfy (1.13).

Let us sum up the foregoing discussion for the case of  $m$  scalar nonlinear units. We consider system (1.9), (1.7) (in more exact terms, system (1.9), (1.8)) with  $\varphi_j(\sigma, t)$  being multiple-valued functions which obey (i) and (ii). When using a vector notation, this system takes the form (1.2), (1.3). More explicitly, (1.3) can be written as

$$\xi(t) \in \varphi[\sigma(t), t], \tag{1.15}$$

where  $\varphi(\sigma, t) = \varphi_1(\sigma, t) \times \dots \times \varphi_m(\sigma, t)$  is a topological product of sets.

For any  $t_0 \in \mathbb{R}^1$ ,  $x_0 \in \mathbb{R}^n$  with  $\sigma_0 = r^*x_0$  there exists a solution  $x(t)$ ,  $\xi(t)$  of this system, such that  $x(t_0) = x_0$ . Moreover, the usual theorems of the qualitative theory are valid; these theorems will be presented in the next section.

System (1.2), (1.15) can be written in a form of a differential inclusion  $\frac{dx}{dt} \in f(x, t)$  with a multivalued function  $f(x, t) = Px + q\varphi(\sigma, t)$  where  $\sigma = r^*x$ . It is easy to verify that this function is upper semicontinuous and the set  $f(x, t)$  is bounded and convex. Therefore, this differential inclusion has a solution with  $x(t_0) = x_0$ . If  $\det q^*q \neq 0$  then  $\xi(t)$  can be found from the equation  $dx/dt = Px + q\xi$ , namely, we put  $\xi(t) = (q^*q)^{-1}q^*(dx/dt - Px)$ . It is easily seen that then (1.15) is satisfied. The case  $\det q^*q = 0$  is discussed in Section 1.2, where the existence of a function  $\xi(t)$ , satisfying (1.2), (1.15) with a found function  $x(t)$ , is established by some more refined reasoning. In both cases the determined functions  $x(t)$ ,  $\xi(t)$  meet system (1.2), (1.15) and the condition  $x(t_0) = x_0$ .

Obviously, the components  $\xi_j(t)$  of the thus determined vector function  $\xi(t)$  satisfy the inclusion  $\xi_j(t) \in \varphi_j[\sigma(t), t]$ . Hence, at the points where  $\varphi_j[\sigma(t), t]$  is single-valued we have  $\xi_j(t) = \varphi_j[\sigma(t), t]$ . The function  $\xi_j(t)$  will be called an extended nonlinearity of  $\varphi_j(\sigma, t)$ . (Accordingly, while the function  $\varphi_j[\sigma(t), t]$  is multivalued, the function  $\xi_j(t)$  is single-valued.) A value  $\xi_j(t)$  is an output at a time  $t$  of the nonlinear unit, described by multivalued function  $\varphi_j(\sigma, t)$ .

So far it has been assumed that outputs of the nonlinear units are all scalar. Let us turn now to the case when outputs of all or some units are vectors. Suppose there are  $k$  nonlinear units, whose exact equations are of

the form

$$\xi_j \in \varphi_j(\sigma, t), \quad j = 1, \dots, k, \quad (1.16)$$

where  $\xi_j$  is an  $m_j$ -dimensional vector; the multiple-valued functions  $\varphi_j(\sigma, t)$  (with values from  $\mathbb{R}^{m_j}$ ) satisfy (i) and (ii). (Incorrectly, but following a common practice, equations of nonlinear units can be written as  $\xi_j = \varphi_j(\sigma, t)$ ,  $j = 1, \dots, k$ .) Inclusion (1.16) again can be represented in the form of (1.15) where  $\varphi(\sigma, t)$  is a multivalued vector function of dimension  $m$ . The way of obtaining a solution  $x(t)$ ,  $\xi(t)$ , which was described above, remains true. Only the form of the sets  $\varphi(\sigma, t)$  is changed. In the foregoing case, when outputs  $\xi_j$  were scalars, for any point  $(\sigma, t)$  the set  $\varphi(\sigma, t)$  was a rectangle (namely, a product of  $m$  segments  $\xi_j^- \leq \xi_j \leq \xi_j^+$ ). Now  $\varphi(\sigma, t)$  is a convex set of more complicated nature (a product of  $k$  convex, bounded, and closed sets). In the same way, it can be shown that there exist vector functions  $x(t)$ ,  $\xi(t)$  satisfying (1.2), (1.16) and the condition  $x(t_0) = x_0$ . A vector function  $\xi_j(t)$  (an output of  $j$ th nonlinear unit) is called *an extended function of  $\varphi_j(\sigma, t)$* .

Note that now, when writing the system in the form of (1.2), (1.15), we should additionally fix values  $m_1, \dots, m_k$ , i.e., point out which components of  $\xi(t)$  are outputs of the given units (e.g.,  $\xi_1, \xi_2$  form an output of the first unit,  $\xi_3$  is an output of the second unit,  $\xi_4, \xi_5, \xi_6$  form an output of the third unit, and so on).

Thus, the rule in italics on page 10 remains true also for the vector case.

### 1.1.3 *Relation to some other definitions of a solution to a system with discontinuous right-hand side*

One of the most popular definitions of a solution of a discontinuous system is that given in [Filippov (1960)]. This definition is justified by a rich theory developed by A.F. Filippov. We will not formulate it in the most general case, but restrict ourselves to the simplest case when the right-hand side  $f(x, t)$  of (1.1) is a function discontinuous on some differentiable surface  $\mathcal{R}_t$  in the space  $(x, t)$ , continuous in a neighborhood of this surface, and there exist limit values  $f_+(x, t)$  and  $f_-(x, t)$  of  $f(x, t)$  as a point  $(x, t)$  approaches the surface, remaining on one of its sides. Consider the most interesting case, when the direction fields  $f_+(x, t)$  and  $f_-(x, t)$  connect with each other on the discontinuity surface, i.e., the case shown in Fig. 1.1 (b). This case occurs for system (1.2), (1.3) with a single nonlinearity ( $m = l = 1$ ), when, e.g.,  $\varphi(\sigma, t)$  is a piecewise continuous function of  $\sigma$  (then the surface  $\mathcal{R}_t$

is a hyperplane  $r^*x = \sigma_0$  where  $\sigma_0$  is a discontinuity point of the function  $\varphi(\sigma, t)$ ). According to A.F. Filippov, the direction field on the discontinuity surface is defined in the following way. Consider an arbitrary point  $x$  at a surface  $\mathcal{R}_t$ . Let us construct a segment connecting the endpoints of the vectors  $f_-(x, t)$  and  $f_+(x, t)$ , and also construct a tangential plane to the surface  $\mathcal{R}_t$  at the point  $x$ . Suppose that  $f_0(x, t)$  is a vector with the origin at  $x$  and the end at the intersection point of the segment with the tangential plane (see, Fig. 1.6 (a)). This is the vector of the required direction field at the point  $x$ .

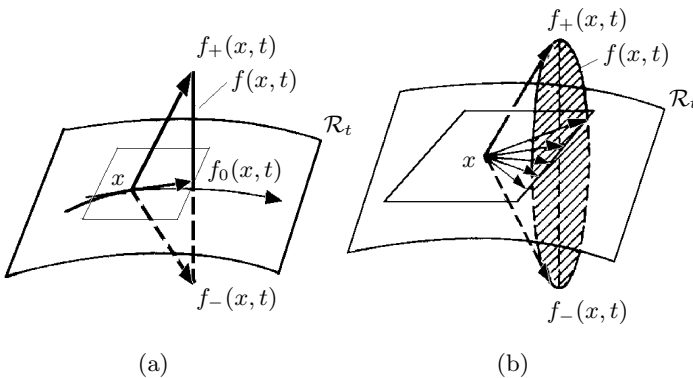


Fig. 1.6

Consider this case from the point of view adopted in our book. On a surface  $\mathcal{R}_t$  we have to extend the function  $f(x, t)$  to a multivalued function with a set  $f(x, t)$  being bounded, convex, and such that  $f_+(x, t) \in f(x, t)$ ,  $f_-(x, t) \in f(x, t)$ . (The latter conditions guarantee that the multiple-valued function  $f(x, t)$  is upper semicontinuous.) The least set with these properties is the segment  $\mathcal{F} = \{\lambda f_+(x, t) + (1 - \lambda)f_-(x, t), \lambda \in [0, 1]\}$ . Indeed, we can put  $f(x, t) = \mathcal{F}$  for  $x \in \mathcal{R}_t$  (see Fig. 1.6 (b)). Since the velocity vector of a sliding mode solution has to lie in the tangential plane,  $f_0(x, t) \in \mathcal{F}$  is the only vector from the range that can be tangent to the trajectory of a sliding mode. Thus we arrived at the definition given in [Filippov (1960)].

At the same time, we could define the set  $f(x, t)$  in another way. It can be chosen as an arbitrary bounded and convex set containing the segment  $\mathcal{F}$  (see Fig. 1.6 (b)). Then some vectors other than  $f_0(x, t)$  emerge on the tangential plane; this leads us to some other sliding mode solutions,

different from that of Filippov. (Note that for any one of such definitions the basic theorems of the qualitative theory, presented in Section 2.2, are valid, although a solution is generally not unique.)

We see that Filippov's definition corresponds to the minimal set  $f(x, t)$  among all the admissible sets. It is convenient in that the uniqueness of a solution occurs more frequently for a solution in the sense of Filippov. However, there are many situations when physically justified solutions are not solutions in the sense of Filippov. Let us cite one example of such a situation.

Consider the problem of a choice of the controls  $u_1, u_2$  in the system

$$\frac{dx_1}{dt} = x_2 u_1, \quad \frac{dx_2}{dt} = u_2 \quad (1.17)$$

which move any initial state to zero in minimal time. The controls obey the constraints  $|u_1| \leq 1, |u_2| \leq 1$ . It has been well known (see, [Pontryagin *et al.* (1964)]) that such controls may be designed for any point of the  $x_1 x_2$ -plane. E.g., in the first quadrant optimal controls take the form

$$u_1 = \begin{cases} +1, & x_1 < x_2^2/2, \\ -1, & x_1 \geq x_2^2/2, \end{cases} \quad u_2 = \begin{cases} -1, & x_1 \leq x_2^2/2, \\ +1, & x_1 > x_2^2/2. \end{cases} \quad (1.18)$$

In particular, the trajectory  $x_1 = x_2^2/2$  is optimal, and  $dx_1/dt = -x_2, dx_2/dt = -1$  on it.

Take a point  $x$  with coordinates  $x_1, x_2$  on this optimal trajectory, and approach it from the side where  $x_1 < x_2^2/2$ . Then the limit value of the right-hand sides of the system considered is  $f_+(x) = (x_2, -1)$ . If we approach  $x$  from the side where  $x_1 > x_2^2/2$  then for the limit values we get  $f_-(x) = (-x_2, +1)$ . Since  $f_+(x) = -f_-(x)$ , in this case the segment  $\mathcal{F}$  passes through the point  $x$ , i.e.,  $f_0(x) = 0$ , and the sliding mode solution in the sense of Filippov is  $x(t) \equiv \text{const}$ . At the same time, for the optimal trajectory, which is also a sliding mode solution, the velocity vector is  $f_*(x, t) = (-x_2, -1)$ . One sees that the optimal trajectory is not a Filippov solution.

However, when the nonlinearities are properly extended, the optimal trajectory is a solution in the sense of our definition. Indeed, system (1.17), (1.18) can be written in the form of (1.2), (1.3) with  $P = 0, \sigma = x, \varphi_1 = x_2 u_1, \varphi_2 = u_2$ . In line with the above assumption, these discontinuous nonlinearities are to be extended with respect to the multivalued functions, satisfying conditions (i) and (ii). This can be done in many ways, depending on whether the nonlinearities are considered as a single block with two

outputs ( $m_1 = m = 2$ ), or as two blocks, each with a single output ( $m_1 = m_2 = 1, m = 2$ ). Let us examine the second case. Then the values of  $\varphi_1$  and  $\varphi_2$  at the discontinuity point are some segments which include limit values. Take the minimal segments  $\varphi_1 = [-x_2, x_2], \varphi_2 = [-1, +1]$  as the values at  $x_1 = x_2^2/2$ . Then the value  $f(x) = \varphi(x)$  at the discontinuity point is a rectangle  $[-x_2, x_2] \times [-1, +1]$ , i.e., the rectangle shown in Fig. 1.7, with corners  $(-x_2, -1), (x_2, -1), (x_2, 1), (-x_2, 1)$ , which includes the vector  $f_*(x) = (-x_2, -1)$ .

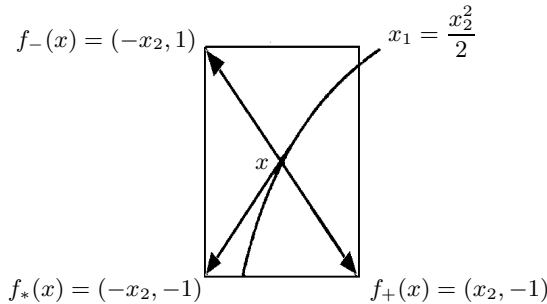


Fig. 1.7

Hence the optimal solution is a solution in the sense adopted here. (In the first case  $m_1 = m = 2$  we can take for the set  $f(x)$  any set containing  $f_-(x)$  and  $f_+(x)$ , e.g., a triangle whose corners are the extremities of the vectors  $f_-(x), f_*(x)$ , and  $f_+(x)$ . If this set contains the vector  $f_*(x)$ , the optimal solution is a solution in the accepted sense.) Some other examples of the same kind can be constructed with the help of the results of the book [Matveev and Yakubovich (2003)].

Turn now to the definition introduced by M.A. Aizerman and E.S. Pyatnitskii (1974). In general terms, the approach by Aizerman and Pyatnitskii is as follows (details can be found in [Aizerman and Pyatnitskii (1974a); Aizerman and Pyatnitskii (1974b)]). Discontinuous nonlinearities  $\varphi_j(\sigma, t)$ , describing some actually existing blocks, are substituted for continuous ones  $\varphi_j^{(\varepsilon)}(\sigma, t)$  which depend on a parameter  $\varepsilon$ . A function  $\varphi_j^{(\varepsilon)}(\sigma, t)$  has to be such that  $\varphi_j^{(\varepsilon)}(\sigma, t) \rightarrow \varphi_j(\sigma, t)$  as  $\varepsilon \rightarrow 0$  for any  $(\sigma, t)$  which is not a discontinuity point of  $\varphi_j(\sigma, t)$ . Consider the system obtained when substituting all the functions  $\varphi_j(\sigma, t)$  for  $\varphi_j^{(\varepsilon)}(\sigma, t)$ . Since the right-hand sides of this system are continuous functions, it has a solution  $x^{(\varepsilon)}(t)$ . We can always

find a sequence  $\varepsilon_k \rightarrow 0$  such that there exists a limit

$$\lim_{\varepsilon_k \rightarrow 0} x^{(\varepsilon_k)}(t) = x(t).$$

In general, the sequence  $\varepsilon_k$  and the limit  $x(t)$  are not uniquely defined. Any such limit is taken as *a solution of the initial system with a discontinuous right-hand side*.

The informal meaning of this definition is obvious. The definition is justified by the fact that solutions so defined have the same conventional properties as solutions of systems with continuous right-hand sides.

