has important physical consequences that will be brought up in Chapter 6. Although the minimum value of q^2 can be zero, this corresponds to no scattering; the maximum value (a rare occurrence, indeed!) is $4P^2$. Although Eq. (1.73) was obtained using nonrelativistic kinematics, it also holds, in fact, as $v^2 \to c^2$ (see, however, our comments in the section "Sizes of Nuclei" in Chapter 2).

1.7 Quantum Treatment of Rutherford Scattering

We arrived at Eq. (1.73) through a rather circuitous classical route. We will now end this section by sketching how the Rutherford cross section can be calculated using quantum mechanics. This will be done through an application of Fermi's Golden Rule,[4] according to which the transition probability to continuum states per unit time in perturbation theory is given by

$$P = \frac{2\pi}{\hbar} |H_{fi}|^2 \rho(E_f) \tag{1.74}$$

where $\rho(E_f)$ is the density of final states and H_{fi} denotes the matrix element of the perturbation Hamiltonian between the initial and the final states

$$H_{fi} = \langle f|H|i\rangle = \int d^3r\, \psi_f^*(\vec{r})H(r)\psi_i(\vec{r}). \tag{1.75}$$

For the case of elastic Rutherford scattering, the wave functions are plane waves, corresponding to free-particle states approaching (i) and leaving (f) the scattering center, and the perturbation Hamiltonian is the Coulomb potential energy given in Eq. (1.23). For the incident and outgoing momenta $\vec{p}$ and $\vec{p}\,'$, respectively, we can define the wave vectors $\vec{k} = \frac{\vec{p}}{\hbar}$ and $\vec{k}' = \frac{\vec{p}\,'}{\hbar}$, and a momentum transfer that results from the scattering $\vec{q} = \hbar(\vec{k}' - \vec{k})$. Except for an overall normalization of the wave functions, our matrix element H_{fi} can now be written as

$$H_{fi} \approx \int_{\substack{\text{all}\\ \text{space}}} d^3r\, e^{i\vec{k}'\cdot\vec{r}}\, V(r)\, e^{-i\vec{k}\cdot\vec{r}} = \int_{\substack{\text{all}\\ \text{space}}} d^3r\, V(r)\, e^{\frac{i}{\hbar}\vec{q}\cdot\vec{r}}. \tag{1.76}$$

[4]A discussion of this famous result for transitions between states can be found in standard texts on quantum mechanics.

The integral on the right is the Fourier transform of $V(r)$, and can be thought of as the potential energy in momentum space. Doing the integration,[5] we find that

$$V(\vec{q}) = \int_{\substack{\text{all} \\ \text{space}}} d^3r\, V(r)\, e^{\frac{i}{\hbar}\vec{q}\cdot\vec{r}} = \frac{(ZZ'e^2)(4\pi\hbar^2)}{q^2}. \tag{1.77}$$

Evaluating the density of final states,[6] substituting into Eq. (1.74), and relating the transition probability to the scattering cross section, leads to the same expression as obtained in Eq. (1.73). Thus Rutherford's result, without any apparent reference to $\hbar$, is also in agreement with quantum mechanics (when effects of intrinsic spin are ignored).

Problems

1.1 Using Eq. (1.38) calculate the approximate total cross sections for Rutherford scattering of a 10 MeV α-particle from a lead nucleus for impact parameters b less than 10^{-12}, 10^{-10} and 10^{-8} cm. How well do these agree with the values of πb^2?

1.2 Prove that Eq. (1.55) follows from the relations in Eqs. (1.53) and (1.54).

1.3 Sketch $\cos\theta_{\mathrm{Lab}}$ as a function of $\cos\theta_{\mathrm{CM}}$ for the nonrelativistic elastic scattering of particles of unequal mass, for the cases when $\zeta = 0.05$ and $\zeta = 20$ in Eqs. (1.52) and (1.53).