We will not present appropriate theorems (a reader is referred to [Aizerman and Pyatnitskii (1974a); Aizerman and Pyatnitskii (1974b)]). It is important to emphasize that a statement in [Aizerman and Pyatnitskii (1974a); Aizerman and Pyatnitskii (1974b)] implies that any solution in the sense of Aizerman and Pyatnitskii is also a solution in the sense adopted by us, i.e., it can be obtained as a solution of the appropriate system with a multiple-valued right-hand side, after extending nonlinearities at discontinuity points for some multivalued functions satisfying (i) and (ii)<sup>5</sup>. However, the converse is not true: not every solution in the sense that we consider is also a solution in the sense of Aizerman and Pyatnitskii. E.g., in the case of a single scalar nonlinearity ( $m = l = 1$ ) we can take a nonlinearity  $\varphi(\sigma, t)$  in (1.2) in the form shown in Fig. 1.8, i.e.,  $\varphi(\sigma) = \text{sgn } \sigma$  for  $\sigma \neq 0$  and  $\varphi(0) = [-\xi_0, +\xi_0]$  where  $\xi_0 > 1$  ( $\xi_0$  may depend on  $t$ ). Nonlinearities

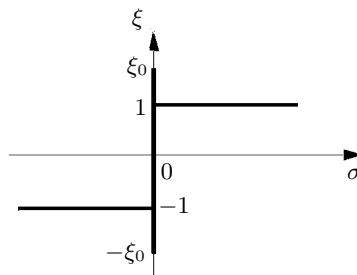


Fig. 1.8

<sup>5</sup>Observe that this claim is true only for systems (1.2), (1.3). M.A. Aizerman and E.S. Pyatnitskii have also studied the systems for which the first equation of (1.2) has the form  $dx/dt = g(x, \xi)$  with a continuous function  $g(x, \xi)$ . Such systems are not considered in this book.

of this type describe “detachment friction,” i.e., the case for which static friction exceeds dynamic friction. Since the definition by Aizerman and Pyatnitskii deals only with those values of a nonlinearity for which  $\sigma \neq 0$ , the system (1.2), (1.3) with this nonlinearity has the same solutions (in the sense of Aizerman and Pyatnitskii) as the system with  $\varphi(\sigma) = \text{sgn } \sigma$ . The same is true for the definition in the sense of Filippov. As for our definition, the range of  $\varphi(\sigma)$  at a discontinuity point plays a crucial role here. E.g., let a nonlinearity be of the form shown in Fig. 1.9. All the stationary solutions

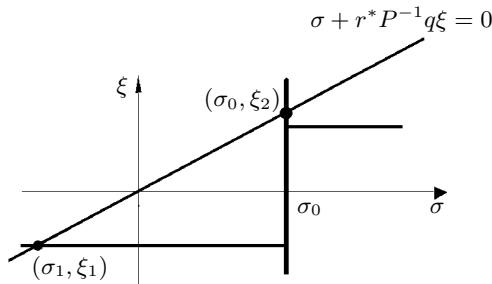


Fig. 1.9

of system (1.2), (1.3) ( $m = l = 1$ ) have the form  $x = -P^{-1}q\xi$  where  $\xi$  is an ordinate of any point  $(\sigma, \xi)$  of intersection of the nonlinearity graph and “the characteristic line”  $\sigma + r^*P^{-1}q\xi = 0$ . (It is supposed that  $\det P \neq 0$ .) Let there be two such points  $(\sigma_1, \xi_1)$  and  $(\sigma_0, \xi_2)$  (see Fig. 1.9). Then there exists a unique stationary solution  $x_1 = -P^{-1}q\xi_1$  in the sense of Aizerman and Pyatnitskii, as well as in the sense of Filippov. As for our definition, there are two solutions:  $x_1 = -P^{-1}q\xi_1$  and  $x_2 = -P^{-1}q\xi_2$ .

At the same time, observe that if a system has a single stationary nonlinearity  $\xi = \varphi(\sigma)$  with only isolated points of discontinuity of the first kind and its values at a discontinuity point  $\sigma_0$  are defined as  $\langle \varphi(\sigma_0 - 0), \varphi(\sigma_0 + 0) \rangle$ , then all three definitions (by Filippov, by Aizerman and Pyatnitskii, and ours) provide the same set of solutions  $x(t)$ . (Our definition requires additional knowledge of  $\xi(t)$ .) If we take two or more nonlinearities of the same type, the sets of solutions in the sense we use and in the sense of Aizerman and Pyatnitskii coincide, but some of these solutions may not be solutions in the sense of Filippov.

In conclusion of this subsection, notice that there is a number of other

definitions of solutions of discontinuous systems and differential inclusions (see [Viktorovskii (1954); Matrosov (1967)]).

#### 1.1.4 *Sliding modes. Extended nonlinearity. Example*

Consider once again the system

$$\frac{dx}{dt} = Px + q\xi, \quad \sigma = r^*x, \quad (1.19)$$

$$\xi = \varphi(\sigma, t) \quad (1.20)$$

where  $x$ ,  $\xi$ ,  $\sigma$  are vectors of dimensions  $n$ ,  $m$ ,  $l$  respectively. Suppose that the previous assumptions about the function  $\varphi(\sigma, t)$  are made, i.e., at any discontinuity point  $(\sigma_0, t_0)$  a set of its values  $\varphi(\sigma_0, t_0)$  is defined and conditions (i), (ii) of Subsection 2.1.2 are satisfied. Let  $D = \{(\sigma_0, t_0)\}$  be the set of all discontinuity points of  $\varphi(\sigma, t)$ .

In line with what was said previously, by a *solution of system* (1.19), (1.20) we mean a pair of functions  $\{x(t), \xi(t)\}$  defined on a segment  $\Delta = [t_0, t_1]$  of nonzero length and such that  $x(t)$  is absolutely continuous,  $\xi(t)$  is integrable, and the relationships

$$\frac{dx(t)}{dt} = Px(t) + q\xi(t), \quad \sigma(t) = r^*x(t), \quad \xi(t) \in \varphi[\sigma(t), t] \quad (1.21)$$

hold almost everywhere on  $\Delta$ .

As was pointed out previously, the function  $\xi(t)$  is called an *extended nonlinearity*  $\varphi[\sigma(t), t]$ . If the matrix  $q^*q$  is nonsingular, the extended nonlinearity is uniquely determined from  $x(t)$  and (1.21):

$$\xi(t) = (q^*q)^{-1} \left[ \frac{dx}{dt} - Px \right].$$

However, it may be that there exist different solutions  $\{x(t), \xi_1(t)\}$ ,  $\{x(t), \xi_2(t)\}$  which are distinct only in an extended nonlinearity. Of course, if  $\varphi(\sigma)$  is a continuous (and therefore single-valued) function, then  $\xi(t) = \varphi[\sigma(t), t]$  is determined uniquely, so in this case a function  $x(t)$  by itself may be called a solution. Following tradition, a function  $x(t)$  will be called a solution even when the function  $\varphi(\sigma)$  is discontinuous. But it should be taken into account that, when speaking about a solution, we always mean the existence of an appropriate extended nonlinearity. A vector  $x(t)$  is frequently called a state of the system (at a time  $t$ ).

Assume that  $[\sigma(t), t] \in D$  for  $t \in \Delta^0$ , where  $\Delta^0 = [t_1^0, t_2^0] \subset \Delta$  is some segment of nonzero length and  $\{x(t), \xi(t)\}$  is a solution defined on  $\Delta_0$ ,  $\sigma(t) = r^*x(t)$ . Then it is said that the solution  $\{x(t), \xi(t)\}$  with  $t \in \Delta_0$  is in a *sliding mode*. For the cases shown in Fig. 1.1 ((b) and (c)) a point  $x(t)$  lies on the discontinuity surface for  $t \in \Delta_0$  (it is said that “the solution  $x(t)$  slides along the discontinuity surface”).

Let in (1.20)  $\det P \neq 0$  and the nonlinearity be stationary, i.e.,  $\varphi(\sigma, t) = \varphi(\sigma)$ . Let us find stationary solutions of the system (1.19), (1.20). (The corresponding states  $x$  are named equilibria or rest points, while the set of all equilibria is called a stationary set of the system (1.19), (1.20).) From what was said above, it follows that all the stationary solutions  $\{x, \xi\} = \text{const}$  are given from

$$Px + q\xi = 0, \quad \sigma = r^*x, \quad \xi \in \varphi(\sigma). \tag{1.22}$$

Thus  $(\sigma, \xi)$  is found from the relationships

$$\sigma + r^*P^{-1}q\xi = 0, \quad \xi \in \varphi(\sigma),$$

and  $x = -P^{-1}q\xi$ . If, in addition, the function  $\varphi(\cdot)$  is discontinuous at the point  $\sigma = \text{const}$  obtained from (1.22), then  $\{x, \xi\} = \text{const}$  is a stationary solution of a sliding mode. For the case when  $m = l = 1$  and a nonlinearity is shown in Fig. 1.9, there are two stationary solutions  $\{x_1, \xi_1\}, \{x_2, \xi_2\}$  with  $x_j = -P^{-1}q\xi_j$ , and the points  $(\sigma_1, \xi_1)$  and  $(\sigma_0, \xi_2)$  are arranged in a way shown in the figure. In this case  $\{x_1, \xi_1\}$  is an ordinary solution, while  $\{x_2, \xi_2\}$  is a sliding mode solution.

Earlier, it was stated without detailed explanation that, in general, a direction field, which defines a sliding mode, cannot be set arbitrarily on a discontinuity surface. Let us examine this issue in more detail. Along the way, we will illustrate the definitions of solution and sliding mode, and the technique for their determination. For simplicity, consider a scalar nonlinearity

$$\xi = \varphi(\sigma) \tag{1.23}$$

( $m = l = 1$ ). In this case in (1.19)  $q$  and  $r$  are column vectors. Assume that  $r^*q \neq 0$ ,  $\sigma = 0$  is the unique point of discontinuity (of the first kind) of the function  $\varphi(\sigma)$ , and the right-hand limit ( $\varphi(+0)$ ) and the left-hand limit ( $\varphi(-0)$ ) satisfy the condition

$$\varphi(-0) < 0 < \varphi(+0). \tag{1.24}$$

(For example,  $\varphi(\sigma) = \text{sgn } \sigma$ , then  $\varphi(\pm 0) = \pm 1$ .) Let us find sliding mode solutions of the system (1.19), (1.23). Suppose  $\sigma(t) \equiv 0$  over  $\Delta^0 = [t_1^0, t_2^0]$ . Then  $\dot{\sigma} = r^*(Px + q\xi) = 0$  for  $t \in \Delta^0$ . Hence  $\xi = -(r^*q)^{-1}r^*Px$ , and for a sliding mode solution we get

$$x(t) = e^{P_0(t-t_1^0)} x(t_1^0), \quad \xi(t) = -(r^*q)^{-1}r^*Px(t) \tag{1.25}$$

with  $P_0 = [I_n - q(r^*q)^{-1}r^*]P$ . (In addition, the vector  $x(t_1^0)$  has to satisfy the relationship  $r^*x(t_1^0) = \sigma(t_1^0) = 0$ ; it is easily checked that then (1.25) implies  $\sigma(t) \equiv 0$ , as it must.) Moreover, since the inclusion  $\xi(t) \in \varphi(0)$  has to hold, a segment  $t_1^0 \leq t \leq t_2^0$  is found from the condition

$$\varphi(-0) \leq \xi(t) \leq \varphi(+0). \tag{1.26}$$

If a vector  $x(t_1^0)$  is such that there are no values  $t > t_1^0$  which meet inequalities (1.26), then there are no sliding mode solutions. If such values exist and they fill the segment  $\Delta^0 = [t_1^0, t_2^0]$ ,  $t_2^0 > t_1^0$ , then there is a sliding mode solution defined on  $\Delta^0 = [t_1^0, t_2^0]$  and given by (1.25).

Denote

$$s^* = -(r^*q)^{-1}r^*P. \tag{1.27}$$

Then  $\xi(t) = s^*x(t)$ . Thus sliding modes fill “a lamina”

$$\Pi = \{x : r^*x = 0, \quad \varphi(-0) \leq s^*x \leq \varphi(+0)\}. \tag{1.28}$$

in the space  $\{x\}$ . Let us explore how the direction field  $f(x) = Px + q\varphi(r^*x)$  behaves in a neighborhood of the discontinuity surface  $r^*x = 0$  and, in particular, in a neighborhood of the lamina  $\Pi$ . Let

$$f_+(x) = Px + q\varphi(+0), \quad f_-(x) = Px + q\varphi(-0) \tag{1.29}$$

be limit values of the function  $f(x)$ . Examine a behavior of the function  $\dot{\sigma} = r^*f(x)$  in a neighborhood of the discontinuity surface  $\sigma = r^*x = 0$ . Suppose that a point  $x$  approaches the discontinuity surface not intersecting it, and  $x_0$  is a limit value,  $r^*x_0 = 0$ . From (1.27) we get

$$r^*Px_0 + r^*q s^*x_0 = 0. \tag{1.30}$$

In view of (1.29), the limit values of  $\dot{\sigma} = r^*f(x)$  are

$$\dot{\sigma}_+ = r^*Px_0 + r^*q\varphi(+0), \quad \dot{\sigma}_- = r^*Px_0 + r^*q\varphi(-0) \tag{1.31}$$

when the point  $x$  moves in the regions  $\sigma > 0$  and  $\sigma < 0$  respectively. Suppose that  $x_0$  does not belong to the lamina  $\Pi$ , i.e., either  $s^*x_0 < \varphi(-0)$ ,

or  $s^*x_0 > \varphi(+0)$ . Let  $s^*x_0 < \varphi(-0)$  and, for the sake of definiteness,  $r^*q > 0$ . Then from (1.30), (1.31) it follows that  $0 < \dot{\sigma}_- < \dot{\sigma}_+$ . Hence on this part of the discontinuity surface the trajectories behave as in Fig. 1.1 (a), i.e., they pass through the hyperplane  $r^*x = 0$  “from left to right.” (When saying this, we mean that in Fig. 1.1 (a) the axis  $\sigma$  is orthogonal to the discontinuity surface and is directed from left to right.) If  $r^*q > 0$  and  $s^*x_0 > \varphi(+0)$  then  $0 > \dot{\sigma}_+ > \dot{\sigma}_-$  and trajectories pass through the hyperplane  $r^*x = 0$  “from right to left.” A similar behavior takes place when  $r^*q < 0$ , provided that  $x_0$  does not lie on the lamina  $\Pi$ . This conforms with the above conclusion that a sliding mode does not exist in this case.

Suppose now that a point  $x_0$  lies in the interior of the lamina  $\Pi$  :  $\varphi(-0) < s^*x_0 < \varphi(+0)$ . Then from (1.30), (1.31) we get

$$\dot{\sigma}_+ > 0 > \dot{\sigma}_- \text{ for } r^*q > 0, \tag{1.32}$$

$$\dot{\sigma}_+ < 0 < \dot{\sigma}_- \text{ for } r^*q < 0. \tag{1.33}$$

In the case of (1.33) the trajectories connect with each other as shown in Fig. 1.1 (b), while in the case (1.32) they behave as shown in Fig. 1.1 (c). In the last case sliding modes arise; as we have seen previously, they are described by formula (1.25). We have already ascertained that on the surface of a sliding mode (i.e., on the lamina  $\Pi$ ) the direction field is uniquely defined from the condition for the existence of a solution; this condition is  $f(x) = Px + q\xi$ ,  $\xi = -(r^*q)^{-1}r^*Px$ .

In the foregoing we assumed that  $m = l = 1$  and  $r^*q \neq 0$ . Both these conditions can be eliminated. It is easily seen that the formulas of a sliding mode (1.25) remain valid also for  $m = l > 1$ ,  $\det r^*q \neq 0$ . The case  $m = l = 1$ ,  $r^*q = 0$  will be examined in detail in Subsection 2.2.2.

In conclusion of this section, consider a specific example of system (1.19), (1.23), for which the behavior described above takes place. Namely, consider a system thoroughly studied in [Andronov and Maier (1947)]:

$$\begin{aligned} \frac{dx_1}{dt} &= -Ax_2 + x_3 - Bx_1 - \xi, & \xi &= \operatorname{sgn} x_1, \\ \frac{dx_2}{dt} &= x_1, & \frac{dx_3}{dt} &= -x_2. \end{aligned} \tag{1.34}$$

(Here all the variables are scalar,  $A$  and  $B$  are constant parameters.) Equations (1.34) describe a system for automatic control of an engine when the mass of the controller as well as liquid and dry friction are not neglected. The first three equations represent dynamics of the controller, while the

last equation describes dynamics of the controlled engine. When considering this system, we will be interested only in its features arising out of the discontinuous nonlinearity.

It is obvious that (1.34) can be rewritten in the form of (1.19), (1.23) with

$$P = \begin{bmatrix} -B & -A & 1 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}, \quad q = r = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad (1.35)$$

and  $\varphi(\sigma) = \text{sgn } \sigma$ . In line with what was said previously, we have to define the set  $\varphi(0)$ . Taking into account that in [Andronov and Maier (1947)] the friction was supposed to be dry, we set  $\varphi(0) = \text{sgn } 0 = [-1, +1]$ .

In this case (see, (1.27))  $s^* = [B, -A, -1]$ . According to the above discussion, in the state space  $\mathbb{R}^3$  we have the previously described behavior of trajectories in a neighborhood of the discontinuity surface  $\sigma = x_1 = 0$  with the lamina of sliding modes (1.28), i.e., with the lamina

$$\Pi = \{x : x_1 = 0, \quad |Ax_2 - x_3| \leq 1\}. \quad (1.36)$$

From (1.22) (or, more simply, directly from (1.34)) we deduce that system (1.34) has stationary solutions and they fill the segment

$$x = \begin{bmatrix} 0 \\ 0 \\ x_3 \end{bmatrix}, \quad x_3 = [-1, +1]. \quad (1.37)$$

It is natural that A.A. Andronov and A.G. Maier arrived at the same conclusions when they studied the specific system (1.34).

Solutions of a sliding mode can be obtained by formulas (1.25), or, more easily, directly from (1.34), according to the scheme given above. Namely, if we set  $x_1 \equiv 0$  and replace the equality  $\xi = \text{sgn } x_1$  for  $|\xi| \leq 1$  in (1.34), then these solutions take the form

$$x_1 \equiv 0, \quad x_2 \equiv x_2^0, \quad x_3 = x_3^0 - tx_2^0, \quad \xi = x_3^0 - Ax_2^0 - tx_2^0. \quad (1.38)$$

Here  $x(0) = \begin{bmatrix} 0 \\ x_2^0 \\ x_3^0 \end{bmatrix}$  and  $t \in \Delta^0 = [0, T]$ , where  $\Delta^0$  is a domain where solution (1.38) exists. This domain is given from the condition  $\xi(t) \in [-1, +1]$ , i.e.,

$$\Delta^0 = \{t : |x_2^0 - Ax_3^0 - tx_2^0| \leq 1\}. \quad (1.39)$$

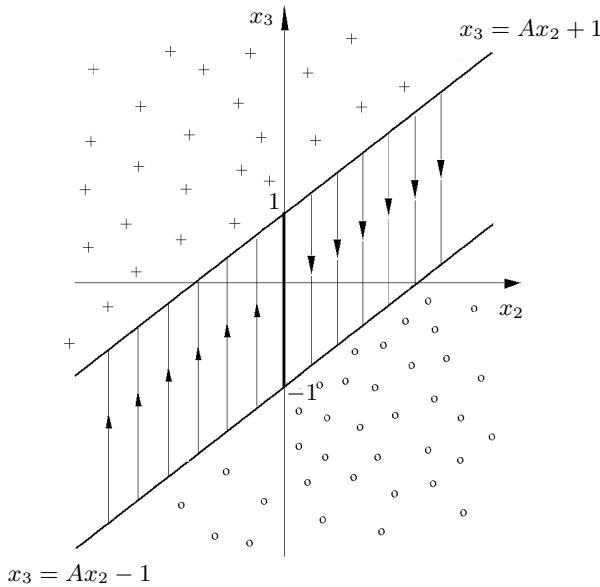


Fig. 1.10

The behavior of the trajectories in a neighborhood of the plane  $x_1 = 0$  is illustrated with Fig. 1.10, where those trajectories that go away from the reader are shown with crosses, while the trajectories that go towards the reader are shown with circles. The rest segment is represented with a bold line.

The above example also enables us to illustrate some other concepts used in this book. Formulas (1.38) demonstrate that the smaller the value of  $x_2^0$  (i.e., the closer a trajectory to the rest segment  $x_1 = 0, x_2 = 0, x_3 = \text{const} \in [-1, +1]$ ), the slower the movement along this trajectory is when  $t \in \Delta^0$ . Furthermore, any equilibrium  $x_1 = 0, x_2 = 0, x_3 = \text{const}, |x_3| \leq 1$ , is Lyapunov unstable, because for any neighborhood of this equilibrium, taken arbitrarily small, there are trajectories which go through this neighborhood and move from the equilibrium to a fixed finite distance. However, one can speak about the stability of the entire rest segment, i.e., of the entire set of equilibria. Namely, a given set is called *Lyapunov stable* (in the small) if for any  $\varepsilon > 0$  those trajectories that start in a  $\delta$ -neighborhood of this set do not leave its  $\varepsilon$ -neighborhood, provided

that  $\delta$  is chosen sufficiently small. A given set is called *globally stable*, if it is Lyapunov stable and all the trajectories approach it as  $t \rightarrow +\infty$ . It will be demonstrated in Section 3.2 that in the example considered the rest segment  $x_1 = 0, x_2 = 0, x_3 = \text{const} \in [-1, +1]$  is globally stable when  $AB > 1$ .

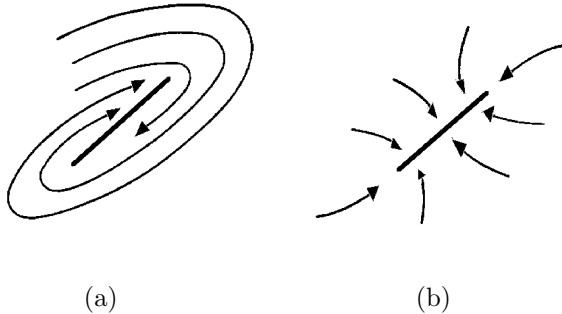


Fig. 1.11

Some systems with multiple equilibria have a stronger stability property. Namely, a stationary set can be not only globally stable, but can also have a property that for any solution the vector  $x(t)$  tends to some point of this set as  $t \rightarrow +\infty$ . Then the stationary set is called *pointwise globally stable*. In Fig. 1.11 is sketched a globally stable set (Fig. 1.11 (a)) and a pointwise globally stable set (Fig. 1.11 (b)).

Further in the book, along with some other results, we establish criteria for the stability of a stationary set and present examples of specific systems whose stationary sets are stable in the sense described above.

## 1.2 Systems of Differential Equations with Multiple-Valued Right-Hand Sides (Differential Inclusions)

In this section we discuss some topics from the theory of differential equations with a multiple-valued right-hand side: we formulate the definition of a solution, prove a local theorem for the existence of solutions, and reveal some properties of solutions which will be needed in what follows. In Subsection 1.2.2 we prove the important Theorems 1.7 and 1.9 on existence and properties of “an extended nonlinearity.” These theorems justify the scheme of reduction of a differential equation with a discontinuous right-

hand side to a differential inclusion, which was given in Section 1.1. Sliding modes are also considered here.

### 1.2.1 *Concept of a solution of a system of differential equations with a multivalued right-hand side, the local existence theorem, the theorems on continuation of solutions and continuous dependence on initial values*

In this subsection we consider a vector differential inclusion

$$\frac{dx}{dt} \in f(x, t), \quad (1.40)$$

whose meaning will be discussed later.

Consider a vector-valued function  $f(x, t)$  ( $t \in \mathbb{R}^1$ ,  $x \in \mathbb{R}^N$ ) which maps each point  $(x_0, t_0)$  of some region  $\mathcal{D} \subset \mathbb{R}^{N+1}$  to a set of points  $f(x_0, t_0) \subset \mathbb{R}^N$ .

If the set  $f(x_0, t_0)$  consists of just one point, we say that the function  $f(x, t)$  is *single-valued at the point*  $(x_0, t_0)$ . Otherwise  $f(x, t)$  is called *multivalued at the point*  $(x_0, t_0)$ .

The functions whose graphs are shown in Fig. 1.3 and Fig. 1.4, (a), (b), are single-valued for  $\sigma \neq \sigma_0$  and multivalued at the point  $\sigma = \sigma_0$ ; the function presented in Fig. 1.4 (d) is multivalued for all  $\sigma$ .

When classifying multivalued functions, as distinct from single-valued ones, it makes sense to consider two kinds of discontinuity. Let  $\mathcal{M}$  be a set in  $\mathbb{R}^N$ . The set of points  $y$  satisfying the inequality  $\inf_{x \in \mathcal{M}} \rho(y, x) < \varepsilon$ , where  $\rho(y, x)$  is the Euclidean distance between  $y$  and  $x$ , is called the  $\varepsilon$ -neighborhood of the set  $\mathcal{M}$ .

**Definition 1.1** A function  $f(x, t)$  is called *semicontinuous*<sup>6</sup> at a point  $(x_0, t_0)$  if for any  $\varepsilon > 0$  there exists  $\delta(\varepsilon, x_0, t_0)$  such that the set  $f(x_1, t_1)$  is contained in the  $\varepsilon$ -neighborhood of the set  $f(x_0, t_0)$ , provided that a point  $(x_1, t_1)$  belongs to the  $\delta$ -neighborhood of the point  $(x_0, t_0)$ .

**Definition 1.2** A function  $f(x, t)$  is called *continuous at a point*  $(x_0, t_0)$  if it is semicontinuous and, besides, for any  $\varepsilon > 0$  there exists  $\delta(\varepsilon, x_0, t_0)$  such that the set  $f(x_0, t_0)$  is contained in the  $\varepsilon$ -neighborhood of the set  $f(x_1, t_1)$ , provided that the point  $(x_0, t_0)$  belongs to the  $\delta$ -neighborhood of a point  $(x_1, t_1)$ .

---

<sup>6</sup>In the literature, this property is commonly called *upper semicontinuity* or  *$\beta$ -continuity*. For the sake of brevity, we will use below only the term “semicontinuity.”

The functions, whose graphs are shown in Fig. 1.3 (a)–(d) and Fig. 1.4 (d), are semicontinuous, however these functions are not continuous in the sense of Definition 1.2. The functions shown in Fig. 1.4 (a)–(c) are not semicontinuous at the point  $\sigma_0$ .

It will be shown later that the existence of a solution of the differential inclusion (1.40) (in the sense described below) follows from the semicontinuity of the function  $f(x, t)$ .

**Definition 1.3** A vector function  $x(t)$  is called a *solution of the differential inclusion* (1.40) if it is absolutely continuous and for those  $t$  for which a derivative  $dx/dt$  exists, the inclusion

$$\frac{dx}{dt} \in f[x(t), t] \tag{1.41}$$

holds.

Let us prove a local theorem for the existence of solutions.

**Theorem 1.1** *Suppose that at any point  $(x_1, t_1)$  of a region<sup>7</sup>*

$$\mathcal{D} : |t_1 - t_0| \leq \alpha, \quad |x_1 - a| \leq \rho$$

*a multivalued vector function  $f(x, t)$  is semicontinuous, while the set  $f(x_1, t_1)$  is closed, bounded, and convex; in addition,*

$$\sup |y| = c \quad \text{for } y \in f(x_1, t_1), (x_1, t_1) \in \mathcal{D}. \tag{1.42}$$

*Then for*

$$|t - t_0| \leq \tau = \min(\alpha, \rho/c) \tag{1.43}$$

*there exists at least one solution  $x(t)$  which satisfies the inclusion (1.41) and the initial condition*

$$x(t_0) = a. \tag{1.44}$$

*Proof* of the theorem generally follows the proof of Peano’s theorem for differential equations with single-valued continuous right-hand sides. For the sake of simplicity, assume that  $\alpha > \rho/c$ .

Divide a segment  $[t_0 - \tau, t_0 + \tau]$  into  $2n$  parts limited by the points

$$t_i^{(n)} = t_0 + i \frac{\tau}{n} \quad (i = 0, \pm 1, \dots, \pm n),$$

---

<sup>7</sup>Henceforth  $|x|$  means  $(x^*x)^{1/2}$ . Such a form of  $\mathcal{D}$  is chosen only for simplicity.

and construct on this segment some Euler polygons

$$x_n(t) = x_n(t_i^{(n)}) + (t - t_i^{(n)}) \widehat{f} [x_n(t_i^{(n)}), t_i^{(n)}]$$

for  $t \in [t_i^{(n)}, t_{i+1}^{(n)}]$ ,  $i = 0, 1, \dots, n - 1$ , and for  $t \in [t_{i-1}^{(n)}, t_i^{(n)}]$ ,  $i = 0, -1, \dots, -(n - 1)$ , where  $x_n(t_0) = a$  and  $\widehat{f} [x_n(t_i^{(n)}), t_i^{(n)}]$  denotes any vector from the set  $f [x_n(t_i^{(n)}), t_i^{(n)}]$ . In view of (1.42), the family of vector functions constructed in segment (1.43) is uniformly bounded and equicontinuous. By the Arzelá theorem, there exists a subsequence  $x_{n_k}(t)$  which uniformly converges in (1.43) to some vector function  $x(t)$ . Along with all the functions  $x_{n_k}(t)$ , the function  $x(t)$  clearly satisfies the initial condition (1.44) and the Lipschitz condition with the constant  $c$ . The last fact implies that  $x(t)$  is absolutely continuous. It remains to verify that inclusion (1.41) is valid. To prove this we begin with a simple lemma.

**Lemma 1.1** *If  $x(t)$  is a function absolutely continuous on<sup>8</sup>  $\langle a, b \rangle$  and  $|dx/dt| \leq c$  for almost all  $t \in \langle a, b \rangle$ , then*

$$\frac{1}{b - a} [x(b) - x(a)] \in \text{conv} \bigcup_{t \in \langle a, b \rangle \text{ a.e.}} \frac{dx}{dt}. \tag{1.45}$$

Here *conv* denotes a closed convex hull and the notation “ $t \in \langle a, b \rangle$  a.e.” means that the union is taken over all points of the interval  $\langle a, b \rangle$  where the derivative  $dx/dt$  exists.

To prove this lemma, we take advantage of a formula which follows from the definition of the Lebesgue integral:

$$\frac{1}{b - a} [x(b) - x(a)] = \frac{1}{b - a} \int_a^b \frac{dx}{dt} dt = \lim_{k \rightarrow \infty} y_k$$

where

$$y_k = \sum_{i=1}^k \frac{\mu_i}{|b - a|} \frac{dx(t_i)}{dt}$$

are Darboux–Lebesgue sums,  $\mu_i \geq 0$ ,  $\mu_1 + \dots + \mu_k = |b - a|$ . Obviously, for all  $k$  the vectors  $y_k$  belong to the right-hand side of inclusion (1.45). Hence (1.45) is valid also for the limit vector. The proof of Lemma 1.1 is complete.

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<sup>8</sup>Recall that  $\langle a, b \rangle = [a, b]$  if  $a < b$  and  $\langle a, b \rangle = [b, a]$  if  $b < a$ .

Let us continue with the proof of Theorem 1.1. Suppose at a point  $t$  there exists a derivative  $dx/dt$ . Verify that inclusion (1.41) holds. From Lemma 1.1 and the construction method of Euler polygons, we get

$$\frac{1}{h} [x_{n_k}(t+h) - x_{n_k}(t)] \in \bigcup_{i=-n_k}^{i=n_k} \widehat{f} [x_{n_k}(t_i^{(n_k)}), t_i^{(n_k)}] \tag{1.46}$$

where  $\widehat{f} [x_{n_k}(t_i^{(n_k)}), t_i^{(n_k)}]$  means the vector from the set

$$f [x_{n_k}(t_i^{(n_k)}), t_i^{(n_k)}]$$

which was chosen by the construction of Euler polygons. Clearly, the set of vectors in the right-hand side of (1.46) is contained in the set

$$\text{conv} \bigcup_{\lambda \in (t-\tau/n_k, t+h+\tau/n_k)} f [x_{n_k}(\lambda), \lambda].$$

Since the function  $f(x, t)$  is semicontinuous, the semidistance<sup>9</sup> of this set from the convex hull

$$\text{conv} \bigcup_{\lambda \in (t, t+h)} f [x(\lambda), \lambda] \tag{1.47}$$

tends to zero as  $n_k \rightarrow \infty$ . From the convexity of the set  $f(x, t)$  and semicontinuity of the function  $f$ , it follows that the semidistance of the hull (1.47) from the set  $f(x, t)$  vanishes as  $h \rightarrow 0$ . ■

For convenience, in what follows differential inclusion (1.40) will be written as a vector differential equation

$$\frac{dx}{dt} = f(x, t) \tag{1.48}$$

with a multivalued right-hand side. By a solution of this equation we mean an absolutely continuous vector-valued function  $x(t)$  satisfying the differential inclusion (1.40) in the sense of Definition 1.3.

Henceforth, without additional notice, we assume that the right-hand side of the system (1.48) is semicontinuous and a set  $f(x, t)$  is closed, bounded, and convex at each point  $(x, t)$  of its domain of definition.

---

<sup>9</sup>By the semidistance of a set  $\mathcal{A}$  from a set  $\mathcal{B}$  in  $\mathbb{R}^N$  we mean  $\sup_{x \in \mathcal{A}} \inf_{y \in \mathcal{B}} \rho(x, y)$ .

Theorems on continuation of solutions, similar to the corresponding theorems for systems with continuous right-hand sides, are valid also for system (1.48) with a multivalued right-hand side.