1.4 What would be the approximate counting rate observed in the Rutherford scattering of 10 MeV α-particles off lead foil at an angle of $\theta = \frac{\pi}{2}$ in the laboratory? Assume an incident flux of 10^6 α-particles per second on the foil, a foil 0.1 cm thick, and a detector of transverse area 1 cm × 1 cm placed 100 cm from the interaction point, and density of lead of 11.3 g/cm^3. What would be the counting rate at $\theta = 5°$? By about how much

[5]The Fourier transform corresponds to a generalization of the Fourier decomposition of functions into series. Transforms of different functions can be found in mathematical tables and are useful for a variety of applications in physics. See, for example, L. Schiff, *Quantum Mechanics*, (New York, McGraw Hill, 1968); A. Das and A. C. Melissinos, *Quantum Mechanics*, (New York, Gordon & Breach, 1986); A. Das, Lectures on Quantum Mechanics, (New Delhi, Hindustan Book Agency 2003).

[6]See a discussion of this issue, and matters pertaining to this entire section, in A. Das and A. C. Melissinos, *Quantum Mechanics*, pp 199-204, A. Das, Lectures on Quantum Mechanics, (New Delhi, Hindustan Book Agency 2003).

would your answers change if the above angles were specified for the center-of-mass – be quantitative, but use approximations where necessary. (Why don't you have to know the area of the foil?)

1.5 Sketch the cross section in the laboratory frame as a function of $\cos\theta_{\rm Lab}$ for the elastic scattering of equal-mass particles when $\frac{d\sigma}{d\Omega_{\rm CM}}$ is isotropic and equal to 100 mb/sr. What would be your result for $\zeta = 0.05$ in Eq. (1.52)? (You may use approximations where necessary.)

1.6 Certain radioactive nuclei emit α particles. If the kinetic energy of these α particles is 4 MeV, what is their velocity if you assume them to be nonrelativistic? How large an error do you make in neglecting special relativity in the calculation of v? What is the closest that such an α particle can get to the center of a Au nucleus?

1.7 An electron of momentum 0.511 MeV/c is observed in the laboratory. What are its $\beta = \frac{v}{c}$, $\gamma = \left(1-\beta^2\right)^{-\frac{1}{2}}$, kinetic energy, and total energy?

1.8 What are the approximate values of the kinetic energy for the recoiling lead nucleus and the momentum transfers (in eV units) at the cutoffs specified in Problem 1.1?

1.9 Taking the ultrarelativistic limit of Eq. (1.71), find an approximate expression for $\theta_{\rm Lab}$ at $\theta_{\rm CM} = \frac{\pi}{2}$, and evaluate $\theta_{\rm Lab}$ for $\gamma_{\rm CM} = 10$ and $\gamma_{\rm CM} = 100$. Does the approximation hold best for particles with small or large mass values?

1.10 What is the minimum impact parameter needed to deflect 7.7 MeV α-particles from gold nuclei by at least 1^o? What about by at least 30^o? What is the ratio of probabilities for deflections of $\theta > 1^o$ relative to $\theta > 30^o$? (See the *CRC Handbook* for the density of gold.)

1.11 Consider a collimated source of 8 MeV α-particles that provides 10^4 α/sec that impinge on a 0.1 mm gold foil. What counting rate would you expect in a detector that subtends an annular cone of $\Delta\theta = 0.05$ rad, at a scattering angle of $\theta = 90^o$? Compare this to the rate at $\theta = 5^o$. Is there a problem? Is it serious (see Problem **1.12**). (*Hint*: You can use the small-angle approximation where appropriate, and find the density of gold in the *CRC Handbook*.)

1.12 Consider the expression Eq. (1.41) for Rutherford Scattering of α-particles from gold nuclei. Integrate this over all angles to obtain n. In principle, n cannot exceed N_0, the number of incident particles. Why?

What cut-off value for θ would be required in the integral, that is, some $\theta = \theta_0 > 0$, to assure that n does not exceed N_0 in Problem **1.4**? (*Hint*: After integrating, use the small-angle approximation to simplify the calculation.) Using the Heisenberg uncertainty principle $\Delta p_x \Delta x \approx \hbar$, where Δx is some transverse distance corresponding to a change in transverse momentum of $\Delta p_x = p_{in}\theta_0 \approx \sqrt{2mE}\theta_0$, calculate the distances Δx to which you have to restrict the description of the scattering. Are these distances sufficiently restrictive? Explain!

Suggested Readings

Geiger, H. and E. Marsden, Philos. Mag. **25**, 604 (1913).

Rutherford, E., Philos. Mag. **21** 669, (1911).

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