**Theorem 1.2** *If a right-hand side of system (1.48) is defined for all  $0 < t < +\infty$  ( $-\infty < t < +\infty$ ),  $x \in \mathbb{R}^N$  and a solution lies in some bounded region  $\mathcal{G} \subset \mathbb{R}^N$ , then this solution is continuable over all  $0 < t < +\infty$  ( $-\infty < t < +\infty$ ).*

*Proof.* Let there exist such  $T > 0$  that a solution  $x(t)$  cannot be continued for  $t \geq T$ . Construct a sphere with center at the origin of coordinates which contains the region  $\mathcal{G}$ . Let  $\mathcal{S}$  be a sphere with center at the origin, whose radius is twice the radius  $R$  of the sphere previously constructed. Denote  $c = \sup |z|$ , where  $z \in f(x, t)$ ,  $0 < t < 2T$ ,  $|x| \leq 2R$ , and, using Theorem 1.1, construct a solution  $y(t)$  satisfying the initial condition  $y(T - \varepsilon) = x(t - \varepsilon)$  with  $\varepsilon$  being some fixed number from the interval  $0 < \varepsilon < 2R/c$ . According to (1.43), the solution  $y(t)$  is defined at least for  $T - \varepsilon \leq t \leq T - \varepsilon + 2R/c$ , which contradicts the supposition that  $x(t)$  cannot be continued for  $t > T$ . For the case  $-\infty < t < +\infty$  the theorem is proved similarly. ■

The next statement extends the well-known Wintner theorem on continuation of solutions of differential equations with continuous single-valued right-hand sides [Wintner (1945)] to systems with multivalued right-hand sides.

**Theorem 1.3** *Let the right-hand side of system (1.48) be defined for all  $0 < t < +\infty$  ( $-\infty < t < +\infty$ ),  $x \in \mathbb{R}^N$  and let the estimate*

$$|\xi| \leq L(|x|), \forall \xi \in f(x, t), x \in \mathbb{R}^N, 0 < t < +\infty \quad (-\infty < t < +\infty) \quad (1.49)$$

*hold, where a function  $L(r)$  is positive, continuous, and has the property*

$$\int_0^\infty \frac{dr}{L(r)} = \infty. \quad (1.50)$$

*Then all the solutions of system (1.48) can be continued to  $0 < t < +\infty$  ( $-\infty < t < +\infty$ ).*

*Proof* of this theorem will be presented for the case  $0 < t < +\infty$  (the case  $-\infty < t < +\infty$ ) is considered similarly). Let a solution  $x(t)$  exist for  $t_0 \leq t < T$  and suppose that it cannot be continued for  $t > T$ .

Then, by Theorem 1.2, the value of  $|x(t)|$  is unbounded when  $t \rightarrow T$ . By the definition of a solution of (1.48), it follows that the equality  $dx(t)/dt = \xi(t)$ ,  $\xi(t) \in f[x(t), t]$ , holds almost everywhere on the interval  $t_0 < t < T$ . Premultiplying this inequality by  $2x^*(t)$  and taking advantage of estimate (1.49), we arrive at the inequality

$$\frac{d|x(t)|^2}{dt} \leq 2|x(t)| L(|x(t)|)$$

which implies

$$t - t_0 \geq \int_{|x(t_0)|}^{|x(T)|} \frac{dr}{L(r)}.$$

In view of (1.50), when  $t$  is sufficiently close to  $T$ , this inequality is contradictory. ■

In what follows, a crucial role will be played by the following theorem on continuous dependence of solutions of (1.48) on initial data.

**Theorem 1.4** *Let  $a_n$  be a sequence of points converging to  $a$  as  $n \rightarrow \infty$ , and  $x(t, a_n)$  be a set of the solutions of the equation (1.48) which satisfy the condition*

$$x(t_0, a_n) = a_n. \tag{1.51}$$

*Suppose that all the solutions  $x(t, a_n)$  exist for  $T_- \leq t \leq T_+$  and lie in some bounded region in  $\mathbb{R}^N$ . Then for any segment  $[\tau_-, \tau_+]$ , which is contained in  $[T_-, T_+]$ , there exists a subsequence  $\{a_{n_k}\}$  such that the functions  $x(t, a_{n_k})$  uniformly converge to a solution  $x(t, a)$  of the equation (1.48) with the initial condition*

$$x(t_0, a) = a, \tag{1.52}$$

*as  $n_k \rightarrow \infty$  and  $t \in [\tau_-, \tau_+]$ .*

The proof of this theorem is straightforward. By the Arzelá theorem, there exists a subsequence  $x(t, a_{n_k})$  converging as  $n_k \rightarrow \infty$  to some continuous vector-valued function, which will be denoted by  $x(t, a)$ . Condition (1.52) follows from (1.51). The limit function  $x(t, a)$  meets the Lipschitz condition, and therefore it is absolutely continuous. It remains to verify

that inclusions (1.41) are valid. By Lemma 1.1,

$$\frac{1}{h} [x(t + h, a_{n_k}) - x(t, a_{n_k})] \in \text{conv} \bigcup_{\lambda \in (t, t+h) \text{ a.e.}} \frac{dx(\lambda, a_{n_k})}{d\lambda}.$$

By Definition 1.3, the set in the right-hand side of this relationship is contained in the set

$$\text{conv} \bigcup_{\lambda \in (t, t+h)} f[x(\lambda, a_{n_k}), \lambda].$$

The semidistance of the last set from the set

$$\text{conv} \bigcup_{\lambda \in (t, t+h)} f[x(\lambda, a), \lambda]$$

vanishes as  $n_k \rightarrow \infty$ , since the function  $f$  is semicontinuous. In turn, the semidistance of this convex hull from the set  $f[x(t, a), a]$  vanishes as  $h \rightarrow 0$  because of the semicontinuity of the function  $f$  and convexity of the set  $f[x(t, a), t]$ . ■

For applications it is a very important case when equation (1.48) is autonomous, i.e., when it takes the form

$$\frac{dx}{dt} = f(x). \tag{1.53}$$

We suppose that  $f(x)$  is an  $N$ -dimensional multivalued vector function, which is semicontinuous at any point  $x_0 \in \mathbb{R}^N$ . In addition, at any point  $x_0 \in \mathbb{R}^N$  the set  $f(x_0)$  is closed, convex, and bounded.

Denote by  $x(t, b)$  a solution of equation (1.53) satisfying the condition  $x(0, b) = b$ . When  $t$  runs through all the interval of existence of the solution, the set of points  $x(t, b)$  forms a curve in  $\mathbb{R}^N$  which is called a *trajectory*. Observe, that when the uniqueness of solutions is lacking, there may be many trajectories going through the same point. A point  $a \in \mathbb{R}^N$  will be called  $\omega$ -limiting for a given trajectory  $x(t, b)$  defined for  $t \in [0, +\infty)$ , if there exists a sequence  $\{t_n\}$  which tends to infinity as  $n \rightarrow \infty$  and such that  $x(t_n, b) \rightarrow a$ . The set of all  $\omega$ -limiting points will be called an  $\omega$ -limiting set.

In the next section, when establishing some Lyapunov-type lemmas, we will use the following property of the trajectories of (1.53).

**Theorem 1.5** *Let an  $\omega$ -limiting set  $\Omega$  of a trajectory  $x(t, b)$  of system (1.53) be bounded. Then for every  $\omega$ -limiting point  $a \in \Omega$  there exists*

at least one trajectory  $x(t, a)$  which passes through this point, defined for  $t \in (-\infty, +\infty)$ , and consists entirely of  $\omega$ -limiting points, i.e.,  $x(t, a) \in \Omega$  when  $t \in \mathbb{R}^1$ .

*Proof.* From the boundedness of the  $\omega$ -limiting set  $\Omega$  it follows that the trajectory  $x(t, b)$  is also bounded when  $t > 0$ . Indeed, if there exists a sequence  $t_n \rightarrow \infty$  such that  $|x(t_n, b)| \rightarrow \infty$ , then, because of the continuity of the vector function  $x(t, b)$  and boundedness of  $\Omega$ , there exists a sequence  $t'_n \rightarrow \infty$  such that the vectors  $x(t'_n, b)$  lie in a bounded region and are separated from  $\Omega$ . Then an accumulation point for  $x(t'_n, b)$  will be an  $\omega$ -limiting point which does not belong to  $\Omega$ . This contradicts the definition of an  $\omega$ -limiting set  $\Omega$ .

Let  $\mathcal{G}$  be a bounded region containing the trajectory  $x(t, b)$  for  $t > 0$ . Since  $a \in \Omega$ , there exist times  $t_n$  such that  $0 < t_1 < t_2 < \dots, t_n \rightarrow \infty$ , and  $a_n = x(t_n, b) \rightarrow a$  as  $n \rightarrow \infty$ . Obviously, the vectors  $y(t, a_n) = x(t_n + t, b)$  lie in  $\mathcal{G}$  for  $t \in [-\tau, \tau]$  and for all  $n$ , provided that  $0 < \tau < t_1$ . By Theorem 1.4, there exists a subsequence  $\{a_{n_k}\}$  such that  $a_{n_k} \rightarrow a$  and the vector functions  $y(t, a_{n_k})$ , as  $t \in [-\tau, \tau]$  and  $n_k \rightarrow \infty$ , converge to some solution  $y_0(t, a)$  of equation (1.53) with the initial condition  $y_0(0, a) = a$ . Since  $x(t_{n_k} + t, b) \rightarrow y_0(t, a)$  as  $n_k \rightarrow \infty$ , it is easily seen that  $y_0(t, a) \in \Omega$  when  $t \in [-\tau, \tau]$ .

Repeating the above reasoning for  $\omega$ -limiting points  $y_0(\tau, a)$  and  $y_0(-\tau, a)$ , we can extract from  $n_k$  a subsequence  $n_{k_i}$  such that  $x(t_{n_{k_i}} + t, b)$  converges for  $t \in [\tau, 2\tau]$  to some solution  $y_1(t)$  satisfying the condition  $y_1(\tau) = y_0(\tau, a)$ , and for  $t \in [-2\tau, -\tau]$  it converges to a solution  $y_{-1}(t)$  which meets the condition  $y_{-1}(-\tau) = y_0(-\tau, a)$ .

Continue the vector function  $y_0(t, a)$  over  $[-2\tau, -\tau]$  and  $[\tau, 2\tau]$  with the formulas

$$y_0(t, a) = \begin{cases} y_{-1}(t), & -2\tau \leq t \leq -\tau, \\ y_1(t), & \tau \leq t \leq 2\tau. \end{cases}$$

Then we get a trajectory  $y_0(t, a)$  which passes through the point  $a$  at  $t = 0$  and consists of  $\omega$ -limiting points of the trajectory  $x(t, b)$  when  $-2\tau \leq t \leq 2\tau$ .

We can continue this process to obtain a trajectory which belongs to  $\Omega$  for all  $t$ . ■

Now let us discuss a situation when a function  $f(x)$  in (1.53) is a super-

position of multivalued functions, i.e., when we have an equation

$$\frac{dx}{dt} = p(z), \quad z = g(x)$$

where  $g(x)$  is a  $m$ -dimensional multivalued function which is semicontinuous at any point  $x_0 \in \mathbb{R}^N$ . In addition, the set  $g(x_0)$  is convex, closed, and bounded in  $\mathbb{R}^m$ . Let the multivalued vector function  $p(z)$  be also semicontinuous at any point  $z_0 \in g(x_0)$  and the set  $p(z_0)$  be convex, closed, and bounded in  $\mathbb{R}^N$ . Will the function  $f(x) = p(g(x))$  be semicontinuous at a point  $x_0$ , and will the set  $p(g(x_0))$  be convex, closed, and bounded? The closedness of this set follows from the semicontinuity of the function  $p(z)$ . Since

$$f(x_0) = \bigcup_{z \in g(x_0)} p(z),$$

the set  $f(x_0)$ , being a union of convex and bounded sets, does not need to be convex or bounded itself. However, if it is required that the set  $p(g(x))$  be convex and bounded for all  $x \in \mathbb{R}^N$ , then it can be easily checked that the semicontinuity of the function  $f(x) = p(g(x))$  follows from the semicontinuity of the functions  $p(z)$  and  $g(x)$ . Indeed, denote by  $S_\varepsilon(\omega)$  an  $\varepsilon$ -neighborhood of a set  $\omega$ , and let

$$\omega(x_0) = g(x_0), \quad \Omega(x_0) = \bigcup_{z \in \omega(x_0)} p(z).$$

Since  $p(z)$  is semicontinuous, for any  $\varepsilon > 0$  and any  $z \in \omega(x_0)$  there exists  $\delta = \delta(\varepsilon, z)$  such that  $u \in S_\delta(z)$  implies  $p(u) \subset S_\varepsilon(\Omega(x_0))$ . From the covering

$\bigcup_{z \in \omega(x_0)} S_\delta(z)$  we can choose a finite covering  $\bigcup_k S_{\delta_k}(z_k)$ . Let  $\mu > 0$  be so

small that  $S_\mu(x_0) \subset \bigcup_k S_{\delta_k}(z_k)$ . Using this  $\mu$ , in view of semicontinuity

of  $g(x)$ , choose  $\nu = \nu(\mu, x_0)$  such that  $x \in S_\nu(x_0)$  implies the inclusion

$g(x) \subset S_\mu(\omega(x_0))$ . Therefore  $g(x) \subset \bigcup_k S_{\delta_k}(z_k)$ , whence it follows that

$p(g(x_0)) \subset S_\varepsilon(\Omega(x_0))$ , which is the required inclusion. Thus we arrived at the following result.

**Theorem 1.6** *Suppose in system (1.53)  $f(x) = p(g(x))$ , the functions  $p(\cdot)$  and  $g(\cdot)$  are semicontinuous, and at any point of their domains of definition their values are convex, closed, bounded sets in  $\mathbb{R}^N$  and  $\mathbb{R}^m$  respectively. Let a value of  $f(x)$  be a convex, bounded set. Then the function*

$f(x)$  is semicontinuous and a local theorem for the existence of solutions is valid for system (1.53).

Corollary. Let in system (1.53)  $f(x) = p(g(x))$ ,

$$p(z) = \sum_{j=1}^r a_j \operatorname{sgn}(c_j, z), \quad g(x) = \sum_{i=1}^l b_i \operatorname{sgn}(d_i, x)$$

where  $a_j, d_j \in \mathbb{R}^N, c_i, b_i \in \mathbb{R}^m$ ,

$$\operatorname{sgn} \sigma = \begin{cases} 1, & \sigma > 0, \\ -1, & \sigma < 0, \\ [-1, +1], & \sigma = 0. \end{cases}$$

(If some of the vectors  $c_j$  or  $d_i$  are equal, we suppose that the system contains several equal units.) Then a solution exists for all  $t > 0$ .

Indeed, the semicontinuity of  $p(\cdot)$  and  $g(\cdot)$  is obvious, as well as closedness and boundedness of the sets  $p(z), g(x), f(x)$  and convexity of the sets  $p(z), g(x)$ . Let us verify that the set  $f(x)$  is convex. Let  $x$  be fixed and

$$(d_i, x) \begin{cases} > 0, & i \in I_+, \\ < 0, & i \in I_-, \\ = 0, & i \in I_0. \end{cases}$$

Then

$$z = \sum_{i \in I_+} b_i - \sum_{i \in I_-} b_i + \sum_{i \in I_0} b_i \xi_i, \quad -1 \leq \xi_i \leq +1.$$

Introduce a notation  $\alpha_{ij} = (c_j, b_i), \eta_{ij} = \alpha_{ij} \xi_i, \alpha_j = \sum_{i \in I_+} \alpha_{ij} - \sum_{i \in I_-} \alpha_{ij}, \alpha_{ij}^- = \alpha_j - |\alpha_{ij}|, \alpha_{ij}^+ = \alpha_j + |\alpha_{ij}|$ . Then  $(c_j, z) = [\mu_j^-, \mu_j^+]$  where  $\mu_j^- = \sum_{i \in I_0} \alpha_{ij}^-, \mu_j^+ = \sum_{i \in I_0} \alpha_{ij}^+$ . Hence  $p(z) = \sum_{j=1}^r a_j \operatorname{sgn} \mu_j$  where  $\mu_j^- \leq \mu_j \leq \mu_j^+$ . Let  $j \in J_+$  if  $\mu_j^- > 0, j \in J_-$  if  $\mu_j^+ < 0$ , and  $j \in J_0$  if  $0 \in [\mu_j^-, \mu_j^+]$ . Then

$$p(z) = \sum_{j \in J_+} a_j - \sum_{j \in J_-} a_j + \sum_{j \in J_0} a_j \lambda_j, \quad -1 \leq \lambda_j \leq +1.$$

Obviously, the set  $p(z)$  is convex. Therefore, by Theorem 1.6, a solution exists on some time interval. The continuity of the solution for all  $t > 0$  follows from Theorem 1.3.

### 1.2.2 “Extended” nonlinearities

Turn now to the system

$$\frac{dx}{dt} = Px + q\xi, \quad \sigma = r^*x, \quad (1.54)$$

$$\xi = \varphi(\sigma, t) \quad (1.55)$$

examined in Section 2.1. Here  $P$ ,  $q$ , and  $r$  are matrices of dimensions  $n \times n$ ,  $n \times m$ , and  $n \times l$  respectively. Equations (1.54) describe a linear part of the system, while equations (1.55) describe its nonlinear part with  $\xi$  being an  $m$ -dimensional vector of outputs of nonlinear blocks. Assume that a multivalued  $m$ -dimensional vector function  $\varphi(\sigma, t)$  is such that the above local theorem for the existence of solutions is valid for system (1.54), (1.55) (we mean Theorem 1.1).

By Definition 1.3, an absolutely continuous  $n$ -dimensional vector function  $x(t)$  is called a solution of system (1.54), (1.55) if an inclusion

$$\frac{dx}{dt} \in Px(t) + q\varphi[\sigma(t), t] \quad (1.56)$$

is valid for almost all  $t$ .

As was noted in Subsection 2.2.1, the information on a solution given by formula (1.56) is incomplete for two reasons. Firstly, for practical applications it is more convenient to use not an inclusion but an equality. Secondly, it is often necessary to know the signals at the outputs of the nonlinear blocks. In other words, we are interested only in those solutions  $x(t)$  to inclusion (1.56) for which there exists an  $m$ -dimensional vector function  $\xi(t)$  satisfying the relationships

$$\frac{dx}{dt} = Px(t) + q\xi(t) \quad \text{for almost all } t, \quad (1.57)$$

$$\xi(t) \in \varphi[r^*x(t), t]. \quad (1.58)$$

From these formulas it is seen that the vector  $\xi(t)$  can be treated as though it were an extension of a multivalued function  $\varphi[r^*x(t), t]$  on a solution  $x(t)$ , but, unlike  $\varphi[\sigma(t), t]$ ,  $\xi(t)$  is a single-valued function of  $t$ .

**Definition 1.4** A vector function  $\xi(t)$  is called *an extended nonlinearity* if (1.57), (1.58) are valid.

Clearly, to any solution  $x(t)$  corresponds its own extended nonlinearity  $\xi(t)$ . Sometimes, for the convenience of reasoning, which uses explicitly an

extended nonlinearity  $\xi(t)$  corresponding to a solution  $x(t)$ , a solution will be denoted by a pair of vector functions  $x(t), \xi(t)$ , implying that they are related by (1.57) and (1.58).

If the matrix  $q^*q$  is nonsingular, the extended nonlinearity  $\xi(t)$  is determined uniquely from (1.57) for almost all  $t$ :

$$\xi(t) = (q^*q)^{-1} \left[ \frac{dx}{dt} - Px(t) \right].$$

Obviously, such  $\xi(t)$  is a measurable function of  $t$ .

In other cases, for a solution  $x(t)$  there may be many functions  $\xi(t)$  which satisfy relationships (1.57), (1.58). There is nothing surprising in it, because even a solution  $x(t)$  of the inclusion (1.56) with the fixed initial condition  $x(t_0) = x_0$  is not determined uniquely. It is known that such a multiplicity may take place even for a classical solution of system (1.54), (1.55), in the case when  $\varphi(\sigma, t)$  is a single-valued continuous vector function. Most of the results following in this book will be established without any assumptions about the uniqueness of solutions. So, rather tight constraints, which guarantee the uniqueness of solutions, will not be imposed on the system at hand.

We pose a question: is there a measurable extended nonlinearity  $\xi(t)$  in the case when  $\det q^*q \neq 0$ ? The following theorem gives an affirmative answer under very general assumptions.

Let  $F(t, x, \xi)$  be a vector function with values in  $\mathbb{R}^n$  defined for  $t \in [a, b]$ ,  $x \in \mathbb{R}^N$ ,  $\xi \in \mathbb{R}^m$ . Moreover, let  $\sigma(t, x)$  be a continuous vector function with values in  $\mathbb{R}^l$  defined in  $[a, b] \times \mathbb{R}^N$  and  $\varphi(\sigma, t)$  be a multivalued function defined in  $[a, b] \times \mathbb{R}^l$  whose values are subsets of  $\mathbb{R}^n$ .

**Theorem 1.7** [V.M. Makarov] *Assume that the function  $F$  is continuous, the function  $\varphi$  is semicontinuous and its values are closed bounded subsets of  $\mathbb{R}^n$ . Let  $x_0(t)$  be a vector function, absolutely continuous on  $[a, b]$ , which satisfies a relationship*

$$\frac{dx_0(t)}{dt} \in \{ F[t, x_0(t), \xi] \mid \xi \in \mathcal{A}(t) \} \tag{1.59}$$

for almost all  $t \in [a, b]$  with

$$\mathcal{A}(t) = \varphi[t, \sigma(t, x_0(t))].$$

Then there exists a Lebesgue measurable in  $[a, b]$ , vector-valued function

$\xi_0(t)$  such that the relationships

$$\frac{dx_0(t)}{dt} = F[t, x_0(t), \xi_0(t)], \quad \xi_0(t) \in \mathcal{A}(t)$$

are valid for almost all  $t \in [a, b]$ .

The proof of this theorem is based on the following lemma whose proof is given below.

**Lemma 1.2** [V.M. Makarov] *Let  $\mathcal{H}, \mathcal{Q}$  be metric compact sets,  $\Psi$  be a continuous mapping of  $\mathcal{H}$  onto  $\mathcal{Q}$ . Then there exists a Borel measurable<sup>10</sup> cut<sup>11</sup>  $\theta$  of the mapping  $\Psi$ .*

*Proof* of Theorem 1.7. Let  $\mathcal{H}$  be “a graph” of the multivalued vector function  $\mathcal{A}(t)$ , i.e., a set

$$\{ (t, \xi) \in \mathbb{R}^{m+1} \mid t \in [a, b], \xi \in \mathcal{A}(t) \}.$$

Since the function  $\mathcal{A}(t)$  is semicontinuous and its values are compact sets, it is easy to verify that the set  $\mathcal{H}$  is compact. Introduce a mapping  $\Psi : \mathcal{H} \rightarrow \mathbb{R}^{N+1}$  with the formula

$$\Psi(t, \xi) = [t, F(t, x_0(t), \xi)] \quad ((t, \xi) \in \mathcal{H})$$

and set  $\mathcal{Q} = \Psi(\mathcal{H})$ . Obviously, the mapping  $\Psi$  is continuous. Hence, by Lemma 1.2, there exists a Borel measurable cut  $\theta$  of the mapping  $\Psi$ . It is clear that  $\theta(t, \eta) = (t, \theta_1(t, \eta))$  where a vector function  $\theta_1$  is Borel measurable together with  $\theta$ . Therefore, the superposition  $\theta_1[t, \eta(t)]$  is Lebesgue measurable on  $[a, b]$ , provided that  $\eta(t)$  is a vector function, Lebesgue measurable on  $[a, b]$ , and  $(t, \eta(t)) \in \mathcal{Q}$  for almost all  $t \in [a, b]$ . Comparing the equalities

$$\Psi[\theta(t, \eta)] = \Psi[t, \theta_1(t, \eta)] = (t, \eta)$$

and

$$\Psi[\theta(t, \eta)] = \Psi[t, \theta_1(t, \eta)] = (t, F[(t, x_0(t), \theta_1(t, \eta))]),$$

<sup>10</sup>A mapping  $f : X \rightarrow Y$  of a metric space  $X$  into a metric space  $Y$  is called *Borel measurable* if a preimage of any Borel subset of  $Y$  is a Borel subset of  $X$ . An equivalent definition is obtained if we require that preimages of all the sets open in  $Y$  be Borel subsets of  $X$ . Any continuous mapping is Borel measurable. A superposition of a Borel measurable mapping  $f$  and a Lebesgue measurable mapping  $g$  is a Lebesgue measurable mapping [Halmos (1974)].

<sup>11</sup>By the definition, a mapping  $\theta : \mathcal{Q} \rightarrow \mathcal{H}$  is a cut of a mapping  $\Psi : \mathcal{H} \rightarrow \mathcal{Q}$  if  $\Psi[\theta z] = z$  for any  $z \in \mathcal{Q}$ . The terms “lifting” or “selector” are also used.

we see that

$$F[t, x_0(t), \theta_1(t, \eta)] = \eta \quad \text{if } (t, \eta) \in \mathcal{Q}. \tag{1.60}$$

In view of (1.59),  $(t, dx_0(t)/dt) \in \mathcal{Q}$  for almost all  $t \in [a, b]$ . Using (1.60), we obtain

$$F \left[ t, x_0(t), \theta_1 \left( t, \frac{dx_0(t)}{dt} \right) \right] = \frac{dx_0(t)}{dt}.$$

In addition,  $\theta_1(t, dx_0(t)/dt) \in \mathcal{H}$ , i.e.,  $\theta_1(t, dx_0(t)/dt) \in \mathcal{A}(t)$  for almost all  $t \in [a, b]$ . It remains to set  $\xi_0(t) = \theta_1(t, dx_0(t)/dt)$ . Theorem 1.7 is proved. ■

*Proof* of Lemma 1.2. Let us construct a required mapping  $\theta : \mathcal{Q} \rightarrow \mathcal{H}$  as a pointwise limit of Borel measurable step-mappings  $\theta_n$ . The mappings  $\theta_n$  will be constructed with the help of sufficiently fine partitions of the sets  $\mathcal{Q}$  and  $\mathcal{H}$ .

Fix an arbitrary sequence of numbers  $\varepsilon_n, \varepsilon_n \rightarrow 0$ . Since the mapping  $\Psi$  is uniformly continuous, there exist numbers  $\delta_n > 0$  such that  $\text{diam } \Psi < \varepsilon_n$ , provided that  $e \subset \mathcal{H}$ ,  $\text{diam } e < \delta_n$ . Obviously, without loss of generality, we can suppose that  $\delta_n \rightarrow 0$  as  $n \rightarrow \infty$ .

Consider closed sets  $\mathcal{F}_j \subset \mathcal{H}$  with the properties

$$\bigcup_{j=1}^{\infty} \mathcal{F}_j = \mathcal{H}, \quad \text{diam } \mathcal{F}_j < \delta_1 \quad (j = 1, 2, \dots).$$

Set  $\mathcal{E}_j = \Psi(\mathcal{F}_j)$ . Then

$$\bigcup_{j=1}^{\infty} \mathcal{E}_j = \mathcal{Q}, \quad \text{diam } \mathcal{E}_j < \varepsilon_1 \quad (j = 1, 2, \dots).$$

Consider sets

$$e_1 = \mathcal{E}_1, \quad e_j = \mathcal{E}_j \setminus \bigcup_{i=1}^{j-1} \mathcal{E}_i \quad (j = 2, 3, \dots).$$

Clearly, the sets  $e_j$  are pairwise disjoint,  $\bigcup_{j=1}^{\infty} e_j = \mathcal{Q}$ ,  $\text{diam } e_j \leq \text{diam } \mathcal{E}_j < \varepsilon_1$ . Let us fix a point  $x_j$  in every set  $\mathcal{F}_j$ , and define a mapping  $\theta_1$  by setting  $\theta_1(z) = x_j$  for  $z \in e_j$  ( $j = 1, 2, \dots$ ). Since  $e_j$  are Borel sets, the mapping  $\theta_1$  is Borel measurable. In addition, if  $z \in e_j$  then  $\Psi(\theta_1(z)) \in \mathcal{E}_j$  and hence<sup>12</sup>

<sup>12</sup>We denote by  $\rho_{\mathcal{H}}$  and  $\rho_{\mathcal{Q}}$  the metrics in  $\mathcal{H}$  and  $\mathcal{Q}$  respectively.

$$\rho_{\mathcal{Q}}[z, \Psi(\theta_1(z))] < \varepsilon_1.$$

Turn now to the construction of a set  $\theta_2$ . For every set  $\mathcal{F}_j$  consider closed sets  $\mathcal{F}_{jk}$  with the properties

$$\bigcup_{k=1}^{\infty} \mathcal{F}_{jk} = \mathcal{F}_j, \quad \text{diam } \mathcal{F}_{jk} < \delta_k \quad (j, k = 1, 2, \dots).$$

Set  $\mathcal{E}_{jk} = \Psi(\mathcal{F}_{jk})$ . Then

$$\bigcup_{k=1}^{\infty} \mathcal{E}_{jk} = \mathcal{E}_j, \quad \text{diam } \mathcal{E}_{jk} < \varepsilon_2 \quad (j, k = 1, 2, \dots).$$

Consider sets

$$e_{j1} = \mathcal{E}_{j1} \cap e_j, \quad e_{jk} = \left( \mathcal{E}_{jk} \setminus \bigcup_{i=1}^{k-1} \mathcal{E}_{ji} \right) \cap e_j$$

$$(j = 1, 2, \dots; \quad k = 2, 3, \dots).$$

Clearly, the sets  $e_{jk}$  are pairwise disjoint,  $\bigcup_{k=1}^{\infty} e_{jk} = e_j$ ,  $\text{diam } e_{jk} \leq \text{diam } \mathcal{E}_{jk} < \varepsilon_2$ . Let us fix a point  $x_{jk}$  in every set  $\mathcal{F}_{jk}$  and define a mapping  $\theta_2$  by setting

$$\theta_2(z) = x_{jk} \quad \text{if } z \in e_{jk} \quad (j, k = 1, 2, \dots).$$

Since  $e_{jk}$  are Borel sets, the mapping  $\theta_2$  is Borel measurable.

If  $z \in e_{jk}$  then  $\Psi(\theta_2(z)) \in \mathcal{E}_{jk}$  and hence  $\rho_{\mathcal{Q}}[z, \Psi(\theta_2(z))] < \varepsilon_2$ . In addition,  $\rho_{\mathcal{H}}[\theta_1(z), \theta_2(z)] < \delta_1$ , since the points  $\theta_1(z)$  and  $\theta_2(z)$  are contained in one of the sets  $\mathcal{F}_j$ .

When constructing a mapping  $\theta_3$ , represent each set  $\mathcal{F}_{jk}$  in the form of  $\mathcal{F}_{jk} = \bigcup_{i=1}^{\infty} \mathcal{F}_{jki}$  where the sets  $\mathcal{F}_{jki}$  are closed and  $\text{diam } \mathcal{F}_{jki} < \delta_3$ . The further reasoning is similar to that given above, when we constructed the mapping  $\theta_2$ . If we continue this process, we get Borel measurable mappings  $\theta_n$  with the properties

$$\rho_{\mathcal{Q}}[z, \Psi(\theta_n(z))] \leq \varepsilon_n \quad (n = 1, 2, \dots), \tag{1.61}$$

$$\rho_{\mathcal{H}}[\theta_n(z), \theta_m(z)] \leq \delta_p, \quad p = \min(m, n). \tag{1.62}$$

In view of (1.62), for any  $z \in \mathcal{Q}$  there exists a limit  $\lim_{n \rightarrow \infty} \theta_n(z)$ . By setting  $\theta(z) = \lim_{n \rightarrow \infty} \theta_n(z)$  ( $z \in \mathcal{Q}$ ) and passing to the limit in (1.61), we get

$$\rho_{\mathcal{Q}}[z, \Psi(\theta(z))] = 0, \quad \text{i.e.,} \quad \Psi(\theta(z)) = z.$$

Therefore,  $\theta$  is a cut of the mapping  $\Psi$ . Since it is a limit of Borel measurable mappings  $\theta_n$ , it is also Borel measurable. ■

Let us prove the following statement which will be used repeatedly.

**Lemma 1.3** [Barabanov (1982)] *Let  $x(t)$  and  $\xi(t)$  satisfy the system*

$$\frac{dx}{dt} = Px + q\xi, \tag{1.63}$$

$$\sigma_0 = h^*x + \varkappa\xi \tag{1.64}$$

where  $P, q, h, \varkappa, \sigma_0$  are constant matrices of sizes  $n \times n, n \times m, n \times m, m \times m$ , and  $m \times 1$  respectively. Suppose that

$$\det[\varkappa - h^*(P - \lambda I)^{-1}q] \neq 0, \quad \forall \lambda \in \mathbb{C}, \tag{1.65}$$

and either  $\sigma_0 = 0$ , or  $\varkappa = 0$ .

Then  $x(t)$  is a solution of some linear system with constant coefficients, and the roots of its characteristic equation coincide with the roots of the polynomial  $\det(P - \lambda I) \det[\varkappa - h^*(P - \lambda I)^{-1}q]$ .

*Proof.* At first, consider the case when  $\sigma_0 = 0$ . Let there exist a nonsingular  $m \times m$  matrix  $S$  such that the first  $m_1$  components of the vector  $S^{-1}\xi(t)$ ,  $m_1 < m$ , can be expressed linearly in terms of the components of the vector  $x(t)$ , while the remaining  $m - m_1$  components, which form a vector  $\eta(t)$ , cannot. Then

$$\xi = S \begin{vmatrix} Rx \\ \eta \end{vmatrix} \tag{1.66}$$

where  $R$  is a constant  $m_1 \times n$  matrix. Then (1.64) takes the form

$$\mathbb{O}_{m,1} = \left[ r^* + \varkappa S \begin{vmatrix} R \\ \mathbb{O}_{m-m_1,n} \end{vmatrix} \right] x + \varkappa S \begin{vmatrix} \mathbb{O}_{m_1,m-m_1} \\ I_{m-m_1} \end{vmatrix} \eta.$$

Henceforth  $\mathbb{O}_{kl}$  denotes a null  $k \times l$  matrix,  $I_k$  is the identity  $k \times k$  matrix. Since  $\eta$  cannot be expressed linearly in terms of  $x$ , we get the equality

$$\varkappa S \begin{vmatrix} \mathbb{O}_{m_1,m-m_1} \\ I_{m-m_1} \end{vmatrix} = \mathbb{O}_{m,m-m_1}. \tag{1.67}$$

Therefore, (1.64) takes the form

$$\mathbb{O}_{m,1} = \left[ h^* + \varkappa S \left\| \begin{matrix} R \\ \mathbb{O}_{m-m_1,n} \end{matrix} \right\| \right] x.$$

By differentiating this equality along the solutions of (1.63), we obtain a relationship

$$\begin{aligned} \mathbb{O}_{m,1} &= \left[ h^* + \varkappa S \left\| \begin{matrix} R \\ \mathbb{O}_{m-m_1,n} \end{matrix} \right\| \right] \\ &\times \left[ Px + qS \left\| \begin{matrix} R \\ \mathbb{O}_{m-m_1,n} \end{matrix} \right\| x + \left\| \begin{matrix} \mathbb{O}_{m-m_1,m_1} \\ I_{m-m_1} \end{matrix} \right\| \eta \right]. \end{aligned} \tag{1.68}$$

Then, from the above property of the vector  $\eta$ , we get an equality

$$\left[ h^* + \varkappa S \left\| \begin{matrix} R \\ \mathbb{O}_{m-m_1,n} \end{matrix} \right\| \right] qS \left\| \begin{matrix} \mathbb{O}_{m_1,m-m_1} \\ I_{m-m_1} \end{matrix} \right\| = \mathbb{O}_{m,m-m_1},$$

whence (1.68) takes the form

$$\begin{aligned} \mathbb{O}_{m,m-m_1} &= \left[ h^* + \varkappa S \left\| \begin{matrix} R \\ \mathbb{O}_{m-m_1,n} \end{matrix} \right\| \right] \\ &\times \left[ P + qS \left\| \begin{matrix} R \\ \mathbb{O}_{m-m_1,n} \end{matrix} \right\| \right] x. \end{aligned} \tag{1.69}$$

By differentiating this inequality and continuing the above reasoning, we get a chain of relationships

$$\begin{aligned} &\left[ h^* + \varkappa S \left\| \begin{matrix} R \\ \mathbb{O}_{m-m_1,n} \end{matrix} \right\| \right] \\ &\times \left[ P + qS \left\| \begin{matrix} R \\ \mathbb{O}_{m-m_1,n} \end{matrix} \right\| \right]^j qS \left\| \begin{matrix} \mathbb{O}_{m_1,m-m_1} \\ I_{m-m_1} \end{matrix} \right\| = \mathbb{O}_{m,m-m_1}, \end{aligned}$$

$j = 0, 1, \dots, n$ . From these equalities and (1.67) we obtain an identity

$$\begin{aligned} &\left[ h^* + \varkappa S \left\| \begin{matrix} R \\ \mathbb{O}_{m-m_1,n} \end{matrix} \right\| \right] \left[ \lambda I_n - P - qS \left\| \begin{matrix} R \\ \mathbb{O}_{m-m_1,n} \end{matrix} \right\| \right]^{-1} \\ &\times qS \left\| \begin{matrix} \mathbb{O}_{m_1,m-m_1} \\ I_{m-m_1} \end{matrix} \right\| = \mathbb{O}_{m,m-m_1}, \end{aligned} \tag{1.70}$$

which can be represented in the form

$$[M(\lambda) - \varkappa S] \left\| \begin{matrix} \mathbb{O}_{m_1, m-m_1} \\ I_{m-m_1} \end{matrix} \right\| = \mathbb{O}_{m, m-m_1} \tag{1.71}$$

with

$$M(\lambda) = [h^*(\lambda I_n - P)^{-1}q + \varkappa] S \\ \times \left[ I_m - \left\| \begin{matrix} R \\ \mathbb{O}_{m-m_1, n} \end{matrix} \right\| (\lambda I_n - P)^{-1}qS \right]^{-1}.$$

From (1.71) and (1.67) it follows that  $\det M(\lambda) \equiv 0$ , which contradicts (1.65). Hence, it is proved that

$$\xi(t) = Qx(t) \tag{1.72}$$

where  $Q$  is a constant  $m \times n$  matrix, and therefore the vector  $x(t)$  satisfies a linear system. By (1.63), (1.64) and Schur’s Lemma, the characteristic polynomial of this system takes the form

$$\det \left\| \begin{matrix} P - \lambda I_n & q \\ h^* & \varkappa \end{matrix} \right\| = \det(P - \lambda I_n) \det[\varkappa - h^*(P - \lambda I_n)^{-1}q]. \tag{1.73}$$

Lemma 1.3 is proved for the case  $\sigma_0 = 0$ .

Assume now that  $\sigma_0 \neq 0$ ,  $\varkappa = 0$ . Using the same line of reasoning as in the case  $\sigma_0 = 0$ , introduce representation (1.66), and by differentiating the equality  $\sigma_0 = h^*x$  along the trajectories of (1.63), we arrive at (1.68) with  $\varkappa = 0$ . Whence we obtain a relationship

$$h^*qS \left\| \begin{matrix} \mathbb{O}_{m_1, m-m_1} \\ I_{m-m_1} \end{matrix} \right\| = \mathbb{O}_{m, m-m_1}.$$

Arguing as for the previous case, we ensure that representation (1.72) and formula (1.73) are valid. ■

### 1.2.3 Sliding modes

Consider again system (1.54), (1.55). We assume that  $l = m$  and the  $i$ th component of the vector  $\varphi$  depends only on the  $i$ th component of the vector  $\sigma$ :  $\varphi_i = \varphi_i(\sigma_i)$  ( $1 \leq i \leq m$ ).

**Definition 1.5** We say that a trajectory  $x(t)$  on an interval  $[t_1, t_2]$  is in a partial sliding mode in respect of the coordinate  $\sigma_i$  if  $\sigma_i(t) \equiv \sigma_i^k$  for  $t \in [t_1, t_2]$  with  $\sigma_i^k$  being a point where the function  $\varphi_i(\sigma_i)$  is multivalued.

If in an interval  $[t_1, t_2]$  a trajectory is in a sliding mode by all the coordinates  $\sigma_i$ ,  $1 \leq i \leq m$ , such a sliding mode will be called *complete*.

Evidently, in the case of a single nonlinearity ( $m = 1$ ) any sliding mode is complete, hence we will just use the term “sliding mode.”

Obviously, a complete sliding mode a solution  $(x(t), \xi(t))$  of system (1.54), (1.55) satisfies the system of linear equations

$$\frac{dx(t)}{dt} = Px(t) + q\xi(t), \quad r^*x(t) = \sigma^k \quad (1.74)$$

(here  $\sigma^k$  is a vector with components  $\sigma_1^k, \dots, \sigma_m^k$ ). From Lemma 1.3 we deduce the following result.

**Theorem 1.8** *The characteristic polynomial of the system of linear differential equations which describe a complete sliding mode of system (1.54), (1.55) is equal up to sign to the product of the characteristic polynomial of the linear part (1.54) and the determinant of the transfer matrix.*

*Corollary.* For system (1.54), (1.55) with a single scalar nonlinearity, the characteristic polynomial of a sliding mode is equal up to sign to the numerator of the transfer function. In this case, when the transfer function is degenerate it is supposed that its denominator is the characteristic polynomial of the linear part of the system, i.e., the numerator and the denominator of the rational fraction are not reduced.

Observe, that partial sliding modes are described with nonlinear differential equations, therefore there is no counterpart of Theorem 1.8 for them.

A case important for applications concerns the systems for which an extended nonlinearity can be written as a sum of an absolutely continuous function and a discontinuous saltus function. Before we start the examination of this case, we need to study in more detail some properties of sliding modes. These results will be required below, in the proofs of Theorems 1.9, 1.10, as well as in the subsequent chapters.

In what follows an important role is played by a specific case of the system (1.54), (1.55) when  $\xi = \varphi(\sigma)$  and each component of the vector  $\varphi$  is a multivalued function of one scalar variable, which is a component of the vector  $\sigma$ .

**Definition 1.6** A scalar multivalued function  $\zeta(\eta)$  of a scalar argument  $\eta$  will be called *piecewise single-valued* if it single-valued and continuous everywhere, with the exception of the at-most-countable set of points  $\eta_k$  which are not accumulated at a finite distance ( $\inf_{k \neq l} |\eta_k - \eta_l| > 0$ ). Moreover,  $\zeta(\eta_k) = [\alpha_k, \beta_k]$  and  $\zeta(\eta_k - 0), \zeta(\eta_k + 0) \in [\alpha_k, \beta_k]$ .

In other words, a piecewise single-valued function is obtained from a single-valued piecewise continuous function, if its value at a discontinuity point is identified with the segment which includes its left-hand and right-hand limits. (The last requirement is imposed to ensure semicontinuity of a multivalued function.)

A vector function  $\varphi(\sigma)$  will be called *piecewise single-valued* if each of its components is a scalar piecewise single-valued function of some component of the vector  $\sigma$ .

At first, consider system (1.54), (1.55) with one scalar piecewise single-valued nonlinearity  $\varphi(\sigma)$ , i.e., a system

$$\frac{dx}{dt} = Px + q\varphi(\sigma), \quad \sigma = r^*x \tag{1.75}$$

where  $P$  is a constant  $n \times n$  matrix,  $q$  and  $r$  are constant  $n$ -dimensional columns.

We will be interested in the following two problems:

- (i) What is the extended nonlinearity  $\xi(t)$  on a sliding mode?
- (ii) How can a manifold of a sliding mode be described visually?

Suppose that the function  $\varphi(\sigma)$  is multivalued at a point  $\sigma_0$ , and there exist limit values  $\varphi(\sigma_0 - 0)$ ,  $\varphi(\sigma_0 + 0)$ , and  $\varphi(\sigma_0) = \langle \varphi(\sigma_0 - 0), \varphi(\sigma_0 + 0) \rangle$ .

Suppose that a trajectory lies on the hyperplane  $r^*x = \sigma_0$  when  $t_1 < t < t_2$ , i.e., that the system is in a sliding mode. We will demonstrate that while the system is in a sliding mode, an extended function  $\xi(t)$  is uniquely determined as a linear combination of the components of  $x(t)$ . According to (1.75), for almost all  $t$  we have

$$\frac{dx}{dt} = Px(t) + q\xi(t), \quad \sigma(t) = r^*x(t). \tag{1.76}$$

Hence

$$\frac{d\sigma}{dt} = r^*Px(t) - \rho_1\xi(t) \tag{1.77}$$

with  $\rho_1 = -r^*q$ .

At first, let us examine the case  $\rho_1 \neq 0$ . Since  $\sigma(t) \equiv \sigma_0$  for  $t \in [t_1, t_2]$ ,  $d\sigma/dt = 0$  for the same  $t$ . But, in view of (1.77), it is possible only if

$$\xi(t) = r^*Px(t)/\rho_1.$$

Furthermore, let

$$\begin{aligned} \rho_\nu &= -r^*P^{\nu-1}q = 0, \quad 1 \leq \nu \leq k, \\ \rho_{k+1} &= -r^*P^kq \neq 0. \end{aligned} \tag{1.78}$$

Then from (1.76) we conclude that the functions  $\zeta_i(t) = r^*P^{i-1}x(t)$  satisfy the equations

$$\frac{d\zeta_1}{dt} = \zeta_2, \quad \dots, \quad \frac{d\zeta_k}{dt} = \zeta_{k+1}, \quad \frac{d\zeta_{k+1}}{dt} = r^*P^{k+1}x - \rho_{k+1}\xi.$$

Since  $\zeta_1(t) \equiv \sigma_0$  for  $t_1 < t < t_2$ ,  $\zeta_2(t) = \dots = \zeta_{k+1}(t) = 0$  for the same  $t$ , and therefore  $\xi(t) = r^*P^{k+1}x(t)/\rho_{k+1}$ . Thus a sliding mode of system (1.75) is described in  $\mathbb{R}^n$  by a linear system

$$\frac{dx}{dt} = Px + q \frac{r^*P^{k+1}x}{\rho_{k+1}} \tag{1.79}$$

where  $\rho_{k+1}$  is the first nonzero number in the sequence

$$\rho_{k+1} = -r^*P^kq \quad (k = 0, 1, 2, \dots).$$

What are the conditions under which a trajectory lies in a discontinuity hyperplane? In order to answer this question, we will study the direction field of system (1.75) in a neighborhood of the hyperplane  $\sigma = \sigma_0$ .

First we consider the case when  $\rho_1 \neq 0$  and an inequality

$$\rho_1[\varphi(\sigma_0 - 0) - \varphi(\sigma_0 + 0)] < 0 \tag{1.80}$$

is satisfied. Let a point  $x_0$  lie on the hyperplane  $\sigma = \sigma_0$  and  $r^*Px_0 > \rho_1\varphi(\sigma_0 + 0)$ . Then from (1.77) it follows that for any trajectory, lying in a sufficiently small neighborhood of the point  $x_0$ , the inequality  $d\sigma/dt > 0$  holds for almost all  $t$ . Hence such trajectories pass through the hyperplane  $\sigma = \sigma_0$ , going from the half-space  $\sigma < \sigma_0$  to the half-space  $\sigma > \sigma_0$  (see Fig. 1.1 (a)). Similarly, it can be shown that if  $r^*x_0 = \sigma_0$  and  $r^*Px_0 < \rho_1\varphi(\sigma_0 - 0)$ , then in a sufficiently small neighborhood of the point  $x_0$  trajectories pass through the hyperplane  $\sigma = \sigma_0$ , going from the half-space  $\sigma > \sigma_0$  to the half-space  $\sigma < \sigma_0$ .

A different picture is obtained when

$$r^*x_0 = \sigma_0, \quad \rho_1\varphi(\sigma_0 - 0) < r^*Px_0 < \rho_1\varphi(\sigma_0 + 0).$$

From equality (1.77) it follows that in a sufficiently small neighborhood of the point  $x_0$  when  $\sigma(t) \neq \sigma_0$  the inequality  $[\sigma(t) - \sigma_0] d\sigma(t)/dt < 0$  is valid, i.e., trajectories of the continuous systems, defined on either side of the hyperplane  $\sigma = \sigma_0$ , are directed towards this hyperplane and connect with each other on it (see Fig. 1.1 (b)). Therefore, a trajectory in a sufficiently small neighborhood of the point  $x_0$  does not leave the hyperplane  $\sigma = \sigma_0$  and hence, as was shown above, it satisfies the linear system of differential equations (1.79) for  $k = 0$ .

At the points of manifolds

$$\begin{aligned} \{r^*x = \sigma_0, \quad r^*Px = \rho_1\varphi(\sigma_0 + 0)\}, \\ \{r^*x = \sigma_0, \quad r^*Px = \rho_1\varphi(\sigma_0 - 0)\} \end{aligned} \quad (1.81)$$

trajectories of system (1.75) also satisfy system (1.79) for  $k = 0$ . Moreover, trajectories can leave the first manifold towards the half-space  $\sigma > \sigma_0$  and the second one towards the half-space  $\sigma < \sigma_0$ .

A manifold

$$\{r^*x = \sigma_0, \quad \rho_1\varphi(\sigma_0 - 0) \leq r^*Px \leq \rho_1\varphi(\sigma_0 + 0)\}, \quad (1.82)$$

where trajectories satisfy the linear system (1.79), is naturally called “a manifold of sliding modes.” Manifolds (1.81) will be called *boundaries of the manifold of sliding modes*.

Observe that sliding modes of relay systems were studied in [Neimark (1957)] and other papers. Some authors applied the term “sliding mode” only to the situation when a sliding manifold had a dimension one less than the dimension of the state space.

If a segment  $\varphi(\sigma_0)$  does not coincide with  $\langle \varphi(\sigma_0 - 0), \varphi(\sigma_0 + 0) \rangle$  then, taking into account semicontinuity of  $\varphi(\sigma)$ , we get

$$\langle \varphi(\sigma_0 - 0), \varphi(\sigma_0 + 0) \rangle \subset \varphi(\sigma_0).$$

Then it is easily seen that the trajectories pass through the manifold

$$\{r^*x = \sigma_0, \quad r^*Px/\rho_1 \in \varphi(\sigma_0) \setminus \langle \varphi(\sigma_0 - 0), \varphi(\sigma_0 + 0) \rangle\},$$

but, at the same time, there is a sliding mode on this manifold. Therefore, at the points of this manifold the solutions are not unique.

Similar arguments also apply in the case (1.78). If we take  $\varphi(\sigma_0) = \langle \varphi(\sigma_0 - 0), \varphi(\sigma_0 + 0) \rangle$  and  $\rho_{k+1}[\varphi(\sigma_0 - 0) - \varphi(\sigma_0 + 0)] < 0$ , we conclude that the trajectories pass through those points of the hyperplane  $\sigma = \sigma_0$  where either  $r^*P^i x \neq 0$  for at most one  $i \leq k$ , or  $r^*P^i x = 0$  for all  $1 \leq i \leq k$ , but  $r^*P^{k+1}x \notin [\rho_{k+1}\varphi(\sigma_0 - 0), \rho_{k+1}\varphi(\sigma_0 + 0)]$ . At the same time, the manifold of a sliding mode is described by

$$r^*x = \sigma_0, \quad r^*Px = \dots = r^*P^kx = 0, \tag{1.83}$$

$$\rho_{k+1}\varphi(\sigma_0 - 0) \leq r^*P^{k+1}x \leq \rho_{k+1}\varphi(\sigma_0 + 0). \tag{1.84}$$

The boundary of the manifold of a sliding mode (1.83), (1.84) is given by the formulas (1.83) and by one of the inequalities

$$r^*P^{k+1}x = \rho_{k+1}\varphi(\sigma_0 + 0) \quad \text{or} \quad r^*P^{k+1}x = \rho_{k+1}\varphi(\sigma_0 - 0).$$

It is worth noting that an  $n$ -dimensional system of differential equations (1.79) describes a sliding mode in  $\mathbb{R}^n$ . Since the trajectories of this sliding mode lie on an  $(n - k - 1)$ -dimensional manifold (1.83), (1.84), on this manifold they satisfy some  $(n - k - 1)$ -dimensional linear system of differential equations of a sliding mode (we will not present this system here). By the corollary to Theorem 1.8, the characteristic polynomial of this linear system is equal up to sign to the numerator of the transfer function. Therefore, the dimension of a sliding mode manifold can be judged from a transfer function only.

Assume now that in case of  $\rho_1 \neq 0$  the inequality opposite to (1.80) is true, namely,

$$\rho_1[\varphi(\sigma_0 - 0) - \varphi(\sigma_0 + 0)] > 0.$$

As before, it is easy to verify that trajectories pass through those parts of the hyperplane  $r^*x = \sigma_0$  which additionally satisfy either  $r^*Px < \rho_1\varphi(\sigma_0 + 0)$ , or  $r^*Px > \rho_1\varphi(\sigma_0 - 0)$ . Take an initial point  $x_0$  on the manifold

$$r^*x = \sigma_0, \quad \rho_1\varphi(\sigma_0 + 0) \leq r^*Px_0 \leq \rho_1\varphi(\sigma_0 - 0)\}. \tag{1.85}$$

Since  $[\sigma(t) - \sigma_0]\dot{\sigma} > 0$  in a sufficiently small neighborhood of the point  $x_0$  when  $\sigma(t) \neq \sigma_0$ , trajectories of continuous systems (1.75), defined on both sides of the hyperplane  $\sigma = \sigma_0$ , do not connect, but “diverge” in the vicinity of  $x_0$  (see Fig. 1.1 (c)).

Therefore, we can emit from this point at least three trajectories. The first one is defined by the linear system (1.79) for  $k = 0$  and lies on the manifold (1.85). The second trajectory goes to the half-space  $\sigma > \sigma_0$  and

satisfies the continuous system (1.75) with  $\varphi(\sigma_0) = \varphi(\sigma_0+0)$  in a sufficiently small neighborhood of the point  $x_0$ . The third trajectory goes to the half-space  $\sigma < \sigma_0$  and satisfies the continuous system (1.75) with  $\varphi(\sigma_0) = \varphi(\sigma_0 - 0)$ .

Similar reasoning is valid in the case (1.78). Namely, if  $\rho_{k+1}[\varphi(\sigma_0 - 0) - \varphi(\sigma_0 + 0)] > 0$ , trajectories can leave a point of sliding modes manifold (1.83), (1.84) into the half-space  $\sigma > \sigma_0$ , as well as into the half-space  $\sigma < \sigma_0$ .

Now let us consider the system (1.54), (1.55) under the assumption  $m > 1$ ,  $\xi = \varphi(\sigma)$ . When  $m > 1$ , a qualitative portrait of this system is more complicated and, generally speaking, the above arguments cannot be applied. However, in the specific case when the matrix  $-r^*q$  is diagonal and nonsingular, the analysis is substantially simplified. In this case, even for a partial sliding mode, the corresponding components of the extended nonlinearity  $\xi(t)$  can be expressed in terms of the linear combinations of the components of the vector  $x(t)$ .

Indeed, let  $\rho_1, \dots, \rho_m$  be the diagonal elements of the matrix  $-r^*q$ , and  $h_i^*$  be the  $i$ th row of the matrix  $r^*P$ . By differentiating the function  $\sigma_i(t)$  along the solutions of system (1.54), we get an equality  $\dot{\sigma}_i = h_i^*x - \rho_i\varphi_i(\sigma_i)$ . Apply the above examinations to each of these relationships to verify that if  $\sigma_i^0$  is a point where the function  $\varphi_i(\sigma_i)$ ,  $\varphi_i(\sigma_i^0) = \langle \varphi_i(\sigma_i^0 - 0), \varphi_i(\sigma_i^0 + 0) \rangle$  is multivalued, and  $\rho_i[\varphi_i(\sigma_i^0 - 0) - \varphi_i(\sigma_i^0 + 0)] < 0$ , then the trajectories pass through those parts of the hyperplane  $\sigma_i = \sigma_i^0$  where either  $h_i^*x > \rho_i\varphi_i(\sigma_i + 0)$ , or  $h_i^*x < \rho_i\varphi_i(\sigma_i^0 - 0)$ . At the same time, on the manifold

$$\sigma_i = \sigma_i^0, \quad \rho_i\varphi_i(\sigma_i^0 - 0) \leq h_i^*x \leq \rho_i\varphi_i(\sigma_i^0 + 0),$$

the trajectories are in a partial sliding mode with respect to a coordinate  $\sigma_i$ , where the  $i$ th component  $\xi_i$  of the extended nonlinearity  $\xi$  is given by the formula

$$\xi_i(t) = h_i^*x(t)/\rho_i. \quad (1.86)$$

The last formula will be useful in Chapter 3. It is easily seen that a trajectory can leave the manifold of a partial sliding mode through the boundary

$$\sigma_i = \sigma_i^0, \quad h_i^*x = \rho_i\varphi_i(\sigma_i^0 + 0)$$

only into the half-space  $\sigma_i > \sigma_i^0$ , and through the boundary

$$\sigma_i = \sigma_i^0, \quad h_i^*x = \rho_i\varphi_i(\sigma_i^0 - 0)$$

only into the half-space  $\sigma_i < \sigma_i^0$ .

Even for a diagonal matrix  $r^*q$  the differential equations of a partial sliding mode are nonlinear, and they will not be studied here.

Let us return to the study of properties of an extended nonlinearity.

**Theorem 1.9** *Suppose that in system (1.75) the nonlinear function  $\varphi(\sigma)$  is piecewise single-valued, and at the points, where it is single-valued, there exists a piecewise continuous derivative  $d\varphi/d\sigma$ , with  $|d\varphi/d\sigma|$  bounded when  $|\sigma|$  is bounded. If  $\rho_1 = -r^*q \neq 0$  then for any interval  $[a, b]$ , where a solution of (1.75) exists, the extended nonlinearity  $\xi(t)$  can be expressed in the form of*

$$\xi(t) = g(t) + s(t) \tag{1.87}$$

where  $g(t)$  is an absolutely continuous function, and  $s(t)$  is a saltus function of the form

$$s(t) = \sum_{k=1}^{\infty} \lambda_k 1(t - t_k). \tag{1.88}$$

Here  $\lambda_k$  are some numbers,  $|\lambda_1| + |\lambda_2| + \dots < \infty$ ;

$$1(t) = \begin{cases} 1, & t > 0, \\ 0, & t < 0. \end{cases}$$

If  $\rho_1 = 0$  then a similar formula is valid for the function  $h^*x(t)\xi(t)$  with  $h^* = r^*P$ .

*Proof.* Without loss of generality, we suppose that  $|\sigma(t) - \sigma_0| < \Delta$  for  $t \in [a, b]$  where  $\sigma_0$  is a point where the function  $\varphi(\sigma)$ , is multivalued, and  $\Delta$  is such a number, that all the other points, where  $\varphi(\sigma)$  is multivalued, are offset by a distance greater than  $\Delta$  from  $\sigma_0$ .

Consider first the case  $\rho_1 \neq 0$ . From the above arguments it is seen that  $\xi(t)$  has discontinuities only of the first kind, the discontinuity points  $t_k$  are isolated, and hence they form an at-most-countable set. Denote this set by  $\mathcal{E}$  and its closure by  $\bar{\mathcal{E}}$ . Since  $\bar{\mathcal{E}}$  does not contain any intervals, the set  $\mathcal{M} = [a, b] \setminus \mathcal{E}$  is an everywhere dense subset of  $[a, b]$ . By setting  $\lambda_k = \xi(t_k + 0) - \xi(t_k - 0)$  in (1.88), we ensure that the function  $g(t)$ , given by (1.87), is continuous on  $[a, b]$ . Let us verify that  $g(t)$  meets the Lipschitz condition on  $[a, b]$  and therefore is absolutely continuous. Since  $g(t)$  is continuous, it suffices to check that

$$|g(\tau') - g(\tau'')| \leq L_1 |\tau' - \tau''| \tag{1.89}$$

where  $\tau', \tau'' \in \mathcal{M}$  ( $\tau' < \tau''$ ). Henceforth  $L_i$  are positive constants whose values are of no significance to us.

From the preceding analysis of a qualitative portrait in a vicinity of the hyperplane  $\sigma = \sigma_0$  ( $\sigma_0$  is a point where the function  $\varphi(\sigma)$  is multivalued) it follows that if  $t^*$  is a point of accumulation of the discontinuity times  $t_k$  then  $x(t^*)$  lies on the frontier (1.81) of the sliding modes manifold. For sake of certainty, assume that (1.80) is satisfied and

$$r^*x(t^*) = \sigma_0, \quad r^*Px(t^*) = \rho_1\varphi(\sigma_0 + 0).$$

Let  $t'$  be the nearest to  $\tau'$  from the right point of the set  $\bar{\mathcal{E}}$ , and  $t''$  be the nearest to  $\tau''$  from the left point of the same set. The existence of these points follows from the closedness of  $\bar{\mathcal{E}}$ .

On the intervals, which do not contain any points of  $\bar{\mathcal{E}}$ ,  $\xi(t)$  satisfies the Lipschitz condition with some constant which is common for all such intervals. The reason is that, as was shown above, either  $\xi(t) = \varphi(\sigma(t))$  and  $d\varphi/d\sigma$  is piecewise continuous at the point  $\sigma(t)$ , or  $\xi(t) = r^*Px(t)/\rho_1$  and the functions  $\sigma(t)$ ,  $x(t)$  satisfy the Lipschitz condition on the interval  $[a, b]$ . Therefore, to prove estimate (1.89) it is sufficient to verify that

$$|g(t') - g(t'')| \leq L_2|t' - t''|. \tag{1.90}$$

Consider first the case when  $t', t'' \in \mathcal{E}$ , i.e., when  $t'$  and  $t''$  are the discontinuity points of the function  $\xi(t)$ . Let  $\bar{t}$  be a point of discontinuity of  $\xi(t)$  which is nearest to  $t''$  from the left. From (1.87) it follows that

$$g(t') - g(t'') = \xi(t' - 0) - \xi(t'' - 0) + s(t'' - 0) - s(t' - 0).$$

Without loss of generality, we can assume that at the points of discontinuity of  $\xi(t)$ , which range from  $t'$  to  $t''$ , the trajectory approaches the sliding mode manifold from the half-space  $\sigma > \sigma_0$  (see Fig. 1.12 (a)) and hence

$$\xi(t' - 0) - \xi(t'' - 0) = \xi(\bar{t} - 0) = \varphi(\sigma_0 + 0).$$

Therefore, to prove (1.90) it suffices to check the inequality

$$|s(t'' - 0) - s(t' - 0)| \leq L_3|t'' - t'|. \tag{1.91}$$

Since  $s(t'' - 0) = s(\bar{t} + 0)$ , we have

$$s(t'' - 0) - s(t' - 0) = \sum_{t' \leq t_k \leq \bar{t}} \lambda_k. \tag{1.92}$$

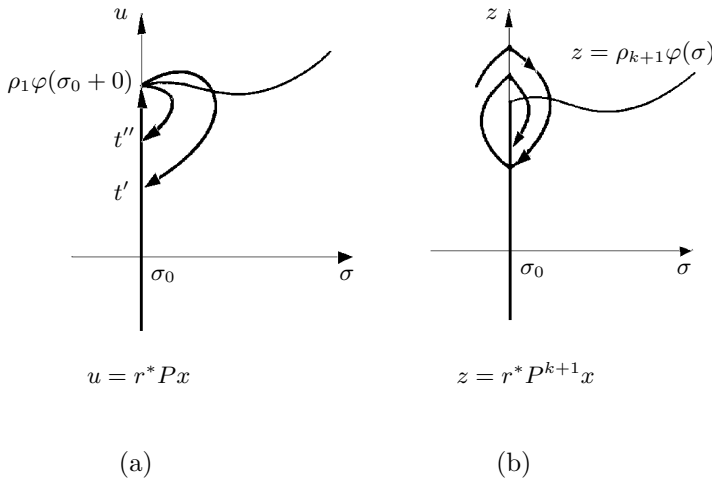


Fig. 1.12

As was shown above,  $\xi(t) = r^*Px/\rho_1$  on the sliding mode. Hence

$$\lambda_k = r^*Px(t_k)/\rho_1 - \varphi(\sigma_0 + 0) = [u(t_k) - u(t'_k)]/\rho_1$$

where  $u(t) = r^*Px(t)$  and  $t'_k$  is the first time, following  $t_k$ , when the trajectory reaches the boundary of the sliding mode. Denote by  $t_j$  the time of discontinuity of  $\xi(t)$  which follows  $t_k$ . Then, obviously<sup>13</sup>,  $t_k < t'_k < t_j$ . Since the function  $u(t)$  satisfies the Lipschitz condition on  $[a, b]$ ,

$$|\lambda_k| \leq L_4 |t_k - t'_k| < L_4 |t_k - t_j|. \tag{1.93}$$

From this inequality it follows that the series  $|\lambda_1| + |\lambda_2| + \dots$  converges, and (1.91) is valid with  $L_3 = L_4$ .

Let us turn now to the case when at least one of the time instants  $t', t''$  is an accumulation point for discontinuity times. Obviously, the times  $t_k$  can accumulate to  $t'$  only from the right and to  $t''$  only from the left. For the sake of certainty, let both the times  $t', t''$  be points of accumulation. Using  $L_4, t'$  and  $t''$ , we can choose a discontinuity time  $t_k > t'$  so close to  $t'$  that the inequality

$$|g(t_k) - g(t')| \leq L_4 |t'' - t'| \tag{1.94}$$

<sup>13</sup>The time instants  $t_k$  may be enumerated not in increasing order.

holds. This can be done owing to the continuity of the function  $g(t)$ . Similarly, let us find a discontinuity time  $t_i \in (t_k, t'')$  such that the estimate

$$|g(t'') - g(t_i)| \leq L_4|t'' - t'| \tag{1.95}$$

is valid. We can write

$$g(t') - g(t'') = [g(t') - g(t_k)] + [g(t_k) - g(t_i)] + [g(t_i) - g(t'')].$$

In the same way as estimate (1.91) was obtained, the second expression in square brackets can be majorized in absolute magnitude with  $L_3|t' - t''|$ . Whence and from (1.94), (1.95) we get (1.90). The theorem is proved for the case  $\rho_1 \neq 0$ .

If  $\rho_1 = 0$ , the preceding arguments fail in the point where we majorized the value  $|\lambda_k|$ . The reason is that when  $\rho_1 = 0$  the point  $t^*$  of accumulation of the discontinuities of the function  $\xi(t)$  may be a point of discontinuity of the second kind, because it is possible that  $t_k \rightarrow t^*$  while  $|\lambda_k| = |\varphi(\sigma_0 + 0) - \varphi(\sigma_0 - 0)| \neq 0$ . Such is the case for a trajectory which “curls” around the boundary of a sliding mode manifold as  $t \rightarrow t^*$ , and passes through the hyperplane  $\sigma = \sigma_0$  at times  $t_k$  (see Fig. 1.12 (b)). However, the function  $v(t) = u(t)\xi(t)$  will be continuous at the point  $t^*$ , because a trajectory is on the boundary of the sliding mode at time  $t^*$ , so  $u(t^*) = 0$ . Therefore, it can be shown that the function  $v(t)$  can be written in the form  $v(t) = g(t) + s(t)$  where  $g(t)$  is an absolutely continuous function and  $s(t)$  is a saltus function given by the formula (1.88) with  $\lambda_k = v(t_k + 0) - v(t_k - 0)$ .

The proof of the fact that the function  $g(t)$  satisfies the Lipschitz condition follows the scheme given above (see the case  $\rho_1 \neq 0$ ). The only difference is in the method of obtaining the estimate for  $|\lambda_k|$ . Let us discuss this in more detail. Since  $\dot{\sigma} = u(t)$ , there is a time instant  $\tau_k \in (t_k, t_j)$ , lying between the consecutive times  $t_k$  and  $t_j$  at which a trajectory is on the hyperplane  $\sigma = \sigma_0$ , and such that  $u(\tau_k) = 0$ . Whence

$$\begin{aligned} |\lambda_k| &= |v(t_k + 0) - v(t_k - 0)| = |u(t_k)| \cdot |\xi(t_k + 0) - \xi(t_k - 0)| \\ &\leq |\varphi(\sigma_0 + 0) - \varphi(\sigma_0 - 0)| \cdot |u(t_k) - u(\tau_k)| \leq L_5|t_k - t_j|, \end{aligned}$$

because the function  $u(t)$  satisfies the Lipschitz condition. Thus, the estimate (1.93) is proved and the proof of Theorem 1.9 is complete. ■

**Theorem 1.10** Consider system (1.54), (1.55) with  $l = m$ . Let the  $i$ -th component  $\varphi_i$  of the vector  $\varphi(\sigma)$  be a piecewise single-valued function of  $\sigma_i$ , i.e., of the  $i$ -th component of the vector  $\sigma$ . Suppose that at the points

where the function  $\varphi_i$  is single-valued there exists a piecewise continuous derivative  $d\varphi_i/d\sigma_i$  with  $|d\varphi_i/d\sigma_i|$  bounded when  $|\sigma_i|$  is bounded.

If the matrix  $r^*q$  is diagonal and nonsingular, then for any interval  $[a, b]$ , where a solution of system (1.54), (1.55) exists, the extended nonlinearity  $\xi(t)$  can be written in the form

$$\xi(t) = g(t) + s(t)$$

where  $g(t)$  is an absolutely continuous  $m$ -dimensional vector function and

$$s(t) = \sum_{k=1}^{\infty} \lambda_k 1(t - t_k)$$

with  $m$ -dimensional constant vectors  $\lambda_k$  having the property  $|\lambda_1| + |\lambda_2| + \dots < \infty$ .

The proof of this theorem follows the same line of reasoning as that of Theorem 1.9 in the case  $\rho_1 \neq 0$ . It is based on the above discussion of the behavior of the trajectories of the system (1.54), (1.55) in a neighborhood of a manifold of a partial sliding mode. It is worth noting that from Theorem 1.10 with  $m = 1$  follows the statement of Theorem 1.9 with  $\rho_1 \neq 0$ .

Theorems 1.9 and 1.10 will be used in the next chapter when we investigate the discontinuous Lyapunov functions.

### 1.3 Dichotomy and Stability

In this section the notions of dichotomy, quasi-gradient-like behavior, global asymptotic stability and piecewise global asymptotic stability are introduced. The Lyapunov-type lemmas, which contain sufficient conditions for these properties to hold, are presented. These lemmas are used in the subsequent chapters.

#### 1.3.1 Basic definitions

Consider a differential inclusion

$$\frac{dx}{dt} \in f(x, t). \quad (1.96)$$

Here  $f(x, t)$  is a multiple-valued semicontinuous vector function, defined for  $-\infty < t < +\infty$ ,  $x \in \mathbb{R}^n$ , which associates every point  $(x_0, t_0)$  with a bounded, closed, convex set  $f(x_0, t_0)$ . A local theorem on the existence of solutions to system (1.96) was proved in Section 2.2.

**Definition 1.7** A vector  $c$  is called *stationary* for the system (1.96) if  $x(t) \equiv c$  is a solution of this system. The set all stationary vectors of (1.96) is called a *stationary set*.

From Theorem 1.4 it follows that the stationary set of the system (1.96) is closed.

**Definition 1.8** System (1.96) is called *dichotomic*<sup>14</sup> if any solution, bounded for  $t > 0$ , tends to the stationary set as  $t \rightarrow +\infty$ .

Thus, if a system is dichotomic, then for any of its solutions there is an alternative: either it is unbounded for  $t > 0$ , or tends to the stationary set as  $t \rightarrow +\infty$ .

**Definition 1.9** System (1.96) is said to be *pointwise dichotomic* if all its bounded solutions approach an equilibrium as  $t \rightarrow +\infty$ .

Dichotomy is a weaker property than stability, because it does not presume the lack of unbounded solutions. However, which is very important for applications, there cannot be any auto-oscillations in a dichotomic system.

Following [Hale (1987)], let us introduce the following notions.

**Definition 1.10** System (1.96) is said to be *gradient-like* if all of its solutions tend to an equilibrium as  $t \rightarrow +\infty$ .

**Definition 1.11** System (1.96) is called *quasi-gradient-like* if any solution tends to the stationary set as  $t \rightarrow +\infty$ .

**Definition 1.12** A stationary set  $\Lambda$  of system (1.96) is *Lyapunov stable (stable in the small)* if for any  $\varepsilon > 0$  there exists  $\delta > 0$  such that for any solution  $x(t)$ , satisfying the condition  $\rho[x(t_0), \Lambda] < \delta$ , the inequality  $\rho[x(t), \Lambda] < \varepsilon$  holds for all  $t > t_0$ .

In other words, a stationary set  $\Lambda$  is Lyapunov stable if all the trajectories, starting at  $t = t_0$  from the points sufficiently close to  $\Lambda$ , for all  $t > t_0$  remain as close to  $\Lambda$  as desired.

**Definition 1.13** A stationary set of system (1.96) is called *globally stable* if it is Lyapunov stable and, in addition, the system is quasi-gradient-like.

<sup>14</sup>Sometimes, following R.E. Kalman [Kalman (1957)], this property is called *monostability*.

**Definition 1.14** A stationary set of system (1.96) is *pointwise globally stable* if it is globally stable and any solution tends to some stationary vector as  $t \rightarrow +\infty$ .

The concept of gradient-like behavior is important for those systems, for which not all stationary points are stable in the small. E.g., most of the pendulum-like systems, considered in the last chapter of the book, have such a peculiarity.

As for the difference between global stability and pointwise global stability, it is straightforward. When a system is globally stable, it is possible that a trajectory “curls” around the stationary set but does not have a specific limit (see Fig. 1.11 (a)). When a system is pointwise globally stable, each trajectory either reaches a stationary point in a finite time, or tends to a stationary point as  $t \rightarrow +\infty$  (see Fig. 1.11 (b)). In Chapter 3 we will examine a number of systems whose stationary sets are globally stable or pointwise globally stable.

Observe that different types of stability of manifolds for systems with continuous right-hand sides were studied in [Yoshizawa (1963)].

### 1.3.2 Lyapunov-type lemmas

When studying dichotomy and all the types of stability described above, the second method of Lyapunov will be used. Here we present a number of auxiliary statements which will be useful in the subsequent chapters.

Consider the case when system (1.96) is autonomous, i.e., has the form

$$\frac{dx}{dt} \in f(x). \quad (1.97)$$

**Lemma 1.4** *Let there exist a continuous function  $V(x)$  defined in  $\mathbb{R}^n$  and having the properties:*

- (i)  $V[x(t)]$  is nonincreasing in  $t$  for any solution  $x(t)$  of (1.97);
- (ii) if an identity  $V[x(t)] = \text{const}$  is valid for all  $-\infty < t < \infty$  and for some solution  $x(t)$ , bounded when  $-\infty < t < \infty$ , then the solution  $x(t)$  is a stationary vector.

*Then system (1.97) is dichotomic.*

*Proof.* Let  $x(t, x_0)$  be any trajectory of system (1.97), which is bounded when  $t > 0$  and passes through the point  $x_0$  at  $t = 0$ , i.e.,  $x(0, x_0) = x_0$ .

The function  $V[x(t, x_0)]$  is bounded for  $t > 0$  and, according to the first

condition of the lemma, is nonincreasing in  $t$ . Hence there exists a limit

$$\lim_{t \rightarrow +\infty} V[x(t, x_0)] = \gamma(x_0). \quad (1.98)$$

Since the trajectory  $x(t, x_0)$  is bounded for  $t > 0$ , the set  $\Omega$  of its  $\omega$ -limiting points is not empty. By Theorem 1.5, for any point  $y$  from the set  $\Omega$  there is a trajectory  $x(t, y)$  which passes through this point and consists of  $\omega$ -limiting points of the trajectory  $x(t, x_0)$ . From (1.98) it follows that  $V[x(t, y)] = \gamma(x_0)$  for all  $t$ . Then, from the second condition of the lemma, the trajectory  $x(t, y)$  coincides with some stationary vector of system (1.97) for all  $t$ , including  $t = 0$ . So the inclusion  $\Omega \subset \Lambda$  is proved.

Assume now that  $x(t, x_0)$  does not tend to  $\Lambda$  as  $t \rightarrow +\infty$ . Then there exist a number  $\alpha > 0$  and a sequence  $t_k \rightarrow +\infty$  such that  $\rho[x(t_k, x_0), \Lambda] > \alpha$ . So, an accumulation point of the set  $\{x(t_k, x_0)\}$  is an  $\omega$ -limiting point of the trajectory  $x(t, x_0)$  and lies outside  $\Lambda$ , which contradicts the inclusion  $\Omega \subset \Lambda$  proved previously. ■

**Lemma 1.5** *Suppose that there exists a continuous function  $V(x)$  defined in  $\mathbb{R}^n$  which satisfies the conditions (i) and (ii) of Lemma 1.4 and the additional condition*

$$(iii) \quad V(x) \rightarrow \infty \text{ as } |x| \rightarrow +\infty.$$

*Then system (1.97) is quasi-gradient-like.*

*Proof.* From condition (i) of Lemma 1.4,  $V[x(t)] \leq V[x(0)]$ . Whence and from property (iii), the solution  $x(t)$  is bounded for  $t > 0$ . Then, by Lemma 1.4 and Definitions 1.7 and 1.8, system (1.97) is quasi-gradient-like. ■

**Lemma 1.6** *Suppose that the stationary set  $\Lambda$  of (1.97) is bounded and there is an  $\varepsilon$ -neighborhood  $\Lambda_\varepsilon$  of the set  $\Lambda$  such that there exists a continuous function  $V(x, c)$  defined for all  $x \in \Lambda_\varepsilon$ ,  $c \in \Lambda$  and having the properties:*

$$(i) \quad V(x, c) > 0 \text{ for } x \in \Lambda_\varepsilon \setminus \Lambda,$$

$$(ii) \quad V(c, c) = 0,$$

*(iii) for any solution  $x(t)$  of system (1.97),  $V[x(t), c]$  is nonincreasing in  $t$  when  $x(t) \in \Lambda_\varepsilon$ .*

*Then the stationary set  $\Lambda$  is Lyapunov stable.*

*Remark.* The function  $V(x, c)$  can satisfy all the conditions of Lemma 1.6 and be nonvanishing when  $c \neq x \in \Lambda$ . If a function  $V(x, c)$  satisfies the conditions of Lemma 1.6 and does not depend on  $c$ , then, by the

property (ii), it has to vanish in all the stationary set. In the latter case, Lemma 1.6 is an extension of the well-known Lyapunov theorem on stability in the small for systems with multivalued right-hand sides.

*Proof* of Lemma 1.6. Let us demonstrate that the set  $\Lambda$  is Lyapunov stable. Suppose the contrary. Then there exist a number  $\eta$ , satisfying the inequality  $0 < \eta < \varepsilon$ , and sequences  $\delta_k \rightarrow 0$ ,  $x_k \in \Lambda_{\delta_k}$ ,  $t_k$  such that  $\rho[x(t, x_k), \Lambda] < \eta$  for  $0 \leq t < t_k$  and

$$\rho[x(t_k, x_k), \Lambda] = \eta.$$

For any point  $x_k$  there exists a point  $c_k$  from the set  $\Lambda$  such that  $\rho(x_k, c_k) < \delta_k$ . By the third condition of the lemma,

$$V[x(t_k, x_k), c_k] \leq V(x_k, c_k). \tag{1.99}$$

Let  $\mu = \inf V(x, c)$  over  $c \in \Lambda$  and over  $x$  belonging to the closed manifold described by the equation

$$\rho(x, \Lambda) = \eta.$$

By the first condition of the lemma,  $\mu > 0$ . From (1.99), the inequality  $\mu \leq V(x_k, c_k)$  follows. The right-hand side of this inequality approaches zero as  $\delta_k \rightarrow 0$ , whereas its left-hand side is positive and does not depend on  $k$ . The contradiction obtained proves the Lemma 1.6. ■

**Lemma 1.7** *Assume that system (1.97) is quasi-gradient-like, all the hypotheses of Lemma 1.6 are satisfied, and the equality  $V(x, c) = 0$  can be valid only for  $x = c$ . Then the stationary set  $\Lambda$  of system (1.97) is pointwise globally stable.*

*Proof.* To prove the lemma it suffices to show that the limit

$$\lim_{t \rightarrow +\infty} x(t, x_0)$$

exists. Suppose the contrary. Then there exist two different  $\omega$ -limiting points  $c_1$  and  $c_2$  from  $\Lambda$ . Let  $\beta$  be a distance between them, and let  $\tau_k$  be a sequence such that  $x(\tau_k, x_0) \rightarrow c_1$  as  $\tau_k \rightarrow \infty$ , and  $\tau_m$  be so large that  $x(t, x_0) \in \Lambda_\varepsilon$  for  $t > \tau_m$ . From the third condition of Lemma 1.3, the inequality  $V[x(t, x_0), c_1] \leq V[x(\tau_m, x_0), c_1]$  is valid for  $t > \tau_m$ . Since  $V(x, c_1) = 0$  only for  $x = c_1$ , we conclude that the trajectory  $x(t, x_0)$  does not leave a  $\beta/2$ -neighborhood of  $c_1$  when  $\tau_m$  is large enough and  $t > \tau_m$ . Then the point  $c_2$  cannot be  $\omega$ -limiting for this trajectory. ■

It is worth noting that a number of authors obtained a rich variety of Lyapunov-type theorems, i.e., such theorems where the problem of stability of invariant sets is reduced to the problem of finding Lyapunov functions with certain properties. It is well known that the main problem is in constructing such functions.

Based on algebraic results of Chapter 2, further we will obtain frequency-domain criteria necessary and sufficient for the existence of Lyapunov functions of a certain class. These criteria satisfy conditions of Lemmas 1.4–1.7 and hence ensure dichotomy or some type of stability. Since these criteria are necessary, they cannot be improved with the help of Lyapunov functions of a prescribed class.

Observe that Lyapunov type theorems for systems with continuous nonlinearities were proved by different authors. See, e.g., monographs [Malkin (1966); Krasovskii (1963); Lefschetz (1965)] and a review [Rumyantsev (1968)]. Remark also that an important generalization and development of the second Lyapunov method [Lyapunov (1963)] was given in [Matrosov *et al.* (1980)].